Towards Automatic Real-Time Controller Tuning and Robustness

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Abstract—In some industrial applications, real-time controller tuning is a time-consuming exercise; it can be costly and difficult when there are a multitude of control loops acting on a number of processes and where the complexity of the control problem requires robust and effective controller design. An automated tuning procedure could speed up this activity, but the tuning method should have good properties. This paper investigates whether certain families or groups of cost functions can provide an enhanced degree of robust controller design. An optimisation algorithm was developed to provide the optimal controller parameters using a number of integral-error cost functions. The robustness of each cost function was assessed using standard and parametric stability margins. For the class of systems investigated, it seems that exponentially- or time-weighted cost functions can consistently give more robust controller design.

Keywords—PI Control, Controller tuning, Robustness, Stability Margin

I. INTRODUCTION

PID control is still commonplace in most modern industrial applications due to its relatively simple structure and wide scope of use. However, the standard Ziegler-Nichols (Z-N) tuning rules that have been preferred for over 30 years are being increasingly challenged in both the industrial and research sectors by requirements for more stringent control performance specifications. The concept of controller robustness has thus become an integral part of control design. This may be viewed as a twofold problem; that of stability robustness and performance robustness.

Although much of the robust control literature has been dominated by the $H_{\infty}$ cost function design paradigm, [1], for example, the aim of the research reported in this paper is to find out whether some time-domain based cost functions can produce more robust control laws. For example, there is evidence that some PID controller tuning rules commonly used in the process industries produce a more robust controller tuning. Given the conclusions drawn by [2] and [3], the problem can be formulated in such a way as to give the following hypothesis for study.

Define the time weighted error-integral tuning criteria family as (1):

$$J = \int_{0}^{\infty} f(e(t), t) dt$$ (1)

The optimal set of controller parameters: $\rho^* \in \mathbb{R}^n$, with the number of controller parameters given by $n$ (where, for PID controllers, $n = 1, 2$ or $3$) and the cost function value given by the minimization of $J$, to obtain $\rho^*$ is defined as $J_{\text{min}}$. Then the study hypothesis can be stated as:

1. The error-integral tuning criteria can produce enhanced robust controller designs when the integral cost function includes any time-weighted component of the error signal.

The error-integral criteria are used to minimise a function of the feedback control system error signal. The unique and real number obtained from the minimisation process is taken to be a quantification of the controller performance, which can be compared to cost function values obtained from other controller parameter settings. An inherent assumption is the optimal controller settings obtained produce a stable closed loop system. The study hypothesis (1) is concerned with just how robust is this optimal choice and whether more robust results are obtained from a particular family of error-integral criteria.

This paper is organized as follows. Section II provides some background and insight to the controller robustness and stability theory in terms of criteria used to measure these properties. Section III presents an overview of controller tuning and introduces the cost function minimization concept for achieving desired controller performance. The methodology for testing the study hypothesis (1) is given in section IV, with results and analysis in section V. Finally, some remarks on extensions to this work and conclusions are provided in section VI.

II. CONTROL SYSTEM ROBUSTNESS AND STABILITY

A. Robustness Performance

Performance of the controllers for robust control is essentially a question of how well the control system retains the desired performance when the system parameters change to values that are different to the nominal system design values. A set of system indicators and measures are used to quantify the desired performance levels that the control system should attain. System transfer functions that are commonly used in control design for robust performance include the complementary sensitivity function ($T$) for reference tracking performance and sensor noise rejection and the sensitivity function ($S$), which is used for disturbance rejection analysis. In general, robust performance is seeking to retain features of the transfer functions, $S$ and $T$, over a given uncertainty class, so that even the worst case scenario should result in an acceptable control system performance.
A typical example of robust performance is the ability of the PI controller to guarantee steady-state reference tracking performance even if process parameter mismatch occurs. This is because the integral term will always eliminate constant offset in the error signal over time. In another example, PI has the ability to yield good disturbance rejection and attenuate constant disturbance effects acting on the system output.

The difficulty in achieving good robust performance arises from the relationship between the two sensitivity functions; namely, $S + T = 1$. Hence, the control system design problem is that of achieving better disturbance rejection or reference tracking despite the coupling between system transfer functions, $S$ and $T$.

**B. Measures of Controller Stability Robustness**

Stability robustness is the ability of the control system to retain closed loop stability over a class of process uncertainty variations. In classical control design there are a number of ways to quantify the process uncertainty and, subsequently, the robustness of the closed loop stability of the control system. For example, interpretations of the Nyquist and Bode plots for the system forward path transfer function leads to some of these stability robustness measures.

The most recognised measures for stability robustness are the *gain margin* (GM) and *phase margin* (PM), which quantify the distance of the closed loop system from being unstable when there are perturbations in the system. The GM calculation yields a measure of the allowable system gain change such that stability of the closed loop is maintained. Alternatively, the GM is the magnitude by which the Nyquist plot may be multiplied before encircling the $(-1 + j0)$ point. Similarly, for the PM calculation, the allowable range of phase in the transfer function $e^{-\theta}G(s)$, which maintains closed loop system stability, is found. The *delay margin* (DM) is defined as the smallest destabilising delay that can be introduced into the closed-loop system. There is a clear relationship between PM and DM given through the formula (2):

$$DM = \frac{PM}{\omega_1} \left( \frac{\pi}{180} \right)$$

where the gain crossover frequency is $\omega_1$.

In real systems there are two sets of parameters, those of the process and those of the controller. The set of process parameters usually arise from complex physical interactions in the process and it is unlikely that the polynomial coefficients of the plant transfer function will perturb independently of each other. This contrasts with the set of controller parameters, which are usually fixed during operation. However, the two sets of parameters have different behavioural characteristics. The plant parameters are subject to indefinable variations due to disturbances, modelling errors and environmental perturbations, whereas the controller could fail or slowly degrade over a period of time if the controller hardware was exposed to an aggressive operational environment.

For controller design, however, the parametric uncertainty must be included in order to get some indication of the likely closed loop stability under system perturbations. This leads to three problems:

i. Closed loop stability robustness regardless of changes in the process parameters;

ii. Closed loop stability robustness regardless of changes in the controller parameters;

iii. Closed loop stability robustness regardless of changes in both process and controller parameters.

In this paper, the study hypothesis $\circ$ is investigated for parametric changes under item i.

The problem of closed loop stability robustness with parametric uncertainty has led to the formulation of the parametric stability margin (PSM) measure. This has been used to improve the stability robustness analysis in the case where perturbations are introduced into the system parameters, [3]. The PSM can then be used, in addition to the classical stability criteria, to fully characterise the system stability robustness for a given tuning rule or a given controller tuning procedure. The PSM is defined as the smallest parametric perturbation that destabilises the closed loop system, as given by a specified norm measure.

Introduce a vector of system parameters, $q \in \mathbb{R}^{n_p}$, where the number of parameters under perturbation is, $n_p$. Define the nominal parameter vector as $q_0 \in \mathbb{R}^{n_p}$, then normalised parameter perturbations can be defined as $\beta_i = (q(i) - q_0(i))/q_0(i), i = 1, ..., n_p$. Hence, $\beta \in \mathbb{R}^{n_p}$ and the PSM is defined by (3):

$$PSM = \min ||\beta||_p$$

where $S$ indicates the stability-instability interface and $p$ denotes the norm type (e.g. 1, 2 or $\infty$). The engineering significance of the $\infty$-norm measure makes this a sensible choice.

**III. COST FUNCTIONS FOR PID TUNING**

The tuning process for PID control involves selecting the values of the three controller gains, $K_p$, $K_I$ and $K_D$ from controller equation (4), so that the controlled variable response is kept close to the desired closed-loop response.

$$C(s) = K_p + \frac{K_I}{s} + K_Ds$$

Calculating these gains may be performed using a number of techniques, such as open-loop, closed-loop and frequency response methods. Reference [4] suggested a method for tuning controllers based on the integral of the closed-loop error, $e(t)$. This was proposed as an advance on the $\frac{1}{2}$ decay ratio condition for the Z-N open-loop method, which may be considered too relaxed for modern controller designs, where faster settling times and smaller overshoots provide tighter control over fast dynamic processes. The error-integral criteria are suitable for the feedback control system, where the output error is to be minimised with respect to time. By integration of the error in time, a unique and real number is obtained by which some assessment of the controller tuning performance
can be made and compared to other controller parameter settings.

There are several criteria based on the error-integral function that have been suggested in the literature [2] and [4]. The standard cost function families are given as follows:

\[ ISE = \int_0^\infty e^2(t)dt \quad (5) \]

\[ IAE = \int_0^\infty |e(t)| dt \quad (6) \]

\[ ITAE = \int_0^\infty t |e(t)| dt \quad (7) \]

The Integral Square Error (ISE) is insensitive to small errors, but is weighted strongly for larger errors. The Integral of the Absolute Error (IAE) is weighted for smaller errors and is less sensitive to larger errors. The Integral of the Time-weighted Absolute Error (ITAE) is, by definition, weighted to error increasingly over time, such that errors occurring over a long time horizon are penalised. Reference [2] indicates that the ITAE criterion is the preferred performance cost function due to its ability to yield more conservative controller parameter settings and this is stated to mean that the ITAE cost function provide more robust controllers. Following this, the study hypothesis was formulated to investigate which time domain cost functions, popularly used in these optimization approaches yield enhanced stability robust control tunings. Section IV explains the methodology for testing the study hypothesis.

IV. METHODOLOGY

The hypothesis study used a common process system model, that of the first-order plus time delay (FOPTD) process model as shown by (8):

\[ G(s) = \frac{Ke^{-\theta s}}{\tau s + 1} \quad (8) \]

where \( K \) is the gain, \( \tau \) is the time constant and \( \theta \) is the time delay.

From (5), (6) and (7), the cost function families were rewritten in generalised form as:

\[ J_{ISE} = \int_0^\infty a^n e^2(t)dt \quad (9) \]

\[ J_{IAE} = \int_0^\infty a^n |e(t)| dt \quad (10) \]

\[ J_{CC} = \int_0^\infty a^n [f e(t)] + \lambda f[u(t)] dt \quad (11) \]

where \( a \) is a constant, exponential or time, \( n \) is a constant or time, \( \lambda \) is a weighting factor on the controller output signal and \( J_{CC} \) is a composite cost function family, with \( u(t) \) being the controller output.

Algorithm 1 Optimal Controller Calculation

Step 1 Initialisation

Choose simulation run time, \( T_f \);
Choose convergence tolerance, \( \varepsilon \);
Set \( R = I \); Choose initial step size, \( \gamma_0 \);
Set loop counter, \( k = 0 \);
Choose initial controller parameter vector, \( \rho(0) \)

Step 2 Loop step

Run simulation;
Calculate cost function, \( J(\rho(k)) \);
Compute gradient;
Introduce perturbation into \( \rho(k) \);
Run simulation;
Compute,

\[ \frac{\partial J}{\partial \rho_i} = \frac{J(\rho_i(k) + \Delta \rho_i) - J(\rho_i(k))}{\Delta \rho_i} \quad (12) \]

for \( i = 1, ..., n \)
Choose update parameter, \( \gamma_k = \chi \times \gamma_0 \);
Calculate parameter vector update:

\[ \rho(k + 1) = \rho(k) - \gamma_k R^{-1} \frac{\partial J}{\partial \rho} \quad (13) \]

Step 3 Convergence test

If \( \|\rho(k + 1) - \rho(k)\| < \varepsilon \) then stop
else \( k := k + 1 \) goto Step 2
nominal step size was set at $10^{-3}$ with the updating parameter ($\chi$) fixed at 1.2.

V. RESULTS

A. Verification of Algorithm 1

An initial simulation was performed to verify that the calculation of the cost function given by Algorithm 1 was correct. The simulation was run for 35 iterations using the $J_{IAE}$ cost function and the FOPTD model with a delay of 1. The cost function was plotted against number of iterations as shown in Fig. 2.

A verification of the initial value of $J_{IAE}$ is provided by evaluating the cost function value using analysis based on the final value theorem.

B. Testing the optimization function

The optimization function of Algorithm 1 was tested using the Linear Quadratic ($LQ$) cost function (a $J_{CC}$ type CF) described by [5] and shown in (14):

$$J_{LQ} = \frac{1}{2T_f} \int_0^{T_f} (e^2(t) + \lambda u^2(t))dt$$

(14)

where $\lambda$ is the controller signal weighting set at $10^{-3}$. As expected, the algorithm performance was almost identical to the iterative feedback tuning (IFT) method presented by [5]. A comparison of the LQ, IFT and Z-N tuning rules revealed that the Z-N gives the best response, but the IFT and LQ ($\lambda = 10^{-3}$) controller parameters, although similar, produced a poor tuning result. However, this does not detract from the fact that the optimisation Algorithm 1 performs in the correct way. In fact, it is the $LQ$ cost function, not the algorithm, that produces the poor tuning result.

The optimisation Algorithm 1 was tested against a number of different tuning methods for the $ISE$ cost function. The optimal controller parameters were taken from [6] for the Cohen-Coon (C-C), Z-N open and closed loop and Shinsky methods. A FOPTD process model was used with a PID controller in order to replicate the methodology described in [6] and the results of the comparison are shown in Fig. 3.

C. Cost function analysis

Sections V. A and V. B verified that Algorithm 1 produced an optimal controller for a given cost function. This section shows the results obtained in testing hypothesis ⊙ by analysis of several cost functions. The cost functions chosen for analysis were:

- Integral Square Error ($ISE$)
- Integral Time-weighted Square Error ($ITSE$)
- Integral Time-squared Square Error ($IT^2SE$)
- Integral Absolute Error ($IAE$)
- Integral Time-weighted Absolute Error ($ITAE$)
- Integral Time-squared Absolute Error ($IT^2AE$)

A second set of cost functions were defined based on an exponential weighting factor. The $\alpha$ term was selected to ensure that the closed loop designs speed up the system. That is, $\alpha$ is chosen to be faster than the slowest of the open loop poles so that the controller output increases the speed of response of the closed loop system.

$$J_{opt} = \frac{1}{2T_f} \int_0^{T_f} (e^2(t) + \lambda u^2(t))dt$$

(14)

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$$J_{opt} = \frac{1}{2T_f} \int_0^{T_f} (e^2(t) + \lambda u^2(t))dt$$

(14)
In addition, the LQ cost function given by (14) was included in the analysis. The $\alpha$ weighting was fixed at 1.5, which is 1.5 times as fast as the open loop pole, where $\tau=1$. The optimal controller parameters for a PI-controller were calculated, using Algorithm 1, for several time delayed systems ($G_\theta, \theta = 0.25, 0.5, 1, 2$), as shown in Tables I and II. The simulation was designed for obtaining good reference tracking, i.e. set-point tracking.

For each set of $\rho_0$, the PSM values and the GM (dB), PM ($^\circ$) and DM (s) were calculated as shown in Tables III, IV, V and VI.

The step responses for all the cost functions with the optimal tuning parameters with step input of 1 and time delays of 0.25 and 2 are shown in Figs. 4, 5, 6 and 7.

The results in Tables I and II show that the size of the controller parameters are reduced as the delay size decreases. In systems with low delay, the proportional and integral controller terms are highest and deviate greatly between cost functions, whereas delay dominant systems produce smaller and more stringent and similar parameters. Examination of

$$J_{IEA_\alpha} = \int_{0}^{\infty} e^{\alpha t} | e(t) | dt$$

(15)

$$J_{IES_{2\alpha}} = \int_{0}^{\infty} e^{2\alpha t} e^2(t) dt$$

(16)

$$J_{IEA_{\alpha-1}} = \int_{0}^{\infty} (e^{\alpha t} - 1) | e(t) | dt$$

(17)

$$J_{IES_{2\alpha-1}} = \int_{0}^{\infty} (e^{2\alpha t} - 1)e^2(t) dt$$

(18)

Fig. 4. Step responses for the ISE and IAE CF families, $\theta=0.25$

Fig. 5. Step responses for the Exp and LQ CF families, $\theta=0.25$

Fig. 6. Step responses for the ISE and IAE CF families, $\theta=2$

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>$G_{0,25}$</th>
<th>$G_{0.5}$</th>
<th>$G_1$</th>
<th>$G_2$</th>
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<td>$K_I$</td>
<td>$K_P$</td>
<td>$K_I$</td>
<td>$K_P$</td>
</tr>
<tr>
<td>ISE</td>
<td>2.15</td>
<td>2.49</td>
<td>1.76</td>
<td>1.96</td>
</tr>
<tr>
<td>ITSE</td>
<td>2.68</td>
<td>2.15</td>
<td>1.47</td>
<td>1.08</td>
</tr>
<tr>
<td>ITAE</td>
<td>2.23</td>
<td>2.12</td>
<td>1.23</td>
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</tr>
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</tr>
<tr>
<td>ITAE</td>
<td>1.91</td>
<td>1.91</td>
<td>1.00</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Table I

$\rho_0$ for several time delayed systems using 2 CF

356
Perform reasonably well. The unweighted functions appear to be conservative controllers. The GM, PM and DM for this delay without being an overly conservative controller. The stability margin values in Tables V and VI yield a number of interesting observations:

- Decreasing GM and increasing PM and DM with increasing delay time for the ISE and IAE CF families;
- Decreasing GM and increasing DM for the IES, IEA and LQ CF families;
- Highly conservative controllers at low delay times and for all exponentially weighted CFs apart from $IE_{S2a-1}$.

More conservative controllers are robustly stable, but are not design optimal. In general, stability robustness would be satisfied with GM $> 2.25$ dB and $20^\circ <$ PM $< 65^\circ$ [3]. The PM constraints are satisfied, or only marginally greater than the Baab rule, for all CF families except the LQ CF for system time delays analysed less than 2. The GM constraint is not satisfied for the ISE, IAE and LQ CFs with a time delay of 2.

The performance of the cost functions was measured with respect to several FOPTD configurations to try and assess the difference between the different tuning rules. The analysis consisted of fixing two plant parameters (delay and either plant gain or time constant) to the nominal value, whilst varying the third parameter through the same range as given in the original methodology for delay. This was used to test whether the CFs perform as well with shifts in the gain and time constants of the FOPTD model. The performance indicator was calculated by assuming that stability robustness is satisfied with the values of GM and PM given above and in [3]. Better performance is achieved with increasing GM and PM, with an upper bound of performance for PM at $65^\circ$. The performance results for all the CFs, excluding LQ, with changing plant delay is shown in Figs. 8. The performance for change in plant gain and time constant was also calculated. The performance index in Fig. 8 clearly shows that at low plant delay times, the exponentially-weighted CFs give excellent performance. However, it appears that the IEA $_a$, IES $_{2a}$ and IEA $_{a-1}$ CFs produce far too conservative results, leading to the conclusion that the IES $_{2a-1}$ CF would be the preferred choice. This was also confirmed by the performance calculations for the change in plant gain and plant time constant, where similar results were observed. The overly conservative designs are not observed for plant delays of 0.5, 1 and 2, although there is a marked increase in performance in the weighted functions.

**Table II**

<table>
<thead>
<tr>
<th>System CF</th>
<th>$G_{0.25}$</th>
<th>$G_{0.5}$</th>
<th>$G_{1}$</th>
<th>$G_{2}$</th>
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<tr>
<td>IEA$_{2a}$</td>
<td>0.93</td>
<td>0.98</td>
<td>0.96</td>
<td>0.97</td>
</tr>
<tr>
<td>IES$_{2a-1}$</td>
<td>2.18</td>
<td>1.96</td>
<td>1.44</td>
<td>1.08</td>
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<tr>
<td>LQ</td>
<td>2.27</td>
<td>1.19</td>
<td>1.39</td>
<td>0.87</td>
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**Table III**

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<th>CF</th>
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<th>$G_{0.5}$</th>
<th>$G_{1}$</th>
<th>$G_{2}$</th>
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<tbody>
<tr>
<td>ISE</td>
<td>0.27</td>
<td>0.28</td>
<td>0.33</td>
<td>0.31</td>
</tr>
<tr>
<td>ITSE</td>
<td>0.32</td>
<td>0.35</td>
<td>0.38</td>
<td>0.52</td>
</tr>
<tr>
<td>IT$^2$SE</td>
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<td>0.42</td>
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</tr>
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<td>0.45</td>
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</tr>
<tr>
<td>ITAE</td>
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<td>0.51</td>
<td>0.35</td>
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<td>IT$^2$AE</td>
<td>0.46</td>
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**Table IV**

<table>
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<th>CF</th>
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<td>IES$_{2a}$</td>
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<tr>
<td>IEA$_{a-1}$</td>
<td>0.72</td>
<td>0.49</td>
<td>0.51</td>
<td>0.48</td>
</tr>
<tr>
<td>IES$_{2a-1}$</td>
<td>0.40</td>
<td>0.35</td>
<td>0.38</td>
<td>0.57</td>
</tr>
<tr>
<td>LQ</td>
<td>0.40</td>
<td>0.32</td>
<td>0.36</td>
<td>0.34</td>
</tr>
</tbody>
</table>

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2.66 weighted outperform the non-weighted CFs. It is also fair to assume that the time-weighted CFs would be tuning rule of choice, when considering the design restrictions to avoid too conservative controllers. The basic calculation was verified for some examples by performing a closed loop step response on the perturbed system with the nominal plant parameters \((\rho_0)\), which are perturbed by the PSM value, \(\beta\), calculated from (3).

The output from the closed loop was analysed and the stability boundary requirement was confirmed if the response showed a steady-state oscillation of constant amplitude as shown in Table III agree favourably with those interpolated from their work. The \(ISE\) and \(IAE\) cost functions give lower PSM values for all delays than other functions. For example, with \(\theta = 0.25\), the \(ISE\) CF gives a PSM of 0.27, which means that a 27% additive perturbation in all the nominal parameters (negative perturbation for \(\tau\)) will cause the closed loop system to become unstable. However, for the same system delay, the \(IAE_{\alpha=1}\) CF gives a PSM of 0.72, which suggests a much more robust design. In general, it can be seen that those cost functions that are time- or exponentially-weighted outperform the non-weighted CFs. It is also fair to

**TABLE V**

<table>
<thead>
<tr>
<th>System</th>
<th>(G_{0.25})</th>
<th>(G_{0.5})</th>
<th>(G_1)</th>
<th>(G_2)</th>
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<td>(ISE)</td>
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<td>(ITSE)</td>
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<td>3.35</td>
<td>57.76</td>
<td>0.42</td>
<td>3.30</td>
</tr>
<tr>
<td>(ITAE)</td>
<td>4.19</td>
<td>63.14</td>
<td>0.58</td>
<td>4.07</td>
</tr>
<tr>
<td>(IT^2AE)</td>
<td>4.34</td>
<td>63.93</td>
<td>0.61</td>
<td>4.71</td>
</tr>
</tbody>
</table>

**TABLE VI**

<table>
<thead>
<tr>
<th>System</th>
<th>(G_{0.25})</th>
<th>(G_{0.5})</th>
<th>(G_1)</th>
<th>(G_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ISE_{\alpha=1})</td>
<td>8.54</td>
<td>74.98</td>
<td>1.37</td>
<td>4.68</td>
</tr>
<tr>
<td>(ISE_{\alpha=2})</td>
<td>8.48</td>
<td>75.13</td>
<td>1.37</td>
<td>4.32</td>
</tr>
<tr>
<td>(ISE_{\alpha=3})</td>
<td>8.51</td>
<td>75.11</td>
<td>1.37</td>
<td>4.00</td>
</tr>
<tr>
<td>(ISE_{\alpha=4})</td>
<td>3.72</td>
<td>62.27</td>
<td>0.51</td>
<td>2.97</td>
</tr>
<tr>
<td>(LQ)</td>
<td>3.75</td>
<td>71.78</td>
<td>0.59</td>
<td>2.83</td>
</tr>
</tbody>
</table>

**Fig. 8.** Performance of the CF families with changes in \(\theta\)**

The PSM is a more representative measure of the robust stability as it takes into account simultaneous parametric uncertainty, stated by [3]. The same authors presented some PSM results for a FOPTD system and comparison of the \(IAE\) and \(ITAE\) values shown in Table III agree favourably with those interpolated from their work. The \(ISE\) and \(IAE\) cost functions give lower PSM values for all delays than other functions. For example, with \(\theta = 0.25\), the \(ISE\) CF gives a PSM of 0.27, which means that a 27% additive perturbation in all the nominal parameters (negative perturbation for \(\tau\)) will cause the closed loop system to become unstable. However, for the same system delay, the \(IAE_{\alpha=1}\) CF gives a PSM of 0.72, which suggests a much more robust design. In general, it can be seen that those cost functions that are time- or exponentially-weighted outperform the non-weighted CFs. It is also fair to

**Fig. 9.** Closed-loop system step response on the stability boundary (\(LQ CF, \theta=0.5\))
A second verification test was to generate the Nyquist plot of the perturbed system and examine whether the plot passes through the \((-1 + j0)\) point. Using the calculated value from Table IV, a comparison was made with PSM values offset by 0.04 and -0.04 to show that the calculated PSM produced a closed-loop system exclusively on the boundary of stability. Any incremental change in the calculated PSM produced a corresponding shift in the Nyquist plot, either in the \(-\infty\) or \(+\infty\) directions depending on the sign of the offset value as shown in Fig 10.

Although it appears that time-weighted and exponentially-weighted cost functions, particularly those weighted against the absolute error, provide inherent robust stability, further validation is still required. Verification of the study hypothesis can be done by increasing the scope of analysis to include SOPTD systems and real industrial plant parameters. This work will be presented in a subsequent paper.

Based on the performance indicator results, it would appear that the exponentially-weighted CFs would provide the best stability robustness design. However, by considering the standard and parametric stability margin results shown, these CFs often result in the design of too conservative controllers.

Figure 8 indicates that the \(IEA_\alpha\), \(IES_{2\alpha}\) and \(IEA_{\alpha-1}\) CFs gives exceptional performance for a plant with low delay, but this is a result of a very conservatively designed controller. The key factors for selecting the best CFs for design must be consistency and conservatism. The most consistent cost functions for FOPTD system stability robustness controller design are assessed to be, from all the results shown, the \(IT^2AE\) and \(IES_{2\alpha-1}\).

VI. CONCLUSIONS

Robust stability analysis was performed on a FOPTD system to examine the hypothesis that there exists a set of cost functions (family) that guarantee robust controller parameters. The Parametric Stability Margin was used together with the standard performance indicators to show that particular cost function designs give better stability robustness than others. Initial results indicated that the inclusion of a time- or exponentially-weighted component in the cost function during tuning improved controller performance. The exponentially-weighted cost functions tended to be more conservative than the time-weighted CFs, suggesting that the latter be the design CF of choice, but consistency assessment suggested that the \(IT^2AE\) and \(IES_{2\alpha-1}\) gave the best overall performance. However, true verification of the hypothesis requires a number of systems to be tested, which are more closely related to real industrial applications. These systems will include second- and higher-order models as well as actual identified plant models.

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REFERENCES


Fig. 10. Nyquist plot for three PSM values using the LQ CF with \(\theta=0.5\)