Abstract:

This paper discusses a common basis for the definition of certain structural parts of Petri nets ("invariants", "siphons", "traps"), namely "open paths". The discussion gives rise to a comprehensive approach to the liveness problem which properly includes both the "variant" method of Lautenbach and the "deadlock" method of Hack.

Dynamical properties of invariants, siphons and traps will be discussed. In particular, it will be shown that the question whether a siphon can be emptied completely under a given marking, forms one central problem for the solution of the liveness problem.

The notion of "equimoving system of open paths" which leads from the filling transitions to the emptying transitions of a siphon is introduced. For the case in which equimoving systems exist, a necessary condition for a siphon to be emptyable is given. As one of the main results, a sufficient liveness condition for general Petri nets is given.

A Step Towards a Solution of the Liveness Problem in Petri Nets

By

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This paper discusses a common basis for the definition of certain structural parts of Petri nets ("invariants", "siphons", "traps"), namely "open paths". The discussion gives rise to a comprehensive approach to the liveness problem which properly includes both the "variant" method of Lautenbach and the "deadlock" method of Hack.

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1. Introduction

The liveness problem is a widely discussed Petri net problem. Given a Petri net and a marking, the problem is to find out whether the net can reach a state in which all transitions or a set of transitions can no longer fire.

The liveness problem is completely solved for a special sub-class of Petri nets [4], [5], [7] called "marked directed graphs" or "synchronisation graphs".

For more general Petri nets, two different approaches have been suggested so far. Commoner and Hack [3], [6] defined deadlocks and traps as sets of places the input and output transitions of which satisfy certain criteria. Lautenbach [8] defined variants as covers covering part of a net with certain systems of circuits. Then certain transitions may be determined the firing of which fills and empties the variant according to the way in which the transitions are covered by the circuits.

Both approaches give rise to a necessary and sufficient condition for the liveness of a subclass of nets. These subclasses, called "free choice nets" and "Petri nets with regulation circuits", are essentially disjoint.

Considered as set of places, every variant is a deadlock, but not vice versa. Thus, deadlocks are more general. On the other hand, variants possess a structure (by the superimposed systems of circuits) that deadlocks lack. In this sense, variants are more powerful. As a consequence, each of the approaches is superior to the other with respect to certain points.

In this paper we will present an approach [1], [2] that encompasses both preceding approaches. We will define a "siphon" as a cover covering part of a net with a system of open paths of a certain type. Considered as sets of places, siphons are equivalent to deadlocks. On the other hand, siphons possess as much structure as variants. Thus, they combine the generality of Commoner's and Hack's approach with the structure of Lautenbach's approach.
After giving some introductory definitions in section 2, we introduce different types of systems of open paths in section 3. Invariants, siphons and traps will be defined as sets of places covered by certain systems of open paths, and their dynamic properties will be derived.

We shall see that the solution of the problem under which conditions a siphon can be emptied is an essential part of the liveness problem. To solve the former problem, we shall generalize the approach suggested by Lautenbach [8] in section 4. In section 5, as one of the main results, we give a sufficient liveness condition that is valid for general Petri nets, and by which practically all cases are covered that arise from the application of Petri nets as models of "real" problems.

In this paper, we do not consider Petri nets where a weight-function is associated to the edges ("thick" edges). However, our results can easily be generalized to cover this case.

2. Definitions

In this section, we will give some basic definitions concerning Petri nets and systems of open paths, assuming that the reader has a little familiarity with Petri nets.

**Definition 1:** A Petri net (compare [12], [13]) is a quadruple \( N = (\Sigma, \Pi, \text{pre}, \text{post}) \), where \( \Sigma \) and \( \Pi \) are finite non-empty sets with \( \Sigma \Pi = \emptyset \), and \( \text{pre}, \text{post} \) are binary relations with \( \text{pre}, \text{post} \subseteq \Sigma \times \Pi \), field \( (\text{pre} \cup \text{post}) = \Sigma \Pi \).

The elements \( p \in \Sigma \) and \( t \in \Pi \) are called places and transitions, respectively. The elements \( e \in \text{pre} \cup \text{post} \) are called edges where edges \( e \in \text{pre} \) lead from places to transitions, and the edges \( e \in \text{post} \) from transitions to places. In the figures, the places are represented by circles, the transition by squares, and the edges by arrows.

A place \( p_1 \) (\( p_2 \)) is called an input place (an output place) of a transition \( t \) if \( (p_1, t) \in \text{pre} \), \( (p_2, t) \in \text{post} \). A transition \( t_1 \) (\( t_2 \)) is called an input transition (an output transition) of a place \( p \), if
\((p, t_1) \in \text{post} ((p, t_2) \in \text{pre})\). A place \(p\) (a transition \(t\)) is called a peripheral place (a peripheral transition), if \(\text{pre}(p) = \emptyset\) or \(\text{post}(p) = \emptyset\) (\(\text{pre}^{-1}(t) = \emptyset\) or \(\text{post}^{-1}(t) = \emptyset\)). For the input and output transitions and places, respectively, of a given \(p \in \Sigma\), \(t \in \Pi\), or of a given set \(S \subseteq \Sigma\) or \(T \subseteq \Pi\), we use the following dot notation (compare [6]) which is illustrated in fig. 2.1:

\[
p^* = \text{pre}(p), \quad S^* = \bigcup_{p \in S} p^*, \quad p = \text{post}(p), \quad S = \bigcup_{p \in S} p
\]

\[
t^* = \text{post}^{-1}(t), \quad T^* = \bigcup_{t \in T} t^*, \quad t = \text{pre}^{-1}(t), \quad T = \bigcup_{t \in T} t
\]

A place \(p \in \Sigma\) is called a shared place if \(|p^*| \geq 2\), or \(|p| \geq 2\), that is if it has more than one input or output transition. (The subclass of Petri nets in which there are no shared places is called marked directed graphs).

**Definition 2:** Given a Petri net \(N\). A mapping \(M : \Sigma \rightarrow \mathbb{N}_0\) is called a marking of \(N\). \(M(p)\) is called the number of tokens on a place \(p\). The number of tokens \(M(S)\) on a set of places \(S \subseteq \Sigma\) is defined as the sum \(\sum_{p \in S} M(p)\). A place \(p\) is called marked (under \(M\)) if \(M(p) \geq 1\); otherwise it is called **blank**.

The marking of a Petri net represents its state. A change of the state is performed by the firing of an activated transition as an elementary event, which is defined as follows:

**Definition 3:** Given a Petri net \(N\) and a marking \(M\) (of \(N\)). Then a transition \(t \in \Pi\) is called activated under \(M\) if \(\forall p \in \cdot t : M(p) > 0\), that is, each of its input places is marked. The firing of a transition \(t \in \Pi\) which is activated under the marking \(M\) is defined as a mapping \(M \rightarrow M'\) with

\[
M'(p) = \begin{cases} 
M(p) - 1 & \text{if } p \in \cdot t/t' \\
M(p) + 1 & \text{if } p \in t'/t \\
M(p) & \text{otherwise}
\end{cases}
\]

In other words, the firing of an activated transition \(t\) removes one token from each input place and adds one token to each output place of \(t\).
Definition 4: Consider a Petri net $N$ and two markings $M, M'$ of $N$. The marking $M'$ is said to be reachable from the marking $M$ if there exists a firing sequence of transitions $t_i \in \Pi$ such that

$$\exists n \geq 1 \text{ } M_0, \ldots, M_n: (M_0 = M) \land (M_n = M') \land (M_{i-1} \overset{t_i}{\rightarrow} M_i) \text{ for } i = 1, \ldots, n$$

$M_0, M_1, \ldots, M_{n-1}, M_n$ is called a marking sequence and denoted by $M_0 \rightarrow M_n$. The set of markings, which consists of a given marking $M$ and all markings that are reachable from $M$, is denoted by $[M]$.

Definition 5: Given a Petri net $N$ and a marking $M$ (of $N$). A transition is called dead under $M$ if it is not activated under any marking $\in [M]$. The marking $M$ is called dead if there is no transition $t \in \Pi$ which is activated. A marking $M$ is called live if $\forall \tilde{M} \in [M]: \tilde{M}$ is not dead.

This means that no dead marking is reachable from a live marking $M$. The definition of live allows certain parts of a system to be dead, that is, there may be a set of transitions under the marking $M$ such that none of them is activated under any marking $\in [M]$. This is excluded by the definition of live. A marking $M$ is called live if there is no transition $t$ such that $t$ is dead under any marking $\in [M]$.

The slack $Sp(U,V,M)$ of a set of transitions $U$ with respect to a set of transitions $V$ under a marking $M$ indicates the maximum number of times $t \in U$ can fire without a $t \in V$ having to fire.

The slack $Sp(U,V,M)$ essentially depends upon the minimal number of tokens on path system(s) leading from $V$ to $U$ (compare [9]). Consider for example fig. 2.2.

$Sp([t_2], \{t_1\}, M) = 0$ since the path leading from $t_1$ to $t_2$ across $p_1, p_2$ contains no tokens. However, all other paths contain tokens.

*We use the liveness notions proposed in [8].
Furthermore, the slack depends on whether the transitions on the path are live or dead. Suppose, in the example of fig. 2.2, that $M(p_1)>0$. However, if $t_3$ is dead, possibly because an input place that does not belong to the path is blank, $Sp([t_2], [t_1], M) = 0$.

Given these introductory definitions, we define open paths and systems of open paths.

**Definition 6:** Given a Petri net $N$ and a sequence $w = (u_0, \ldots, u_n)$ where $n \geq 1$ and $u_i \in \Sigma \cup \Pi$ for $i = 0, \ldots, n$; $w$ is called a (simple, directed) path from $u_0$ to $u_n$ if $\forall i = 0, \ldots, n-1: (u_i, u_{i+1}) \in \text{pre} \cup \text{post}^{-1}$ \land $(\forall i, j \leq n \land (i = 0 \land j = n) \lor (i = n \land j = 0))$: $i \neq j = \Rightarrow u_i \neq u_j$.

We say that the elements $u_i$ and the edges $(u_i, u_{i+1})$ lie on $w$ and call $u_0$ the initial element and $u_n$ the final element of $w$. As we see, the initial point and final point are allowed to be equal, but, apart from this case all places and transitions must be pairwise distinct. A path $w$ in which the initial element is equal to the final element is called a (simple, directed) cycle.

An open path is either a path $w = (u_0, \ldots, u_n)$ whose initial and final element $u_0$ and $u_n$ are places, or a single place. A cycle can be considered as an open path where any of the places may be taken as initial and final elements. An open path will be represented by a line "\[ \hline \]" in the figures.

**Definition 7:** Given a Petri net $N$, and a family of open paths $\{w_i\}_{i=0}^m$. (A family differs from a set in that a single element can be contained several distinct times). Then $c = \{w_i\}_{i=0}^m$ is called a system of open paths that covers a set of places $P = P(c)$, or an open cover of $P$, if every place $p \in P(c)$ lies at least on one path $w_i$.

We say that a place $p \in P(c)$ is covered by $c$. We say that a transition $t \in \Pi$ and an edge $(p, t) \in \text{pre} \cup \text{post}$, respectively, is $k$-covered by $c$ if it lies on exactly $k$ paths of $c$. For a given $p \in P$, $t \in p'$, and $t' \in p$ we denote by $L_c(p, t)$ and $L_c(t', p)$, the covering degree $k$ of the edges $(p, t)$, and $(p, t')$, respectively.
3. Systems of Open Paths and Structural Properties

In the sequel, we define several types of systems of open paths. First, we note some properties of open paths. We see that only by tokens which enter or leave an open path "by its places" the number of tokens on the path can be modified. Therefore, tokens which enter or leave a path "by its transitions" are not of interest for our subsequent considerations.

Now suppose a token appears on some place, e.g., $p_1$ in figure 3.1, that is covered by an open cover. If the token came via an uncovered edge, say $(p_1, t_0)$, then "the cover has been filled" with a new token, but if the token came via a covered edge, say $(p_1, t_1)$, then the token has simply "remained" in the cover. Conversely, the cover can "lose" tokens via uncovered edges, say $(p_3, t_4)$.

In the case of covered edges, it is also of importance how many open paths cover the edge. Consider the place $p_2$ (in figure 3.1). If a token leaves this place over the 2-covered edge, $(p_2, t_3)$, then two tokens may return to $p_2$ from $t_3$ and $t_5$. We draw the intuitive conclusion that a cover is filled in case a token leaves the place via an edge which is covered by more paths that the number of paths covering the edge by which the token appeared on the place. The conclusion about emptying a cover is similar.

In definition 8 we introduce essentially three types of systems of open paths or open covers and distinguish them according to the number of paths by which edges of covered places are covered.

Definition 8: Given a Petri net $N$, $P \subseteq \Sigma$, and a system of open paths $c$ covering $P$. The cover $c$ is called

a) an I-cover of $P$ if $\forall p \in P \forall t, \hat{t}_2 \in \hat{p}, t_1, t_2 \in p'$:

$$L_c(\hat{t}_1, p) = L_c(\hat{t}_2, p) = L_c(p, t_1) = L_c(p, t_2) = L_c(p) > 0$$
b) an S-cover of \( P \) if \( \forall p \in P \ \forall t_1, t_2 \in \Gamma \):
\[
L_c(t_1, p) = L_c(t_2, p) =: K_c(p) > 0
\]

c) an 1-S-cover of \( P \) if \( c \) is an S-cover of \( P \) and \( K_c(p) = 1 \) for every \( p \in P \).
d) a T-cover of \( P \) is \( \forall p \in P \ \forall t_1, t_2 \in \Gamma' \):
\[
L_c(p, t_1) = L_c(p, t_2) =: K_c(p) > 0
\]
e) a 1-T-cover of \( P \) if \( c \) is a T-cover of \( P \) and \( K_c(p) = 1 \) for every \( p \in P \).

\( K_c(p) \) is called the covering degree of a place \( p \in P \). For places \( p \in \Sigma / P \), \( K_c(p) \) is defined as zero.

If there is an \( I_1, (S-, T-) \) cover covering \( P \) then there is an infinity of \( I_1, (S-, T-) \) covers that cover \( P \). Therefore we introduce simple covers. We call an \( I_1, (S-, T-) \) cover simple if no (systems of) open path(s) can be removed from it such that it remains an \( I_1, (S-, T-) \) cover. An 1-S-cover and 1-T-cover is simple in this sense. It can easily be seen that for every \( S_1, (T-) \) cover of \( P \), there is at least one 1-S-, (1-T-) cover that covers \( P \).

We classify sets of places as structural parts of a Petri net according to their possible covering. In general, a given set of places may be covered by several covers.

**Definition 9:** Given a Petri net \( N \) and \( \Sigma \subseteq \Sigma \).

a) \( P \) is called a (simple) invariant if there exists a (simple) \( I \)-cover of \( P \).

b) \( P \) is called a siphon if there exists a S-cover of \( P \).

c) \( P \) is called a trap if there exists a T-cover of \( P \).

In Fig. 3.2, some I-covers and 1-S-covers are shown. An \( I \)-cover requires the covering degree of edges which are incident to a place to be constant. Therefore, it either extends to include peripheral places, as the I-covers \( \{w_1, w_4, w_2\} \) and \( \{w_1, w_4, w_3\} \) shown in Fig. 3.2b), or it
consists of cycles, possibly including open paths from the cycles to peripheral places, such as the 1-cover \( \{w_7, w_8, w_9, w_{10}\} \) of Fig. 3.2c.

Intuitively, we notice that an I-cover cannot acquire tokens from outside nor lose them. Nor can it generate or delete tokens inside. These observations will be formalized in theorem 1.

S-covers or T-covers need not cover a net in such a complete way, as an I-cover does. An S-cover and an 1-S-cover may "end" at non-peripheral places, but "begin" only at peripheral places. For example, in Fig. 3.2b), \( \{w_1, w_2\} \{w_3, w_4\} \{w_5, w_6\} \{w_7, w_8\} \{w_9, w_{10}\} \) is an S-cover, but \( \{w_2, w_3\} \) is not an S-cover; in Fig. 3.2c). \( \{w_7, w_8, w_9\} \) is a 1-S-cover. Since there are non-covered edges that enter a place of an S-cover and of a 1-S-cover, it can never acquire tokens from "outside". But since there may be non-covered edges leaving its places, it may lose tokens to "outside". Moreover, an S-cover may generate tokens inside, e.g. in Fig. 3.2b) over the paths \( w_2 \) and \( w_3 \), or it may delete tokens. T-covers show just the contrary behaviour; they may begin at non-peripheral places, and may acquire tokens from "outside".

Before characterizing the dynamic properties of the different covers in theorem 1, we define the weighted sum of tokens over an invariant, siphon, or trap, also called weighted marking. Let \( P \) be a set of places covered by an I-cover, S-cover or T-cover, respectively, and let \( K_c(p) \) be the covering degree of each \( p \in P \). Then the weighted marking \( W_c(M) \) is:

\[
W_c(M) = \sum_{p \in P} K_c(p)M(p)
\]

Remember, \( M(P) \), the number of tokens on \( P \subseteq \Sigma \) was defined as \( \sum_{p \in P} M(p) \).

**Theorem 1:** Given a net \( N \), a marking \( M \), and \( P \subseteq \Sigma \).

a) If \( P \) is an invariant (covered by the I-cover \( c \)) then

\[
VM \subseteq [M]; \quad W_c(M) = W_c(M).
\]

Conversely, if for \( VM \subseteq [M] \) there is a weight \( w(p), w(p) > 0, \) associated to each place \( p \in P \) such that

\[
\sum_{p \in P} w(p)M(p) = \sum_{p \in P} w(p)M(p)
\]

then there is an I-cover for \( P \) where \( w(p) \) is the covering degree of every place \( p \in P \).
b) If $P$ is an invariant, then there is a lower bound $e$ and an upper bound $u$ for $P$ where $e, u \geq 0$ such that:

$$
\text{VM} \mathcal{M}[M^*]: e \leq M(P) \leq u \Rightarrow \text{VM} \mathcal{M}[M^*]: M(P) = 0
$$

c) $P$ is a siphon iff VM: $M(P) = 0 \Rightarrow \text{VM} \mathcal{M}[M^*]: M(P) = 0$

d) $P$ is a trap iff VM: $M(P) > 0 \Rightarrow \text{VM} \mathcal{M}[M^*]: M(P) > 0$

**Proof:** We only give a short outline of the proof.

a) Let $c$ be an I-cover of $P$. If $c$ is by a firing of $t$ covered by $c$, a given marking $M$ is transformed into $M'$ (i.e., $M \overset{t}{\rightarrow} M'$), then

$$
W_c(M') = W_c(M) + \sum_{p \in t} K_c(p) - \sum_{p \notin t} K_c(p).
$$

According to definition 9a), for every $t \in \Pi$, the number of paths on which $t$ lies is

$$
\sum_{p \in t} K_c(p) = \sum_{p \notin t} K_c(p)
$$

Therefore, $W_c(M') = W_c(M)$ for $M \overset{t}{\rightarrow} M'$. If a transition $t$ not covered by $c$ fires, then $W_c(M') = W_c(M)$, since $K_c(p) = 0$ for $p \notin t, t'$.

The proof of the reverse direction constructs, for every transition $t$, certain paths from its input to its output places where the covering degree of each edge $(p, t)$ $(p \in t, t')$ corresponds to the given weight factor $w(p)$. This can be done for every $t$ covered by $P$, as (*) shows. It is easy to see that the resulting system of open paths is an I-system.

b) The proof is based on the equation (*), and uses the upper and lower bound of $\sum_{p \in P} K_c(p)$.

c) Let $P$ be a siphon, and $c$ a 1-$S$-covering of $P$. From the definition of a 1-$S$-covering it follows for every $t \in \Pi$:

$$
t \in P = t \cap P \neq \emptyset \quad (**)
$$

If $P$ is blank under a given marking, it remains blank since (**), means that no transition of $'P$ can ever fire.
For the reverse direction we show that (**) can be derived from
the proposition of the theorem. Then we construct a 1-S-covering.
For every $p \in P$, we construct one open path per input edge to a
place $p \in P$ which exists according to (**).

d) The proof is done analogous to that of c).

Part a) of theorem 1 says that if we associate the covering degree $K_c$
as a weight to every place of an invariant, then the weighted sum of
tokens on an invariant is constant under all markings that are reach-
able from a given initial marking. The sum of tokens itself does not
remain constant, but is bounded between a lower and an upper bound, as
part b) shows.

Part b) does not hold in the reverse direction, since such lower and
upper bounds exist trivially for all marking sequences of a finite
length, and we may find Petri nets not covered by invariants, which
allow only such sequences.

Note, however, that there may be places in a net which do not
belong to an invariant.

Part c) says that once a siphon is empty of tokens, it remains empty
and cannot acquire any more tokens; that is, all transitions covered
by the S-cover cannot fire anymore.

Part d) says that a trap cannot be emptied anymore, once it contains
at least one token. Thus, siphons are "deadlocks", and traps are
"traps" in the sense of Commoner and Hack [3], [6]. Traps are not
really important for our considerations. Their definition was given
for completeness reasons. Invariants are, obviously, special siphons
and traps.

We have seen that an S-cover, once it is emptied, causes the set
of transitions covered by it, and possibly other transitions, to be
dead. (It does not necessarily cause all transitions of the net to be
dead, as Fig. 3.3 illustrates. The siphon $\{p_1, p_2\}$ is empty. However,
the marking is live-1.) The reverse is not true. Fig. 3.4 shows a net
where the set of transitions \( \{t_1, t_2\} \) is dead; however, no empty siphon exists. Theorem 2 which follows states that there is an empty siphon if all transitions of a Petri net are dead. (This theorem has first been derived by Commoner and Hack [3], [6]).

**Theorem 2:** Let \( N \) be a Petri net, and \( M \) be a dead marking of \( N \). Then there exists a set of places \( P \) such that \( P \) is a siphon and \( M(P) = 0 \) (that is, \( P \) is "empty").

**Proof:** The proof is done by applying a back-tracing method similar to the one developed by Lautenbach [8], extended for general Petri nets: Given a marking \( M \) such that all transitions are dead.

**Step 0:** Choose some place that is blank under \( M \); except for trivial cases, we can find such a place with input transitions.

**Step 1:** Every input transition \( t \) of this place has a blank input place since it is dead under \( M \). Therefore, we can construct an open path from the blank input place to the blank output place of \( t \). This is repeated for every input transition of the given place. The open paths are taken to belong to the cover \( c \) which, as a consequence, covers exactly once every edge that enters the given place.

**Step 2:** For the places that are now newly covered there are two alternatives. If a place has no input transition, or if it had been covered already, then it needs no longer to be considered. Otherwise, step 1 must be repeated for this place.

The construction eventually stops since the net is finite. The covering \( c \) created by Step 1 is a 1-S-cover according to definition 8, and from its construction it follows that it is empty.

**Lemma 2a:** For every marking \( M \) which is not live-1 there is a (possibly empty) firing sequence emptying some siphon.

**Lemma 2a:** follows directly from the fact that if a marking \( M \) is not live-1 then \( 3 \bar{M} \in [M] \) : \( \bar{M} \) is dead. By theorem 2, there is an empty siphon for \( \bar{M} \).
The converse of theorem 2 is not true for general Petri nets, as we have already seen.

4. Emptying of Siphons

In the last section, we saw that empty siphons play an important role for the liveness of a marked Petri net. There must be at least one siphon which is empty or can be emptied under a given marking if the marking is not live - 1. On the other hand, every empty siphon causes a marking to be not live - 5. Therefore, we consider in the sequel the problem under which conditions a siphon is or is not capable of being emptied.

4.1 Other Approaches

So far, there are two approaches which solve this problem, each for a restricted class of Petri nets.

Commoner and Hack investigate the case in which a siphon contains a marked trap ([3], [6]).

In free choice nets, this is a necessary and sufficient condition for the siphons not to be emptyable. Consider for example fig. 4.1. The net which does not include the place $p_4$ and the (broken lined) edges $(p_4, t_4)$ and $(p_4, t_2)$ is a free choice net. The places $\{p_1, p_2, p_3\}$ form a siphon that contains no trap. Therefore, by bringing all tokens of the siphon into the place $p_3$, and then firing the transition $t_2$ repeatedly the siphon can be emptied under any marking.

In Petri nets which are not free choice nets, this condition is only a sufficient condition as Commoner has shown ([3]). Consider again fig. 4.1. The net including $p_4$ and its adjacent edges is not a free choice net. Under the given marking, the siphon cannot be emptied as can easily be seen. However, the siphon contains no marked trap.

Consequently, there must be other reasons not covered by the above condition, which imply that a siphon cannot be emptied. The approach proposed by Lautenbach [8] seems to give an explanation. Lautenbach defines variants by covering a set of places with an incomplete system of
circuits. This covering determines transitions the firing of which empties, and other transitions the firing of which fills the variant. Now, we see that a variant can be emptied if its emptying transitions can fire sufficiently many times (i.e., to empty the variant completely) without the filling ones having to fire. In the subclass of Petri nets called nets with regulation circuits, this depends directly on the marking of the places on the regulation circuit. In [8] and [10], necessary and sufficient conditions are given for a variant (with regulation circuits) to be emptyable.

These conditions are extended in [11] and [9] to nets without regulation circuits. A variant can be emptied if its marking is smaller than the slack of its emptying transitions with respect to its filling transitions. This idea will be illustrated by the following example.

Consider again fig. 4.1. The system of open paths as shown in fig. 4.1 can be considered as an incomplete system of circuits which covers a variant. The variant is filled by the firing of $t_1$ and emptied by the firing of $t_2$ as one can intuitively see.

In the net without the place $p_4$, there is no other path-system leading from $t_1$ to $t_2$ apart from the variant itself. The marking of the variant is $M(p_1) + M(p_2) + M(p_3)$. The slack $Sp(t_2, t_1, M) = M(p_1) + M(p_2) + M(p_3)$ equals the marking of the variant. Consequently, the emptying transition $t_2$ can fire sufficiently many times to empty the variant.

If the place $p_4$ is included in the net of fig. 4.1, there is another path from $t_1$ to $t_2$ which leads across the places $p_4$ and $p_4$. As we have seen, the slack $Sp(t_2, t_1, M)$ is determined by the path-system with the minimal marking. That is, $Sp(t_2, t_1, M) = \min \{ M(p_1) + M(p_4), M(p_1) + M(p_3) \}$. This slack is smaller than the marking of the variant if $M(p_4) < M(p_2) + M(p_3)$. Under all markings for which this condition is true, the variant cannot be emptied. Since, in fig. 4.1, $M(p_4) = 1$, $M(p_3) = 2$, the variant cannot be emptied.

Note that both nets of fig. 4.1 are not nets with regulation circuits. In the sequel, we develop conditions implying that a siphon cannot be emptied (applicable to fig. 4.1).
4.2 Filling and Emptying Transitions

In this section, we show how the filling and emptying transitions of a siphon can be found. A necessary condition for a siphon to be emptyable will be developed. Siphons will be classified according to their filling and emptying properties.

We start by defining formally the notion that a siphon "is emptyable."

Definition 10: Let $M$ be a marking. We say that a siphon $P \in \Sigma$ is emptyable if $\exists M \in [M] : \tilde{M}(P) = 0$.

This definition includes the trivial case that a siphon is already empty under the marking $M$ (i.e., $M(P) = 0$).

Remark: A siphon $P$ covered by an $S$-cover $c$ can be emptied iff $\exists M \in [M] : W_c(M) = 0$. For, $\tilde{M}(P) = 0 \iff W_c(M) = 0$. This follows from: $\forall p \in P, K_c(p) > 0$.

Thus, we can consider the weighted marking of a siphon instead of its marking in order to judge whether it can be emptied or not.

We consider a siphon $P$ and its adjacent transitions $\mathcal{HJP}$ ($= P'$ by theorem 1c).

Definition 11: Let $P \subseteq \Sigma$ be a siphon covered by an $S$-cover $c$.

\begin{align*}
\text{a)} \quad T_c^+ &= \{ t \in P' \mid \sum_{p \in t} K_c(p) - \sum_{p \in t} K_c(p) > 0 \} \\
\text{b)} \quad T_c^- &= \{ t \in P' \mid \sum_{p \in t} K_c(p) - \sum_{p \in t} K_c(p) < 0 \} \\
\text{c)} \quad T_c^0 &= \{ t \in P' \mid \sum_{p \in t} K_c(p) - \sum_{p \in t} K_c(p) = 0 \} \\
&= P \setminus (T_c^+ \cup T_c^-)
\end{align*}

In definition 11, we compare (the sum of) the covering degree over all output places of a transition $t \in P'$ with the covering degree over all input places. If the first one is greater, then we say that the transition belongs to the set of filling transitions $T_c^+$; if the second one is greater, then we say that it belongs to the set of emptying
transitions $T^+_c$. Otherwise, it belongs to the set of neutral transitions $T^0_c$. The filling and emptying factor of a transition $t \in T^+_c$ and $T^-_c$, respectively, is defined by the difference between the covering degree over its input and the covering degree over its output places:

**Definition 12:** Let $P \subseteq \Sigma$ be a siphon covered by an $S$-cover $c$.

a) For $t \in T^+_c$: $K^\text{fill}_c(t) = \sum_{p \in t^+} K_c(p) - \sum_{p \in t^-} K_c(p)$

b) For $t \in T^-_c$: $K^\text{empty}_c(t) = \sum_{p \in t^-} K_c(p) - \sum_{p \in t^+} K_c(p)$

The filling factor $K^\text{fill}_c(t)$ and the emptying factor $K^\text{empty}_c(t)$ are always positive, according to their definition.

The following lemma shows that the weighted marking of a siphon is increased if and only if filling transitions fire, and decreased if and only if emptying transitions fire. Let $c$ be an $S$-cover, $M$ a marking; let $h_{M \rightarrow M'}(t)$ indicate the number of times a transition $t$ has fired during a given marking sequence $M \rightarrow M'$ with $M, M' \in [M]$. The following lemma holds:

**Lemma 3:**

$\forall M, M' \in [M]: W_c(M') = W_c(M) + \sum_{t \in T^+_c} K^\text{fill}_c(t) \cdot h_{M \rightarrow M'}(t) - \sum_{t \in T^-_c} K^\text{empty}_c(t) \cdot h_{M \rightarrow M'}(t)$

**Proof:** Consider the firing of a single transition $t$ which transforms a marking $M$ into $M'$. Then:

$W_c(M') = W_c(M) + \sum_{p \in t^+} K_c(p) - \sum_{p \in t^-} K_c(p)$

This follows directly from the firing rule. With definitions 11 and 12, we get
for \( t \in T^+_c \): \( W_c(M') = W_c(M) + K^\text{fill}_c(t) \)

for \( t \in T^-_c \): \( W_c(M') = W_c(M) - K^\text{empty}_c(t) \)

for \( t \in T^0_c \): \( W_c(M') = W_c(M) \)

for all other \( t, t \not\in P' \): \( W_c(M') = W_c(M) \), since \( K^c_c(p) = 0 \) for \( \forall p \in t', t \).

We get lemma 3 for general marking sequences by adding the above equations.

From lemma 3, we can derive the following theorem:

**Theorem 4:** A siphon (covered by an S-cover c) can be emptied under a marking \( M \) iff there exists a marking sequence \( M \rightarrow \tilde{M} \) such that

\[
\sum_{t \in T^+_c} K^\text{empty}_c(t) h_{M+M^c}(t) = W_c(M) + \sum_{t \in T^+_c} K^\text{fill}_c(t) h_{M+M^c}(t) \quad (4)
\]

The proof follows directly from definition 10 and lemma 3.

Theorem 4 gives a necessary and sufficient condition for a siphon to be emptyable. It says that there must exist a (possibly empty) marking sequence where the firing frequencies of the filling and emptying transitions satisfy condition (4). In the special case in which no filling transition, and only one of the emptying transitions fires, a siphon is emptyable if there is a marking sequence \( M \rightarrow \tilde{M} \) where

\[
h_{M+\tilde{M}}(t) = \frac{W_c(M)}{K^\text{empty}_c(t)}
\]

Obviously, for certain values of \( K^\text{fill}_c, K^\text{empty}_c \) and of the initial weighted marking \( W_c(M) \), there are no marking sequences satisfying condition (4). Consider for example fig. 4.2 where a 1-S-cover c covers a siphon. There is one filling transition \( t^+ \) with \( K^\text{fill}_c(t^+) = 2 \), and one emptying transition \( t^- \) with \( K^\text{empty}_c(t^-) = 2 \). All weight factors of places in the siphon equal one. Hence,
\[ W_c(M) = 3. \] The siphon can never be emptied as can easily be seen.
Thus, we can derive a necessary condition for a siphon to be emptyable.
It is based only upon the values of the initial weighted marking, and
of the filling and emptying factors. Let

\[ T^+_c = \{ t_i \} \text{ where } i = 1, \ldots, |T^+_c| \]

\[ T^-_c = \{ t_i \} \text{ where } i = 1, \ldots, |T^-_c| \]

**Lemma 5:** If a siphon covered by an S-cover \( C \) can be emptied, then

\[ \exists \{ a_i \}_{i=1}^{\left| T^+_c \right|}, \{ b_i \}_{i=1}^{\left| T^-_c \right|}, \quad a_i, b_i \in N_0 \]

\[ W_c(M) \geq \sum_{i=1}^{\left| T^+_c \right|} a_i \#_{c}^{\text{fill}}(t_i) - \sum_{i=1}^{\left| T^-_c \right|} b_i \#_{c}^{\text{empty}}(t_i) \quad (5) \]

The sums are supposed to equal zero if \( T^+_c = \emptyset \) or \( T^-_c = \emptyset \), respectively.

The proof follows with \( W_c(\tilde{M}) = 0 \) and \( a_i = h_{\tilde{M}^f}(t_i), b_i = h_{\tilde{M}^e}(t_i) \) from theorem 4. Condition (5) is trivially satisfied by \( a_i, b_i = 0 \) if \( W_c(M) = 0 \).

Another necessary condition for a siphon to be emptyable follows
also from theorem 4.

**Lemma 6:** If a siphon covered by an S-cover \( c \) can be emptied under a
marking \( M \), then

\[ \exists t, t \in T^-_c: \sum_{p \in T^+_t} K_c(p) = 0, \quad (6) \]

or \( W_c(M) = 0 \).

**Proof:** Consider the single firing \( M \xrightarrow{t} \tilde{M} \) of a transition \( t \in T^-_c \) which empties the siphon. Thus, \( W_c(M) > 0, W_c(\tilde{M}) = 0 \). With proof of lemma 3,
line 3:

\[ W_c(\tilde{M}) = W_c(M) - \sum_{p \in T^+_t} K_c(p) + \sum_{p \in T^-_c} K_c(p). \]
Since \( t \) is activated under \( M \) we have
\[
\sum_{p \in t} \mathcal{K}_c(p) \leq \mathcal{W}_c(M).
\]
This implies that \( \mathcal{W}_c(M) \) can only equal zero if
\[
\sum_{p \in t} \mathcal{K}_c(p) = 0.
\]

Consider for example fig. 4.3. It shows a siphon, \( \{p_1, p_2\} \), covered by a 1-S-cover. The siphon has a filling transition \( t^+ \) with \( K_c^\text{fill}(t^+) = 1 \). Thus, numbers \( a_i \) and \( b_i \), satisfying condition (5) may be found. However, the siphon cannot be emptied. For its emptying transition \( t^- \), condition (6) is not satisfied since \( K_c(p_2) = 1 \) and \( p_2 \in t^- \). We see that \( t^- \) returns, every time it fires, one token to the siphon. Thus, it can never empty the siphon completely.

The set of places \( \{p_1, p_2\} \) is also a trap. A 1-T-cover is obtained by removing, say, the open path \( (p_1, t^+, p_1) \) from and adding the open path \( (p_1, t^-, p_2) \) to the 1-S-cover. It happens to be not only in this example that a siphon is also a trap, but in all cases where condition (6) is not satisfied. This leads to lemma 7.

**Lemma 7:** A siphon is also a trap iff condition (6) is not satisfied.

The proof is done by considering a place \( p_0 \) of the siphon \( P \).
Every edge \( (p_0, t) \in \text{pre} \) is covered or can be covered by an open path, according to definition 11, if \( t \in T_c^+ \) or \( T_c^0 \). Every edge \( (p_0, t) \in \text{pre} \), where \( t \in T_c^- \), can be covered by an open path leading back into the siphon if there is a \( p \in t^- \) such that \( p \in P \), that is, \( \sum_{p \in t^-} K_c > 0 \). If there is no such place \( p \in P \), that is, if (6) does not hold, then we can construct a T-cover of the siphon as explained.

We now classify siphons which are supposed to be covered by an 1-S-cover \( c \) according to their filling and emptying transitions, and consider whether they can be emptied or not. Trivially, every siphon can be emptied under a marking \( M \) if \( \mathcal{W}_c(M) = 0 \). This will not be mentioned in our subsequent considerations.

Four classes of siphons can be distinguished:

1) \( T_c^- = T_c^+ = \emptyset \).
These siphons are also invariants and traps. Their weighted marking is constant since there are no transitions the firing of which increases or decreases it. Consequently, there exists a simple I-cover covering these siphons. However, c itself needs not necessarily also be an I-cover.

2) $T^-_c = \emptyset$, $T^+_c \neq \emptyset$.

These siphons are also traps, but not invariants. This means, the weighted marking is not necessarily constant for all $\tilde{M} \in [M]$.

3) $T^+_c = \emptyset$, $T^-_c \neq \emptyset$.

These siphons can be emptied if both conditions (5) and (6) are satisfied and if the rest of the net allows the emptying transitions to fire sufficiently many times.

If at least one of the conditions (5) and (6) is not satisfied, then there exists a firing sequence $M \rightarrow \tilde{M}$ such that the transitions $t \in T^-_c$ are dead under $\tilde{M}$ without the siphon being empty. This shows that the problem of transitions that are dead without a siphon being empty is related to the existence of certain types of siphons. But this will not be discussed further in this paper.

4) $T^+_c \neq \emptyset$, $T^-_c \neq \emptyset$.

If condition (6) is not satisfied then the siphon is also a trap. However, it may still increase and decrease its weighted marking.

If both conditions (5) and (6) are satisfied, then it depends on the external circumstances whether the siphon can be emptied or not. External circumstances could cause an emptying transition to be dead or to be capable of firing only a limited number of times before it becomes dead. Then the siphon might not be emptyable. Other external reasons might restrict the slack between emptying and filling transitions, as we have already seen.

4.3 A Criterion Using the Slack

We consider in the sequel siphons of type 4), and we assume that the necessary conditions (5) and (6) for the siphon to be emptyable are satisfied.
The idea is that a siphon cannot be emptied if the slack of the emptying transitions with respect to the filling ones is smaller than the "content" of the siphon. Considering weight factors, this is expressed by the following condition for a siphon with an emptying transition $t^-:
abla^c_p(t^-, t^+_c, M) \leq \frac{W_c(M)}{k^c_{\text{empty}}(t^-)} \tag{*}

In nets with regulation circuits, this condition can be applied directly. If it is satisfied for a siphon, then the siphon cannot be emptied. But in more general nets, certain complications may arise as in the example of Fig. 4.4 a).

The siphon $\{p_0\}$ is covered by the 1-S-cover c as shown, with $T^+_c = \{t^+_1\}$, $k^c_{\text{fill}}(t^+) = 1$, and with $T^-_c = \{t^-\}$, $k^c_{\text{empty}}(t^-) = 1$. The weighted marking $W_c(M) = 3$. The slack $\nabla^c_p(t^-, t^+_c, M) = 0$, since the path-system leading from $t^+$ over $p_1, p_2, p_3$ to $t^-$ contains no tokens. Therefore, by applying condition (*) one would draw the (wrong) conclusion that the siphon $\{p_0\}$ cannot be emptied.

Consider a marking sequence $M \rightarrow \overline{M}$ where $t^+$ fires three times, and $t^-$ does not fire. Then, $W_c(\overline{M}) = 6$ and $\nabla^c_p(t^-, t^+_c, \overline{M}) = 6$ since $\overline{M}(p_1) + \overline{M}(p_2) + \overline{M}(p_3) = 6$. Now, it is obvious that the siphon can be emptied.

As a consequence, condition (*) cannot be applied in general. But where does this problem come from? In our example, the firing of $t^+$ increases $W_c(M)$ by 1 and $\nabla^c_p(t^-, t^+_c, M)$ by 2. The firing of $t^-$ decreases $W_c(M)$ by 1 and $\nabla^c_p(t^-, t^+_c, M)$ by 1. We see that the problem arises because $t^-$ does not decrease the values of $W_c(M)$ and $\nabla^c_p(t^-, t^+_c, M)$ by the same degree as $t^+$ increases them.

Modifications of the net, as shown in fig. 4.4 b) and c), cause the problem to disappear. With the modification of fig. 4.4 b), the firing of $t^+$ increases $\nabla^c_p(t^-, t^+_c, M)$ by 1; with the modification of fig. 4.4 c), the firing of $t^-$ decreases $\nabla^c_p(t^-, t^+_c, M)$ by 2. In both cases, the siphons cannot be emptied if condition (*) is satisfied.
This suggests that condition (*) can be applied if the path-systems leading from $T^-_c$ to $T^+_c$ possesses certain properties; these will be analyzed in section 5.

5. Equimoving Systems of Path

In this section, we analyse under which conditions the slack, under a given marking, between emptying and filling transitions determines for all reachable markings whether a siphon can be emptied or not. To do this, we introduce a special kind of open cover, called a $(T^* \rightarrow T)$-cover. It will be shown that siphons and, very often, also the paths that connect the filling and emptying transitions can be considered as $(T^* \rightarrow T)$-covers. Then we are able to define a particular property, called "equimovingness", as a property of a set of $(T^* \rightarrow T)$-covers connecting the same sets of transitions. A necessary condition for a siphon to be emptyable, and a sufficient condition for a marking to be live-1 will be derived for nets in which there exists at least one equimoving path between the filling and emptying transitions of each siphon.

5.1 $(T^* \rightarrow T)$-covers

A $(T^* \rightarrow T)$-system of open paths covering a set of places $P$, also called a $(T^* \rightarrow T)$-cover of $P$, is defined as follows (see also fig. 5.1):

Definition 13: Let $P \subseteq \Sigma$, $T^*$, $T \subseteq \Pi$, $T^* \cap T = \emptyset$, 

\[ P_0 = \{ p \in P | \ \exists \ \tau^* \ \in \ \tau^* \ \text{Ap} \ \subseteq T \} \]

and $c$ a system of open paths. $c$ is called a $(T^* \rightarrow T)$-cover of $P$ if

\[
\begin{align*}
\forall t \in T^* : & \ t \cap P \neq \emptyset & (i) \\
\forall t \in T : & \ t \cap P \neq \emptyset & (ii) \\
\forall p \in P \setminus P_0, \ \forall t_1, t_2 \in \tau^* \ \text{Ap}, \ \forall t_1, t_2 \in \tau^* \ \text{Ap}: & \ L_c(t_1, p) = L_c(t_2, p) = L_c(p, t_1) = L_c(p, t_2) & (iii) \\
& \ = \ : K_c(p) > 0 & \\
\forall t \in T^* : & \ \sum_{p \in t} K_c(p) < \sum_{p \in t} K_c(p) & (iv) \\
\forall t \in T : & \ \sum_{p \in t} K_c(p) > \sum_{p \in t} K_c(p) & (v)
\end{align*}
\]

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For $p \in P_0$, set $K_c(p)$ any positive number such that (iv) and (v) hold.

According to definition 13, there are two particular sets of transitions, $T^*$ and $T$, connected by the system of open paths as required by (i) and (ii). (iii) requires that, apart from two exceptions described below, all input and output edges of every covered place must have the same covering degree. That is, the set of places $P$ is covered by a cover similar to an I-cover.

Edges from $t$ to $p$ ($p, t \in \text{post}$) where $t \in T^*$, $p \in P$ and edges from $p$ to $t$ ($p, t \in \text{pre}$) where $p \in P$, $t \in T$ need not have the same covering degree as the other edges of a place $\in P$. But note that edges from $p$ to $t$ ($p, t \in \text{pre}$) where $p \in P$, $t \in T^*$, and edges from $t$ to $p$ ($p, t \in \text{post}$) where $t \in T$, $p \in P$ must satisfy the covering condition. This is illustrated in fig. 5.1.

We must distinguish between places $\in P_0$ and $\in P \neq P_0$ as illustrated by fig. 5.2 for the following reason. Places $\in P_0$ have only input and output edges that fall under the exception from rule (iii) described above. Therefore, no covering degree $K_c(p_0)$, $p_0 \in P_0$ is defined by (iii). If, for a place $p_0 \in P_0$, the transitions $p_0$ and $p_0'$ are not covered by an open path, then we can define any positive integer as covering degree. But, if those transitions are covered then $K_c(p_0)$ must be large enough to satisfy (iv) and (v).

Since the firing of transitions covered by an I-cover does not change the weighted marking of the I-cover, we see that the firing of transitions covered by the $(T^* \rightarrow T)$-cover other than transitions $\in T^*$, $T$ does not change its weighted marking. From (iv) and (v), we conclude that the firing of $t \in T^*$ increases, and that of $t \in T$ decreases the weighted marking. In the sequel, we first consider an example of a $(T^* \rightarrow T)$-cover, then we formalize these ideas. Consider fig. 5.3. It shows a $(T^* \rightarrow T)$-cover with $T^* = \{t^*\}$, $T = \{t\}$ covering $P = \{p_1, p_2, p_3\}$. Since $P_0 = \emptyset$, the covering of the edges of every place $\in P$ satisfies condition (iii). Thus, we obtain $K_c(p_1) = 2$, $K_c(p_2) = K_c(p_3) = 1$. The transition $t^*$ leads from $p_2$ with $K_c(p_2) = 1$ to
with $K_c(p) = 2$, satisfying (iv); $t$ leads from $p_2$ with $K_c(p_2) = 1$ to $p_4$ with $K_c(p_4) = 0$, satisfying (v).

**Definition 14:** Let $P \subseteq E$, $T^* \subseteq \Pi$ and $T^* \cap T = \emptyset$. $P$ is called a $(T^* \rightarrow T)$-system if there exists a cover $c$ such that $c$ is a $(T^* \rightarrow T)$-cover of $P$.

Definition 14 says that a set of places $P$ which can be covered by a $(T^* \rightarrow T)$-cover $c$ is called a $(T^* \rightarrow T)$-system. The weighted marking of a $(T^* \rightarrow T)$-system is defined in the same way as that of an invariant, siphon or trap:

$$W_c(M) = \sum_{p \in P} K_c(p) \times M(p)$$

The filling and emptying factors of transitions $t^* \in T^*$ and $t \in T$, respectively, are defined in the same way as in definition 12:

$$K_c^{\text{fill}}(t^*) = \sum_{p \in t^*} K_c(p) - \sum_{p \in t} K_c(p) \quad (>0 \text{ according to (iv)})$$

$$K_c^{\text{empty}}(t) = \sum_{p \in t} K_c(p) - \sum_{p \in t^*} K_c(p) \quad (>0 \text{ according to (v)})$$

In the example of Fig. 5.3, $W_c(M) = 2M(p_1) + M(p_2) + M(p_3)$, and $K_c^{\text{fill}}(t^*) = 1$, $K_c^{\text{empty}}(t) = 1$.

**Theorem 8:** Let $P \subseteq E$, $T^* \subseteq \Pi$, $T^* \cap T = \emptyset$, $c$ a $(T^* \rightarrow T)$-cover of $P$, $M$ a marking. Then $\forall \tilde{M} \in [M]: W_c(\tilde{M}) = W_c(M) + \sum_{t \in T^*} K_c^{\text{fill}}(t) \times h_{\tilde{M}+\tilde{M}}(t) - \sum_{t \in T} K_c^{\text{empty}}(t) \times h_{\tilde{M}+\tilde{M}}(t)$. The proof is similar to that of Lemma 4.4.

Theorem 8 says that only the firing of transitions $t^* \in T^*$ and $t \in T$ can increase and decrease, respectively, the weighted marking of a $(T^* \rightarrow T)$-system. As a consequence of Lemma 8, the construction of a $(T^* \rightarrow T)$-cover $c$ yields an upper bound for the slack $Sp(t,T^*,M)$ with $t \in T$. With $\tilde{M} \in [M]$, the upper bound for the slack is:

$$Sp(t,T^*,M) = h_{\tilde{M}+\tilde{M}}(t)$$

where $\tilde{M}$ is such that $h_{\tilde{M}+\tilde{M}}(t) = 0$ for every $t \in T^*$ and $W_c(\tilde{M}) = \min$. 

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Hence, with theorem 8
\[
Sp(t, T^*, M) \leq \frac{w_c(M)}{K^\text{empty}(t)}
\]

(**)

Consider for example Fig. 5.4a. A \((T^* \rightarrow T)\)-cover \(c\) is shown in this figure with \(T^* = \{t^*\}\), \(T = \{t\}\), \(K_c^\text{fill}(t^*) = 2\), \(K_c^\text{empty}(t) = 2\). Under the marking \(M\) as indicated, \(w_c(M) = 5\). Therefore, \(Sp(t, t^*, M) \leq [5/2] = 2\).

We can derive only an upper bound for the slack, but not the slack itself, by considering a \((T^* \rightarrow T)\)-system. This is because a transition of the \((T^* - T)\)-system might be dead due to circumstances exterior to the cover. A transition might also be "dead" as long as no \(t^* \in T^*\) fires due to "internal" reasons, as Fig. 5.4b shows. This example shows a \((T^* \rightarrow T)\)-cover, where \(t_0\) is dead as long as \(t^*\) does not fire. According to (**), we obtain the upper bound of 2 for \(Sp(t, t^*, M)\). However, \(Sp(t, t^*, M) = 1\).

Theorem 9: Let \(T^*, T \subseteq \Sigma\), \(T^* \cap T = \emptyset\), \(t \in T\), and \(M\) be a marking.
\[
Sp(t, T^*, M) \leq \min \left\{ \frac{w_c(M)}{K^\text{empty}(t)} \mid c \text{ is a } (T^* \rightarrow T)\text{-cover} \right\}
\]
The proof follows immediately from (**), and definition 13.

Theorem 9 says that in case there are several \((T^* \rightarrow T)\)-covers between the sets of transitions \(T^*, T\), an upper bound for the slack between \(t \in T\) and \(T^*\) is determined by the \((T^* \rightarrow T)\)-cover with the "minimal content".

5.2 Siphons and \((T^* \rightarrow T)\) systems

In the preceding section we saw that both \((T^* \rightarrow T)\)-systems and siphons behave in a similar way. But, obviously, they are not equivalent since there are \((T^* \rightarrow T)\)-systems \(P\), where all \(p \in T^*\) \(\notin P\). Such \((T^* \rightarrow T)\)-systems are not siphons.

Theorem 10: Let \(c\) be an \(S\)-cover of \(P \subseteq \Sigma\). Construct a cover \(c'\) by removing as many elementary paths covering a transition \(t \in T^+\) as the difference \((\Sigma_{p \in T^+} K_c(p) - \Sigma_{p \in T^+} K_c(p))\) states. Then, \(c'\) is a \((T^* \rightarrow T)\)-cover of \(P\).
Proof: (i) and (iv) follows from the definition of $T^+_c$.
(ii) and (v) follows from the definition of $T^-_c$.
(iii) is guaranteed by the construction of $c^i$.

Consider for example Fig. 5.5 which shows a siphon $P = \{p_1, p_2, p_3\}$ covered by a 1-S-cover $c$ where $T^+_c = \{t_1\}$, $T^-_c = \{t_2\}$. Except $p_3$, all places already satisfy the condition (iii). By removing one open path covering $t_1$ (i.e., either $(p_3, t_1, p_2)$ or $(p_3, t_1, p_1)$), (iii) becomes true for $p_3$. The edges $(p_2, t_1)$ and $(p_4, t_4)$ (as post), respectively (whence the path is being removed) are excepted from (iii).

In many cases, not only a siphon can be considered as a $(T^* \rightarrow T)$-system, but also other paths which lead from $T^*$ to $T$. Consider for example Fig. 4.1. The path leading from $t_1$ to $t_2$ across $p_1$ and $p_4$ can be considered as a $\{t_1 \rightarrow t_2\}$-cover $c$ which covers $[p_1, p_4]$ by an open path $(p_1, t_4, p_4)$. Consider also Fig. 4.4. In case a) and c), a $\{t^* \rightarrow t^-\}$-cover that covers $[p_1, p_2, p_3]$ can be constructed; in case b) one that covers $[p_1, p_3]$.

This suggests that the case in which there are several $(T^* \rightarrow T)$-systems connecting $T^*$ and $T$, is of particular interest. Theorem 9 shows that, in this case, the upper bound for the slack under a given marking is determined by the $(T^* \rightarrow T)$-system with "minimal content". But the example of Fig. 4.4. a) shows, that, in general, the system having the "minimal content" under a particular marking $M$ not necessarily has the "minimal content" under all $\tilde{M} \in [M]$.

5.3 Equimoving $(T^* \rightarrow T)$-Systems

Suppose there are several $(T^* \rightarrow T)$-systems between $T^*$ and $T$. We call two or more such systems equimoving if their respective filling and emptying factors are, in a certain sense, proportional to each other.

Then we show that the weighted markings of equimoving $(T^* \rightarrow T)$-systems leave certain relations invariant.

Definition 15: Let $U, \tilde{U} \subseteq \Sigma$, $T,T^* \subseteq \Pi$, $T^* \cap T = \emptyset$, and $c$ a $(T^* \rightarrow T)$-cover of $U$, $\tilde{c}$ a $(T^* \rightarrow T)$-cover of $\tilde{U}$.
U and \( \hat{U} \) are called equimoving if \( g, g \) is rational, \( \forall t^x \in T^x, \forall t \in T \):

\[
\frac{K^\text{fill}(t^x)}{K^\text{fill}(t^x)} = \frac{K^\text{empty}(t)}{K^\text{empty}(t)} = g(-g(U, \hat{U}))
\]

\( g(U, \hat{U}) \) is called equimoving ratio.

Consider, for example the siphon c and the path-system between \( T^x_c \) and \( T_c \) of fig. 4.4. In case a), these two \( (T^x_c \rightarrow T^{-1}_c) \)-systems are not equimoving, in case b) and c), they are equimoving with the equimoving ratio \( g=1 \) and \( 1/2 \), respectively.

**Theorem 11:** Let \( x \) be an integer, \( U, \hat{U} \subseteq \Sigma, T, T^x \subseteq \Pi, T^x \cap T = \emptyset, M \) a marking, \( c, \hat{c} \) a \( (T^x \rightarrow T) \)-cover of \( U, \hat{U} \) respectively and \( U, \hat{U} \) equimoving with an equimoving ratio \( g(U, \hat{U}) \). Then

\[
\forall \bar{M} \in [M]: W^c_c(\bar{M}) = W^c_c(M) + x \iff W^\hat{c}_c(\bar{M}) = W^\hat{c}_c(M) + 1 \cdot \frac{x}{g}
\]

The proof follows from theorem 8 and definition 15 with

\[
x = \sum_{t \in T^x} K^\text{fill}(t) \ast h^{\bar{M}}_{\hat{M}}(t) - \sum_{t \in T} K^\text{empty}(t) \ast h^{\bar{M}}_{\hat{M}}(t).
\]

Theorem 11 says that the weighted markings of two \( (T^x \rightarrow T) \)-systems change "in proportion" if the two systems are equimoving. Consider for example the siphon and the \( (\{t^1_1\} - \{t^2_1\}) \)-system \( \{p_1, p_4\} \) of fig. 4.1. They are equimoving with ratio \( g=1 \). Therefore, for all \( \bar{M} \in [M] \),

\[
W^\text{siphon}(\bar{M}) = W^\text{path}(\bar{M}) + 1, \text{ since } W^\text{siphon}(M)=2, W^\text{path}(M)=1.
\]

We also see that the upper bound for the slack as determined by the siphon and the path, respectively, are not independent; for every successor marking, the upper bound determined by the siphon keeps surpassing by 1 the upper bound determined by the path. In general, we can state the following:

**Corollary 12:** The assumptions are as in theorem 11. Then \( \forall t \in T, \forall \bar{M} \in [M] \):

\[
\frac{W^c_c(M)}{K^\text{empty}(t)} \left\{ \leq \right\} \frac{W^\hat{c}_c(M)}{K^\text{empty}(t)} \Rightarrow \frac{W^c_c(\bar{M})}{K^\text{empty}(t)} \left\{ \leq \right\} \frac{W^\hat{c}_c(\bar{M})}{K^\text{empty}(t)}
\]

The proof follows directly from theorem 11 and definition 15.
Corollary 12 says the following: if several \((T^* \rightarrow T)\)-systems are equimoving then the system with the smallest (or largest) factor \(W_c(M)/K^\text{empty}_c(t)\) under the initial marking, has the smallest (or largest) factor under all reachable markings. This factor gives an upper bound for the slack of the emptying transition with respect to the set \(T^*\) of filling transitions, as theorem 9 shows. Consequently, it is always the same one out of several equimoving systems from which an upper bound for the slack may be derived, regardless whether filling, emptying or neutral transitions fire.

If the \((T^* \rightarrow T)\)-systems in question are not equimoving, this is not generally true. Consider for example Fig. 4.4a). Under the initial marking \(M\), \(W_{\text{sink}}(M)/K^\text{empty}_{\text{sink}}(t^-) = 3\), \(W_{\text{path}}(M)/K^\text{empty}_{\text{path}}(t^-) = 0\). Under \(\tilde{M}\), after \(t^+\) has fired four times, \(W_{\text{sink}}(\tilde{M})/K^\text{empty}_{\text{sink}}(t^-) = 7\), \(W_{\text{path}}(\tilde{M})/K^\text{empty}_{\text{path}}(t^-) = 8\).

Given now a siphon and a path system from the filling to the emptying transition which is equimoving with the siphon. Then, an upper bound for the slack of the emptying transitions with respect to the filling ones is given by the path-system. If this upper bound for the slack is smaller than the number of times emptying transitions must fire to empty the siphon, then it will always be so. That is, the siphon can never be emptied. We obtain the following theorem:

**Theorem 13:** Let \(c\) be an \(S\)-cover of \(P \subseteq \Sigma\) where \(T^+_c \neq \emptyset\), \(T^-_c \neq \emptyset\). If \(P\) can be emptied then \(\forall U \subseteq \Sigma\), \(U\) equimoving with \(P\), \(\hat{c}\) is \((T^+_c \rightarrow T^-_c)\)-cover of \(U\), \(\forall t \in T^-_c:\)

\[
\frac{W_c(M)}{K^\text{empty}_c(t)} \leq \frac{W_c(M)}{K^\text{empty}_c(t)} \tag{13}
\]

The proof follows from corollary 12.

Theorem 13 gives a necessary condition for a siphon to be emptyable. It requires that the upper bound for the slack determined by all equimoving \((T^+ \rightarrow T^-)\)-systems is greater or equal to the number of times the
emptying transitions fire to empty the siphon. The reverse of theorem 13 gives a sufficient condition for a siphon not to be emptyable. A siphon $c$ cannot be emptied if there exists an equimoving $(T_c^+ \rightarrow T_c^-)$-cover $\hat{c}$ such that

$$\forall t \in T_c^-, \quad \frac{W_c(M)}{K_c^\text{empty}(t)} > \frac{W_c(M)}{K_c^\text{empty}(t)}$$

From (14), we get a sufficient condition for a marking to be live-1.

**Theorem 14:** A marking $M$ of a Petri net $N$ is live-1 if there is in $N$ no siphon $c$ that is empty under $M$ (i.e., $W_c(M) = 0$) and for every siphon $c$,

- condition (5) is not satisfied
- or condition (6) is not satisfied
- or there is an equimoving $(T_c^+ \rightarrow T_c^-)$-cover $\hat{c}$ such that condition (14) is satisfied.

Theorem 14 follows directly from lemmas 2a, 5, 6 and theorem 14. In a Petri net where no dead transitions exist without a siphon being empty, as, e.g., in simple nets [3], the above condition is a sufficient condition for a marking to be live-1 and live-5.

### 5.4 Remarks

1) We may use theorem 11 to obtain another kind of invariant: we subtract two equimoving $(T^* \rightarrow T)$-systems (each covered by a $(T^* \rightarrow T)$-cover $c', c''$) with equimoving ratio $g$. The weight factors of the resulting invariants are seen to consist of the difference $K_{c'}(p) - g \neq K_{c''}(p)$. Hence, they may be negative.

The resulting weighted marking is:

$$W_c(M) = W_{c'}(M) - g \cdot W_{c''}(M)$$

Now, $\forall M \in [M]$:

$$W_c(M) = W_c(M)$$

since $W_c(M) = W_{c'}(M) - g \cdot W_{c''}(M)$

$$= W_{c'}(M) + x - g(W_{c''}(M) + x/g)$$

$$= W_{c'}(M) - g \cdot W_{c''}(M)$$
ii) Theorem 13 gives a necessary, but not a sufficient condition for a siphon to be emptyable, for the following reasons. Apart from the fact that there may be dead transitions on a path from the filling to the emptying transitions, there may exist path-systems that lead from the filling to the emptying transitions, that are not equimoving \((T^+ \rightarrow T^-)\)-systems (compare Fig. 4.4). However, they may prevent a siphon from actually being emptied.

iii) To conclude with, we consider an example from Commoner [3] in Fig. 5.6. There is a siphon \(\{a,b,c\}\) with the filling transition \(t^+\) and the emptying transition \(t^-\). The path which leads from \(t^+\) to \(t^-\) over the places b and d forms a \((T^+ \rightarrow T^-)\)-system (covered by the open path \((b,t,d)\)). Since all filling and emptying factors equal 1, both systems are equimoving.

According to (14), the siphon cannot be emptied, because its weighted marking equals 1, but the marking of the path equals 0. There are no other siphons with emptying transitions and no dead transitions. Hence, the marking is live-1 and live-5.

Putting any number of tokens on b increases the weighted marking of both the siphon and the path by the same degree. Thus, the new marking would also be live.

Putting any number of tokens on d increases the weighted marking of the path and, consequently, the slack between \(t^-\) with respect to \(t^+\). Condition (14) is no longer satisfied, and the marking is not live-1.

This example illustrates that, in Petri nets which are more general than free choice nets, there may be a marking \(M\) which is live-1 or live-5. However, there may exist a marking \(M'\geq M\) that is not live-1 or live-5.

iv) It can be shown that "nets with regulation circuits" are a very special subclass of the class of Petri nets we are interested in, namely nets in which, for every siphon, all path systems leading from its filling transitions to its emptying transitions are equimoving with the siphon.
6. Conclusions

In this paper, we have developed a comprehensive approach which encompasses ideas developed by Commoner and Hack and by Lautenbach.

The general notion of a siphon was presented, and it was shown in which way its filling and emptying transitions may be determined. Theorem 2 and theorem 4 indicate that these notions are the central ones for the solution of the live(-1)ness problem.

After determining, for every siphon of a Petri net, its filling and emptying transitions, the "only" thing we still have to do is to find out whether the emptying transitions can fire sufficiently many times to make the weighted marking of the siphon equal zero. For this purpose we must not only consider the Petri net but also its marking (whereas the notions of siphon and filling and emptying transitions are dependent on the structure of the net only).

Then, we have introduced the notion of equimoving \((T^* \to T)\)-systems, and we have shown that the behaviour of a Petri net is comparably simple to analyse if a siphon and path systems from its filling to its emptying transitions are equimoving.

References


10 K. Lautenbach, Liveness in Petri Nets, GMD, IISF, Bericht 02.1/75-7-29.


Fig. 2.1 The dot notation

Fig. 2.2 Slack from \( t_2 \) to \( t_1 \)

Fig. 3.1 A System of open paths

Fig. 3.2 Systems of open paths
Fig. 3.3 Empty siphon \( \{p_1, p_2\} \)

Fig. 3.4 Dead transitions \( t_1, t_2 \)
Fig. 4.1 The siphon \{p_1, p_2, p_3\}

Fig. 4.2 A siphon that cannot be emptied

Fig. 4.3 A siphon that cannot be emptied

Fig. 4.4 Siphon and path system from \( t^+ \) to \( t^- \)
Fig. 5.1 A \((T^\ast + T)\)-system

Fig. 5.2 A \((T^\ast + T)\)-system

Fig. 5.3 \((T^\ast + T)\)-system

Fig. 5.4 Upper bound for the slack \(S\) \((t, t^\ast, M)\)
Fig. 5.5  A siphone

Fig. 5.6  Commoner's example