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The present report shows that postulating density amounts to postulating that (a) each single history of a process is either infinite or has a first cause (but not both), and (b) each single future of a process is either infinite or has a last effect (but not both). This result is interpreted and applied to the question of Turing-computability.

A Theorem on the Characteristics of Non-sequential Processes

By

E. Best

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The present report shows that postulating density amounts to postulating that (a) each single history of a process is either infinite or has a first cause (but not both) and (b) each single future of a process is either infinite or has a last effect (but not both). This result is interpreted and applied to the question of Turing-computability.

About the author
Mr. E. Best received the diploma in Computer Science from the University of Karlsruhe, W. Germany, in 1974. Subsequently he was a research scientist at the GMB/Bonn, Institute for EDP and Law; since June 1976, he has been a Research Associate at the University of Newcastle upon Tyne.
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1) Introduction

Given a piece of hardware and programming primitives, it is a sensible question to ask what kinds of computations can be carried out with them. Now that increasingly versatile hardware can be cheaply manufactured it is a sensible question to ask what kinds of computations can be carried out in principle.

Although this question has already received an answer with the notion of Turing-computability and Church's thesis, it has been approached from a different point of view by Petri. In [1] he investigates real processes in general (that is, processes occurring in the real world, not necessarily "computations" if this notion is used in a constrained sense). He introduces an axiom of density whose significance lies in the hypothesis that it is observed by real processes; the latter phrase is to mean that any (formal) description violating the axiom is not an appropriate model of a real process.

The hypothesis has the characteristics of a law of nature and must therefore be justified. We list five points in defence of its validity.

First, the density hypothesis is meaningful in its own right; as we shall see, it intuitively says that "the real world is everything which is the case".

Second, the density axiom can be proved equivalent to another statement which also provides a meaningful albeit different interpretation: no infinite history (of a real process) is preceded by "first causes" and no infinite future has "last effects". The proof of this equivalence is the main aim of this report.

The author had conjectured the equivalence in a talk given at the workshop "Verification of Parallel Processes" (University of Aarhus, June 1977) in conjunction with an attempt to relate work of Hewitt (appeared in [4]) to
[1]. Our third point is that indeed, under some reasonable assumptions, some of the laws of [4] can be shown to be a direct consequence of the axiom of density. We will only hint this relationship by proving a theorem having the same form as three laws of [4].

Fourth, the density axiom resolves certain paradoxes. As [1] shows, the venerable paradoxon of "Achilles and the tortoise" can only be stated under the assumption that constructions are permitted which violate the axiom of density. We will show that the same reasoning applies to the diagonalization paradoxon which is widely used to prove that certain functions are not Turing-computable. Thus, a link between the computability question and the density axiom is established.

As to the fifth point: all preceding arguments are worthless as defence of the density axiom if all processes observing the axiom are finite; for example, the limits of computability show up only if infinite processes are taken into consideration - at least theoretically. It is true that all finite processes satisfy the axiom; but the converse is not true. The axiom of density is weak enough not to exclude the infinite case which interests us in the first place.

The paper is organized as follows: chapter 2 provides the necessary heuristic and mathematical framework up to the definition of K-density (the density axiom) and some simple observations. The contents of [1] are recalled as far as is suitable for our purposes. Chapter 3 then states and proves our main theorem: a process description is K-dense if and only if it contains certain characteristic causal components. In section 3.4, the theorem which bridges to [4] is proved. The two above-mentioned paradoxes are analysed in chapter 4, and some possible interpretations of our main theorem are deduced in the same chapter.
2. Basic preparations and results (cf [1])

2.1 Heuristics and notation

We choose condition-event structures as our mathematical tool for the description of real processes; it is outside the scope of this report to show that indeed these structures are appropriate for that purpose. We merely note that cause-effect phenomena (causal dependence and concurrency) can be captured, so as to correctly describe non-sequential processes.

When pictorial representation is desired, events are drawn as squares; an event comprises a coincidence class of changes, that is, it is defined by the change(s) it effects. Conditions (drawn as circles) are defined to be those entities which are changed by an event. A relation \( F \) (for "flow") describes this causal connection; \( F \) is represented as a set of arrows:

\[
\begin{align*}
\ & b \quad F \quad e\\
\ & e \quad F \quad b
\end{align*}
\]

indicates that event \( e \) ends condition \( b \), and

indicates that event \( e \) starts condition \( b \).

We assume that events and conditions are unique (occurring and holding, respectively, only once); the description takes a process "as it happens" and does not allow the specification of repetition or alternatives. Concurrency will be defined as causal independence. In [1], condition-event structures with these characteristics are called causal nets.

As far as the notation is concerned, we use: labels (\( \text{Un} \)) to introduce notions and abbreviations; labels (\( \text{An} \)) to identify axioms postulating properties of such notions; labels (\( \text{In} \)) to prefix lemmas and labels (\( \text{Tn} \)) to indicate theorems. Proofs will be done largely by contradiction; the string "\( \text{C! to} \)" links the contradicting statements and the conclusion is preceded by "hence". We use set notation but do not specify index ranges when unambiguous; for example, \( \{ b_i \} \) is an abbreviation for \( \{ b_i \mid i \in \mathbb{N} \} \).
2.2 Causal nets

(D1) Let \( E \) be a set of events and \( B \) a set of conditions; 
\[ X := E \cup B \quad (\text{the set of "elements"}) \]

(A1) \( E \neq \emptyset, \ B \neq \emptyset, \ E \cap B = \emptyset \)

(D2) Let some "start" and "end" relation be given by 
\[ F : \subseteq E \times B \cup B \times E \]

(A2) \( X = \text{field}(F) \)
("there is no unconnected event or condition")

(D3) \[ p^+ := \bigcup_{n \in \mathbb{N}} p^n, \quad \preceq := p^+ \]
("causal dependence", the transitive closure of \( F \))

(D4) \( \forall x \in X: \ x := F^{-1}(x), \ x^* := F(x) \)
(this notation is usual)

(A3) \( \forall b \in B: \ \|b\| \leq 1 \wedge \|b^*\| \leq 1 \)
("no alternatives in a causal net"; if \( b \in B \), we occasionally commit the inaccuracy to denote the single element of \( *b \ (b^*) \) - if it exists - also with \( *b \ (b^*) \))

(A4) \( \preceq \cap \sim^1 = \emptyset \)
("no repetition in a causal net")

(D5) Concurrency is given by: 
\[ \text{co} : \subseteq X \times X \]
\[ \overline{\text{co}} := X \times X \setminus \text{co} \]

(A5) \( \text{id} \big|_{X} \subseteq \text{co} \)
("\( x \) is concurrent to \( x \)"")

(A6) \( \text{co} = \overline{\text{co}}^{-1} \)
("\( \text{co} \) is symmetric"")

**But**: \( \text{co} \) is not necessarily transitive!

(A7) \( \text{co} = X \times X \setminus (\preceq \cup \preceq^{-1}) \)
("\( \text{co} \) means causally independent")

(L1) \( \preceq \) is irreflexive
Proof: from (A7) and (A5) \( \square \)
(L2) \textless is transitive

Proof: from (D3) \hspace{1cm} \square

\textless is thus partial ordering (in terms of causality).

(L3) \overline{co} is symmetric

Proof: from (D5) and (A6) \hspace{1cm} \square

(L4) \forall y \in x: (x \in \overline{y} \land x' \in \overline{y} \Rightarrow x \ co x') \land

\hspace{1cm} \land (x \in y' \land x' \in y' \Rightarrow x \ co x')

Proof: (i) by (A3) and (A5) if y \in B

(ii) y \in \bar{E}:

(ii1) x,x' \in \overline{y};

\hspace{1cm} \text{assume } x \less x';

\hspace{1cm} \text{then (A3) implies } y \less x',

\hspace{1cm} \text{C! to (A4) since } x' \not\sim y,

\hspace{1cm} \text{hence } x \ co x' \text{ by (A7)};

(ii2) x,x' \in y';

\hspace{1cm} \text{assume } x \less x';

\hspace{1cm} \text{then (A3) implies } x \less y,

\hspace{1cm} \text{C! to (A4) since } y \not\sim x,

\hspace{1cm} \text{hence } x \ co x' \text{ by (A7)} \hspace{1cm} \square

The proof of (L4) has been carried out in detail because it shows which kind of steps are omitted in subsequent proofs.

We call a structure \( N = (E,E,F,co) \) a causal net (or simply net) if it satisfies (D1)-(D5) and (A1)-(A7); a first example is given in the next section.

2.3 \( x \)-density; example and simple observations

Assume a causal net \( N = (E,E,F,co) \); we extend relations \( co \) and \( \overline{co} \): a cut is a maximal set of concurrent elements; a line is a maximal set of causally dependent elements.

Terminologically, we follow [2]. Cuts can be interpreted as cases (or "global states"), lines as sequential processes (or "single continuous activities"); we use respective terms as synonyms.
(D6) Let \( R \subseteq X \times X \) with \( R = R^{-1} \); \( R := X \times X \setminus R \);
K is called Ken of \( R \)
\[
\Leftrightarrow (D6a) \forall x,y \in K: x \neq y \implies x R y
\]
(all different elements of \( K \) are in relation \( R \))
\[
\land (D6b) \forall z \notin K \exists x \in K: z R x
\]
(\( K \) is the largest set with \( (D6a) \))
\[
\text{CUTS} := \{ C \mid C \text{ is Ken of } co \}
\]
\[
\text{LINES} := \{ P \mid P \text{ is Ken of } co \}
\]

Given a set of concurrent/causally dependent elements, a cut/a line containing them may always be found:

(L5) (a) Let \( C_0 \subseteq X \) with \( x \text{ co } y \) for \( x,y \in C_0 \);
then \( \exists C \in \text{CUTS} \) with \( C_0 \subseteq C \)

(b) Let \( P_0 \subseteq X \) with \( x \text{ co } y \) for \( x,y \in P_0 \land x \neq y \);
then \( \exists P \in \text{LINES} \) with \( P_0 \subseteq P \)

Proof: (a) Let \( C := \{ D \subseteq X \mid C_0 \subseteq D \land \forall x,y \in D: x \text{ co } y \} \),
then \( C \) partially orders \( C \);

let \( D \) a totally ordered subset of \( C \); then

\[
(1) \bigcup D \subseteq C, \text{ since for any two } x,y \in \bigcup D:
\]
\[
D \in D \quad \forall x \in D_1, y \in D_2, \text{ and } x \text{ co } y
\]

since \( D \) is totally ordered;

define
\[
f(D) := \begin{cases} 
D \cup \{ y \} & \text{if there is a } y \notin D: \\
D & \text{otherwise}
\end{cases}
\]

then

\[
(2) D \in C \implies f(D) \in C
\]

(3) \( D \in f(D) \);

by (1)-(3), a fundamental fixpoint lemma applies (cf. [5]), giving:
\[
\exists C \in C: C = f(C); \text{ this } C \text{ is a cut with } C_0 \subseteq C
\]

(b) similar

Fig. 1 gives an example of a causal net; the dashed line indicates a cut \( C \), the slant arrow a line \( P \):
Fig. 1

We see that $b \in C \cap P$; from (D5), it is immediate that

\[(L6) \forall C \in \text{CUTS} \forall P \in \text{LINES}: |C \cap P| \leq 1\]

If the intersection is never empty, co is called "K-dense":

\[(D7) K\text{-dense}(co) : \iff \forall C \in \text{CUTS} \forall P \in \text{LINES}: |C \cap P| = 1\]

The density axiom then reads:

\[(A8) K\text{-dense}(co)\]

("co is K-dense"; inaccurately, we will also sometimes call the net "K-dense")

From the earlier interpretation of cuts and lines, we derive a first motivation for the hypothesis that real processes observe (A8): in reality, there cannot be a global state (case) in which the progress of some single continuous activity is not determined.

In addition, [1] motivates this hypothesis by showing that it is a generalization of an axiom known from physics.

It is not accidental that the net of Fig. 1 is K-dense; all finite nets have this property:

\[(T1) |X| < \omega \Rightarrow K\text{-dense}(co)\]

This theorem can be found in [1] and in [3].

We shall delay the proof of (T1) until theorem (T2) enables us to have the following stronger form as an immediate corollary:
(T1') \( |E| < \infty \lor |B| < \infty \Rightarrow \) K-dense(co).

In "cuts" and "lines" we can re-discover the dualism conditions/events, in the sense that conditions are the relevant elements of cuts while events are the relevant elements of lines, at least in the context of lemmas (L7) and (L8) which follow:

(L7) \( \forall P \in \text{LINES}: \forall C \in \text{CUTS}: C \cap P = \emptyset \iff \forall C' \in \text{CUTS}: C' \cap P = \emptyset \land C' \subseteq B \)

Proof: (\( \Rightarrow \)): clear
(\( \Leftarrow \)): Given \( P \in \text{LINES}, C \in \text{CUTS}, C \cap P = \emptyset \);
choose \( e \in C \cap E \) and define

\[
C' := \begin{cases} 
  C \setminus \{e\} \cup e \text{ if } e \neq \emptyset \\
  C \setminus \{e\} \cup e^* \text{ if } e = \emptyset
\end{cases}
\text{(} => e^* \neq \emptyset \text{ by (A2)}\text{)}
\]

using (L4) and (L6) for C, it is easy to prove:

\( C' \in \text{CUTS} \);
using (L6) for P, we have:

\( C' \cap P = \emptyset \);
(L7) follows from elimination of all \( e \in C \cap E \)

(L8) \( \forall P \in \text{LINES} \forall b \in P \cap B: b \subseteq P \land b^* \subseteq P \).

3) Characterization of K-density

3.1 Examples of non-K-dense nets

![Fig. 2a](image1)

![Fig. 2b](image2)
By (11), non-$k$-dense nets must be infinite; if we regard nets $N_1$ of Fig. 2a and $N_2$ of Fig. 2b infinite as suggested by the dots, we have the simplest non-$k$-dense nets.

Fig. 2a carries some labelling for further reference; in Fig. 2b, the cut $C$ and the line $P$ with $C \cap P = \emptyset$ are shown.

3.2 Causal components and subnets

It is plain that there are more complicated non-$k$-dense nets; in a sense given in precise terms by theorem (T2) below, however, nets violating (A8) are not substantially different from $N_1$ or $N_2$: they contain $N_1$ or $N_2$ (or both) as causal component. The reverse is also true: if a net has such components then it violates (A8).

A causal net $N'$ may be a causal component of another net $N$ without being a subnet of $N$ in the sense of [1]. First, we illustrate the intended difference between these two notions using Figs. 2c and 2d; then they are defined in (D8).

As suggested by the shading, we wish to consider $N_1$ a causal component of this net. However, the arrow from $e_1$ to $b'_1$ of $N_1$ is not part of the net of Fig. 2c; therefore, $N_1$ is not a subnet of it.

In Fig. 2d, the shading and the thick arrows show that $N_1$ is a subnet of net $N$; $N_1$ is not a causal component, because we have $b_1 \leq e_3$ in $N_1$ but $b_1 \not< e_3$ in $N$. 
Formally,

\((D8)\) Let \(N' = (B', E', f', \text{co}')\) and \(N = (B, E, F, \text{co})\) two nets;

\((D8a)\) \(N' \subseteq N\) : \(\iff\) \(B' \subseteq B\)
\(\land\ E' \subseteq E\)
\(\land\ \forall x', y' \in X' : x' \text{ co} y' \iff x' \text{ co} y'\)
\(\land\ \forall x', y' \in X' : x' \prec y' \iff x' \prec y'\)

("\(N'\) is causal component of \(N\")

\((D8b)\) \(N'\) is subnet of \(N\) : \(\iff\) \(B' \subseteq B\)
\(\land\ E' \subseteq E\)
\(\land\ f' = f \cap (B' \times E' \cup E' \times B')\)

Obviously, both relations are reflexive and transitive.

3.3 Characterization of \(K\)-density in terms of causal components

As already mentioned, we investigate the case in which nets \(N_1\) or \(N_2\) are causal components of some other causal net \(N\):

\((\star)\) \(N_1 \subseteq N\) or
\((\star\star)\) \(N_2 \subseteq N\).

We are now ready to state

\((T2)\) Let \(N = (B, E, F, \text{co})\) a causal net;

let \(N_1\) and \(N_2\) the nets of Figs. 2a and 2b;

then:

\(K\text{-dense(co)} \iff \neg (\star) \land \neg (\star\star)\)
Proof:

(⇒): We distinguish two cases, (⇒1) and (⇒2).

(⇒1): Assume (¬∀); we construct \(P \in \text{LINES}\) and \(C \in \text{CUTS}\) with \(C \cap P = \emptyset\).

Let \(e_1, b_i, b_j\) as in Fig. 2a.

By (¬∀) and (D8a), \(\{b_i\} \cup \{b_j\} \subseteq B\)
\[\{e_i\} \subseteq E\]
\[b_i \text{ co } b_j\]
\[i \neq j \Rightarrow e_i \text{ co } e_j \land b_i \text{ co } b_j\]

By (D5), there exists \(P \in \text{LINES}\) with \(\{e_i\} \subseteq P\); this line \(P\) can be partitioned into two subsets, \(P_1\) and \(P_2\):

\[P_1 := \{y \in P \mid \exists i: y \lessdot e_i\}\]
\[P_2 := \{z \in P \mid \forall i: e_i \lessdot z\} .\]

We show that

\[(I) (a) \forall y \in P_1 \forall z \in P_2: y \lessdot z\]

(b) \(P = P_1 \cup P_2\)

Proof of (I): (a) \(\exists i: y \lessdot e_i \lessdot z\);

(b) let \(x \in P\);

by def. of \(P\), \(\forall i: x \neq e_i \Rightarrow x \text{ co } e_i\),
by infinity of \(\{e_i\}, x \in P_1 \cup P_2\) \(\square\)(I)

\(P\) is thus partitioned as follows:

\[\begin{array}{c}
P_1 \\
\{ e_1 \} \lessdot \{ e_2 \} \lessdot \{ e_3 \} \lessdot \ldots \end{array} \]

"gap" \(\overset{b_1}{\triangle}\) \(\overset{b_2}{\triangle}\) \(\overset{b_3}{\triangle}\)
Since the $b_i$ cannot belong to $P_2$, they indicate the existence of a certain "gap" between $P_1$ and $P_2$ (justifying the term "dense").

The cut $C$ to be constructed will be placed in this gap; if $P_2 = \emptyset$ we simply apply (L5) and define

$C \in \text{CUTS}$ such that $\{b_i^T\} \subseteq C$,

which is possible since $b_i \not< b_j$.

Clearly, $C \cap P_1 = C \cap P = \emptyset$.

If $P_2$ is not empty the construction is a little bit more sophisticated:

Let $P_2 \neq \emptyset$;

by (L8), this implies $P_2 \cap E \neq \emptyset$;

let $z \in P_2 \cap E$ arbitrarily;

by definition, $\forall i: e_i^T \not< z$;

by (D3), there is a finite chain from $e_i^T$ to $z$:

$$e_i^T \rightarrow \cdots \rightarrow z$$

(possibly there are infinitely many chains);

for each pair $(e_i^T, z)$, we choose one such chain and denote by $B_i^z$ the set of conditions on the chain;

define

$$A_i^z := \{ b \in B_i^z \mid \forall j>i: \neg b < e_j^T \} ;$$

clearly, $z \cap A_i^z \neq \emptyset$ otherwise $e_j^T \not< z$;

we define $b_i^z$ to be the minimal element of $A_i^z$,

that is, $\forall b \in A_i^z: b \neq b_i^z \Rightarrow b_i^z < b$;

$b_i^z$ is well-defined since $|A_i^z| < \infty$, $A_i^z \neq \emptyset$, and $\not<$ totally orders $A_i^z$;

we need the following properties of $b_i^z$:

$$\exists j>i: b_i^z < e_j^T \land \neg \exists j>i: b_i^z < e_j^T ;$$

let $C_0$ be the set of all $b_i^z$:

$$C_0 := \{ b_i^z \mid i \in \mathbb{N} \land z \in P_2 \cap E \} ;$$
then

\( \forall b, b' \in C_0: b \text{ co } b' \)

Proof of (II): assume \( b \prec b' \);

by (A3), \( b' \prec 'b' \),

C! to the properties of \( b_1^z \) or

C! to (A4);

hence \( b \text{ co } b' \)

\( \square \) (II)

By (II) and (L5), we find a cut \( C \) with \( C_0 \in C \);

for this cut the following is true:

(III) \( C \cap P = \emptyset \)

Proof of (III): (i) suppose \( x \in P_1 \);

then \( \exists j, z: x \prec e_j \prec b_j^z \), since \( P_2 \neq \emptyset \);

hence \( x \notin C \);

(ii) suppose \( x \in P_2 \);

(ii1) \( x \in P_2 \cap E \);

then \( \exists i: b_i^x \prec x \);

hence \( x \notin C \);

(ii2) \( x \in P_2 \cap B \);

then \( \star x \in P_2 \cap E \) by (L8)

and \( x \notin C \) follows

from (ii1);

(1), (ii), (I), (b) \( \Rightarrow C \cap P = \emptyset \)

\( \square \) (III)

\( \square \) (\( \Rightarrow 1 \))

\( \Rightarrow 2 \): Assume \( (\forall x) \); then \( \exists C \in \text{CURS}, P \in \text{LINES} \): \( C \cap P = \emptyset \)

(from \( \Rightarrow 1 \) and arrow-reversal).

\( \square \) (\( \Rightarrow 2 \))

\( \square \) (\( \Rightarrow \))
(≤): Assume ∃C ∈ CUTS, P ∈ LINES: C ∩ P = ∅;

by (L7), assume C ∈ B;

we show: (†) or (‡†).

For the proof which follows, it suffices to restrict one's attention to the set P ∩ E; this set can be partitioned into two subsets, P₁ and P₂, with properties similar to (⇒1)(I). The fact that P₁ has no maximal element implies (†), and the fact that P₂ has no minimal element implies (‡†). The proof is completed by showing that P₁ and P₂ cannot have a maximal (a minimal) element at the same time.

\[ P₁ := \left\{ y ∈ P ∩ E \mid \exists x ∈ C: y < x \right\} \]
\[ P₂ := \left\{ z ∈ P ∩ E \mid \exists x ∈ C: x < z \right\} \]

(I) (a) \( ∀ y ∈ P₁ \forall z ∈ P₂: y < z \)

(b) \( P ∩ E = P₁ ∪ P₂ \)

Proof of (I): (a) by (D6a) of C

(b) by (D6b) of C

We define

\[ \text{max}(P₁) := \left\{ y_{\text{max}} ∈ P₁ \mid \forall y ∈ P₁: y < y_{\text{max}} \Rightarrow y < y_{\text{max}} \right\} \]
\[ \text{min}(P₂) := \left\{ z_{\text{min}} ∈ P₂ \mid \forall z ∈ P₂: z < z_{\text{min}} \Rightarrow z < z_{\text{min}} \right\} ; \]

obviously, \( |\text{max}(P₁)| ≤ 1 \) and \( |\text{min}(P₂)| ≤ 1 \).

(II) \( \text{max}(P₁) = ∅ \Rightarrow (†) \)

Proof of (II): let \( \text{max}(P₁) = ∅ \);

choose \( \{ y_i \} \), \( i = 1, 2, ... \)

with \( y_i ∈ P₁ \) and \( y_i < y_j \) for \( i < j \);

by definition of \( P₁ \), there exist conditions \( x_i ∈ C \) with \( y_i < x_i \) for all \( i \);

if these \( x_i \) are mutually distinct we can define \( b_i := x_i \), \( e_i := y_i \) and immediately derive (‡†); but not necessarily \( i ≠ j \Rightarrow x_i ≠ x_j \);
therefore we pick some appropriate 
b_i on chains from y_i to x_i, as follows:
for each pair (y_i, x_i), choose a chain

\[ y_1 \rightarrow \cdots \rightarrow x_i \]
define

A_i as the set of conditions "a" on the 
chain from y_i to x_i, for which:
\[ \neg \exists j > i: a < y_j ; \]
A_i \neq \emptyset since x_i \in A_i;
let a_i the minimal element of A_i
\[ (\forall a \in A_i: a \neq a_i \Rightarrow a_i < a) ; \]define (e_i, b_i) recursively as follows:
e_i := y_i, b_i := a_i;
assume e_i, b_i already defined; by def.
of b_i, there exists j > i with "b_i < y_j;
set e_{i+1} := y_j, b_{i+1} := a_j;
by this definition and (A4), we have:

(i) i \neq j \Rightarrow b_i \neq b_j
(ii) b_i \text{ co } b_j
(iii) \forall j > i: b_i \text{ co } e_j ;
since e_i < e_{i+1}, we can find conditions
b_i on chains from e_i to e_{i+1};
thus, e_i, b_i and b_i together with
(i)-(iii) define the net N_1,
and clearly N_1 \subseteq \mathcal{N}
\[ \Box \text{(II)} \]

By symmetry,
(III) min(P_2) = \emptyset \Rightarrow (\star \star)

It remains to show that
(IV) max(P_1) = \emptyset \lor \min(P_2) = \emptyset
Proof of (IV): by contradiction:
let $y_{\text{max}} \in \max(P_1)$, $z_{\text{min}} \in \min(P_2)$;
by (I), $y_{\text{max}} < z_{\text{min}}$,
that is, $\exists n: y_{\text{max}} \leq n \leq z_{\text{min}}$;
assume an event $e \in B$ with
$y_{\text{max}} < e \leq z_{\text{min}}$;
e $\in P_1$ C! to $y_{\text{max}} \leq \max(P_1)$
e $\in P_2$ C! to $z_{\text{min}} \leq \min(P_2)$,
hence $e \notin P$;
therefore, there must exist $b \in B$ with:
y_{\text{max}} \leq b \leq z_{\text{min}}$;
obviously, $b \in P$;
also, $b \in C$ because for arbitrary $x \in C$:
x < b C! to $y_{\text{max}} \in P_1$
b < x C! to $z_{\text{min}} \in P_2$,
hence $b \in C$;
C! to $C \cap P = \emptyset$ \hfill (IV)
\hfill (\leq)
\hfill (T2)

Corollary:
(T1') follows directly from the infinity of $B_1$ and $E_1$ (and the corresponding sets for $N_2$), (T2) and (D8a).
3.4 A consequence

Given two events $e_1$ and $e_2$; we show that (A8) excludes the existence of an infinite chain of events from $e_1$ to $e_2$. Formally, if $(E, E, r, co)$ has the following properties:

(i) $|E| = \infty$ ($"infinite"$)
(ii) $\forall x, y \in E: x \neq y \Rightarrow x \leq co y$ ($"chain"$)
(iii) $e_1 \in E \land \forall x \in E\{e_1\}: e_1 \leq x$ ($"from \ e_1"$)
(iv) $e_2 \in E \land \forall x \in E\{e_2\}: x \leq e_2$ ($"to \ e_2"$),

then we have:

(T3) $\rightarrow$ K-dense(co)

Proof: for simplicity's sake, we proceed rather informally;

call $x$ a cluster point if $\forall y > x \exists z: x < z < y$;

using (i)-(iv), we start constructing a sequence of events $\{x_i\}$, $x_1 := e_1$, such that $x_i \leq x_{i+1} \leq e_2$;

if, for some $i$, $x := x_i$ is a cluster point, we stop constructing $\{x_i\}$ and start constructing $\{y_i\}$ as described below;

otherwise, $\{x_i\}$ is infinite and we have established ($\forall$) since for all $i$: $x_i < e_2$;

now suppose there exists a cluster point $x$ with $x < e_2$;

define $\{y_1\}$, $y_1 := e_2$ such that $x < y_{i+1} < y_i$; $\{y_1\}$ exists because $x$ is a cluster point, and is infinite;

thus, we have ($\forall\forall$);

in summary, we have ($\forall$) or ($\forall\forall$), and by (T2),

we have (T3) $\square$

The reverse of (T3) is not true since $N_1$ is not K-dense but no two events with an infinite chain of events between them can be found.
4) Interpretation

4.1 Achilles (cf. [1])

If Achilles was still alive he would probably refuse to run after as many tortoises as suggestions to that effect have been put to him; possibly out of anger, possibly he would after all believe all those people who keep predicting him that he can never reach any of these tortoises, because he has to run half of the distance, then half of the rest, etc... indefinitely.

Of course there is no reason to trust such a prediction; the contradiction is only apparent in our minds – reality is not that paradoxical. In formulating the paradoxon, we have constructed the following causal net:

\[
\begin{align*}
  & e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow ... \\
  & \downarrow \quad \downarrow \quad \downarrow \\
  & e \quad e \quad e \\
\end{align*}
\]

**Fig. 3**

Achilles starts running \((e_1)\), runs half of the distance \((e_2)\), etc... Taking Fig. 3 "literally", he can be seen running along the northern shore of the gap, longing for a chance to bridge it: in case he is just about to reach his tortoise (that is, the cut before event "e"), he is simply "disappeared"; his progress is not determined – another formulation of the same paradoxon.

Hence: **if** \((A8)\) is observed, the paradoxon cannot be formulated. Postulating that Fig. 3 describes a real process means postulating that stones fall upwards; \((A8)\), like the
laws of free fall, explains (but can of course not be formally deduced from) the contradiction between construction and reality.

4.2 Turing computability

In the present context we are not so much interested in Achilles' problems but in the problem of Turing computability. There could be a definite connection between K-density and computability; after all, in the theory of Turing machines, it is shown that a function $F$ can be written down such that there is no Turing machine $M$ computing $F$: $F$ is non-computable.

Nothing is said as to whether this $F$ may "actually exist" or is a construction only; but there is such a statement about $M$: $M$ did not, does not, and will never, exist. If we assume the contrary, we can derive the diagonalization paradoxon; but we can also conjecture that (A8) already implies the non-existence of $M$, if we want to follow the track beaten in section 4.1. This is indeed the case, as will be shown in the remainder of this section.

First, it is convenient to agree on some idioms; for Turing machines $X$ and $M$, let $x$ and $m$ their respective G"odel numbers. As usual, G"odel numbers are unique and effectively analysable. We say that a Turing machine $X$ loops if $X$ does not stop. For an integer $k$, "$X(k)$" is a shorthand for "given $k$, $X$ ... "; thus, "$X(k)$ stops" reads: "given $k$, $X$ stops".

In our informal discussion, we cannot be bothered much by the differences between machines, programs and flowcharts. Let Fig. 4 be a flowchart definition of some machine $M$: 
Fig. 4

In words, if $X(x)$ stops then $M(x)$ loops; if $X(x)$ loops then $M(x)$ stops. $M$ cannot be constructed because $x := m$ (the "diagonalization") immediately gives: $M(m)$ stops iff $M(m)$ loops, which is impossible.

However, a good deal of naivety entitles us to ask: why on earth should it be impossible to implement the algorithm of Fig. 4? It is not difficult to build an interpreter which decomposes $x$, synthesizes $X$ and step by step simulates the execution of $X$, given $x$; if $X(x)$ stops this interpreter might as well enter an infinite loop. If $X(x)$ loops, however, then our simulation also loops and never gets a chance to execute its "stop" command, contrary to the specification of Fig. 4.

As we can see, we have just reconstructed the net of Fig. 3 for the right-hand-side branch of Fig. 4; the event "e" corresponds to the "stop" command and the "e_1" are our simulation steps. We sum up: if $M$ simulates $X$ then

(P1) if $X(x)$ loops then $M(x)$ violates (A8).

We could avoid this difficulty by devising a "clever algorithm" to decide whether $X(x)$ stops, by "looking at X's structure plus input x" rather than simulating $X(x)$. For exceptionally simple cases, such an algorithm could indeed work; take, for example, $X_0$ which is defined to stop if its input $k$ is odd and to loop if its input $k$ is even. Clever analysis may lead to the desired result, and $M$ executes its
"stop" command even though \( X_0(k) \) loops, provided \( k \) is even. Hence (P1) is no longer true if \( M \) uses structural analysis.

Suppose, however, \( x := m \); then \( \tilde{M} \) has to find out, by structural analysis, whether or not \( M(m) \) loops, which involves finding out whether or not \( M(m) \) loops, and so on indefinitely. Again we have Fig. 3; this time, the "e_i" correspond to the steps of the structural analysis which was supposed to exist, and "e" corresponds to the first command of \( M \) after this analysis. To sum up: if \( M \) analyses \( X \) then

(P2) \( M(m) \) violates (A8).

Together, (P1) and (P2) show that any machine \( M \) which performs the task described by the flow-chart of Fig. 4, violates (A8) for some input \( x \): (A8) implies the non-existence of \( M \).

From our point of view, the fact that a violation of (A8) ensues from the construction of \( M \) is consistent with the fact that we have the diagonalization paradoxon: \( M \) is a construction of our mind, and formulating the paradoxon is another way of saying that \( M \) appropriately models a real process - which is invalid according to (A8).

For two reasons, we assert that (A8) is satisfactory an explanation of the phenomenon of non-computability.

First, we can do more than merely deduce the non-existence of \( M \); we are able to pinpoint the exact location where \( M \) fails to observe (A8). If \( m \) simulates \( X \) then the "X(x) loops" branch violates (A8); if \( M \) analyses \( X \) then the violation is located somewhere between "input(x)" and "X(x)?" (for some \( x \)).

Second, we have not used case (\( \Xi \)) of (T2); we have made exclusive use of (\( \lambda \)) which deals with sequential processes which are "infinite towards the future". In our case, (\( \lambda \)) suffices because it is far more important to investigate the kinds of machines we can built, the kinds of - possibly infinite - processes we can give rise to, than to speculate about the possibly infinite history of some or other already existing processes. (A8) treats both cases - infinite futures
and infinite histories – in a symmetric way. Of course, Church's thesis is not touched by these considerations.

4.3 The first cause/last effect interpretation

The last remarks of section 4.2 lead to an interpretation of (T2). Note that, in Fig. 3, event e belongs to the line through all e₁; e can be called a last effect of e₁. The fact that e is after all e₁ depends upon the existence of an infinite cut. In general, we leave open whether this cut actually precedes an event such as e in Fig. 3, or whether some other structure depends upon the cut, and call the cut itself a last effect of the infinite future in question. Similarly, we call the corresponding cut of (E) a first cause (of an infinite history), and derive the following interpretation of (T2):

K-density is equivalent to the absence of first causes (of infinite histories) and last effects (of infinite futures).

4.4 The source/sink interpretation

We attempt another interpretation by reconsidering Fig. 2d; let N be the net of this figure. N₁ is a subnet of N but not a causal component, the reason being that the b₁ are redundant in the sense that no new causal dependencies are defined by them. We might say that N is essentially a chain which does not at all "emit information about its progress to the outside"; all conditions serve to define causal dependencies within N.

\[ N₁ \subseteq N \] can then be interpreted as saying that any infinite future emits a finite amount of information to its outside; from some point on, it ceases telling the rest of the net: "I've progressed such and such far". In this sense, (T2) says that the infinite future of a net can be seen as a sink – and, likewise, the infinite history as a source –, of progress information, if and only if (A8) holds.
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