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We completely characterise transition systems generated by ENCL-systems after extending the standard notion of a region - defined as a certain set of states - with explicit information about events which, in particular, are responsible for crossing its border. As a result, we are able to construct, for each such transition system, a suitable ENCL-system generating it.

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About the author

Maciej Koutny obtained his MSc (1982) and PhD (1984) from the Warsaw University of Technology. In 1985 he joined the then Computing Laboratory of the University of Newcastle upon Tyne to work as a Research Associate. In 1986 he became a Lecturer in Computing Science at Newcastle, and in 1994 was promoted to an established Readership at Newcastle. In 2000 he became a Professor of Computing Science. His research interests centre on the theory of distributed and concurrent systems, including both theoretical aspects of their semantics and application of formal techniques to the modelling and verification of such systems; in particular, model checking based on net unfoldings. Recently, he has been working on the development of a formal model combining Petri nets and process algebras. He has also investigated non-interleaving semantics of priority systems, and the relationship between temporal logic and process algebras. He is a member of the Steering Committee of the International Conference on Applications and Theory of Petri nets (http://www.daimi.au.dk/PetriNets/).

Marta Pietkiewicz-Koutny received her M.Sc. in Applied Mathematics from the Warsaw University of Technology in 1982. In 1984 she joined the Department of Operational Research, Institute of Econometrics in the Warsaw University of Economics where she worked as a junior lecturer until 1986. In 1987 she joined the Computing Laboratory of the University of Newcastle upon Tyne first as a research associate and then as a demonstrator (1988-1997). In the period 1997-2000 she was a Ph.D. student at the Department of Computing Science of the University of Newcastle upon Tyne, and in December 2000 she was awarded her Ph.D. degree.

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THEORY OF REGIONS,
TRANSITION SYSTEMS,
INHIBITOR ARCS,
ACTIVATOR ARCS,
CONTEXT ARCS

Synthesis of Elementary Net Systems with Context Arcs and Localities

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Keywords: theory of concurrency, Petri nets, elementary net systems, localities, net synthesis, step sequence semantics, structure and behaviour of nets, theory of regions, transition systems, inhibitor arcs, activator arcs, context arcs.

1 Introduction

We are concerned with a class of concurrent computational systems whose dynamic behaviours exhibit a particular mix of *asynchronous* and *synchronous* executions, and are often described as adhering to the 'globally asynchronous locally synchronous' (or GALS) paradigm. Intuitively, actions which are 'close' to each other are executed synchronously and as many as possible actions are always selected for execution. In all other cases, actions are executed asynchronously. Two important applications of the GALS approach can be found in hardware design, where a VLSI chip may contain multiple clocks responsible for synchronising different subsets of gates [1], and in biologically motivated computing, where a membrane system models a cell with compartments, inside which reactions are carried out in co-ordinated pulses [2]. In both cases, the activities in different localities can proceed independently, subject to communication and/or synchronisation constraints.

To formally model GALS systems, [3] introduced *Place/Transition-nets with localities* (PTL-nets), defined as PT-nets where transitions are assigned to explicit localities. Each locality identifies transitions which may only be executed synchronously and in a maximally concurrent manner. The idea of adding localities to an existing Petri net model was taken further in [4], where Elementary Net Systems (EN-systems) replaced PT-nets as an underlying system model. In this paper, we build on the work reported in [4], by considering EN-systems extended with two non-standard kinds of arcs, namely *inhibitor* arcs and *activator* (or *read*) arcs, collectively referred to as *context* arcs following the terminology of [5]. The resulting model will be referred to as *Elementary Net Systems with Context Arcs and Localities* (or ENCL-systems).

It is worth pointing out that both inhibitor arcs (capturing the idea the enabling of a transition depends on a place being *unmarked*) and activator arcs (capturing the idea the enabling of a transition depends on a place being *marked* by more tokens than those consumed when the transition is fired) are presumably the most prominent extensions of the basic Petri net model considered in the literature. Such context arcs can be used to test for a specific condition, rather than producing and consuming resources, and proved to be useful in areas such as communication protocols [6], performance analysis [7], and concurrent programming [8]. More recently, [9] applied context arcs to deal with several salient behavioural features of membrane systems, such as promoters, inhibitors and dissolving as well as thickening of membranes.

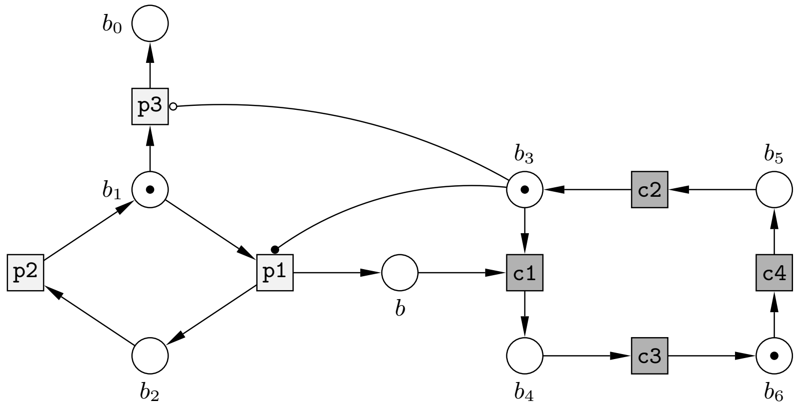

Fig. 1. A producer/consumer system with a business conscious producer.

Consider the ENCL-system in Figure 1 modelling a producer/consumer system consisting of one producer (who can execute events `p1`, `p2` and `p3`), and two consumers (who can execute events `c1`, `c2`, `c3` and `c4`). The buffer-like condition b in the middle holds items produced by the event `p1` and consumed by `c1`. The activator arc between `p1` and b_3 (represented by an edge ending with a small black circle) means that the producer adds a new item to the buffer only if there is a consumer waiting for it, and the inhibitor arc between `p3` and b_3 (represented by an edge ending with a small circle) means that the producer can leave the production cycle only when no customer is eager to get the produced items. It is assumed that all the events executed by the producer belong to one locality, while all the events executed by the consumers belong to another locality. To indicate this in the diagram, we use different shading for the boxes representing events assigned to different localities.

In terms of possible behaviours, adding localities can have a significant impact on both the executability of events and reachability of global states. For example, under the standard net semantics the model in Figure 1 would be able to execute the step sequence {p1}{c1}, but the execution model of ENCL-systems rejects this. The reason is that after {p1}, it is possible to execute the step {c1, c4} consisting of two co-located events, and so executing c1 alone would violate the maximal concurrency execution rule within the locality assigned to the events used by the consumers. A possible way of 'repairing' this step sequence would be to add the 'missing' event, resulting in the legal step sequence {p1}{c1, c4}. Another legal step sequence is {p1}{p2, c1, c4}, where the second step is maximally concurrent in a global sense, as it cannot be extended any further. Note also that the event p3 is not enabled at the beginning since b_3 contains a token and there is an inhibitor arc linking p3 and b_3, and after {p1}{p2, c1, c4} the event p1 is not enabled since b_3 is now empty and there is an activator arc linking p1 and b_3.

The Petri net problem we are going to investigate in this paper, commonly referred to as the *Synthesis Problem*, is in essence concerned with model transformation, from a class of transition systems (sometimes called reachability graphs) to a class of Petri nets. The key requirement is that the Petri net obtained from a given transition system should capture the same behaviour, i.e., its reachability graph should be isomorphic to the original transition system. This problem was solved for the class of EN-systems in [10], using the notion of a region which links nodes of transition systems (global states) with conditions in the corresponding nets (local states). The solution was later extended to the pure bounded PT-nets [11], general Petri nets [12], safe nets [13] and EN-systems with inhibitor arcs [14, 15], by adopting the original definition of a region or using some extended notion of a generalised region [16].

In a previous paper [4], we have solved the synthesis problem for the class of Elementary Net Systems with Localities (ENL-systems). In doing so, we introduced *io-regions*, a generalisation of the standard notion of a region of a transition system, as the latter proved to be insufficient to deal with the class of ENL-systems (and hence also for ENCL-systems considered in this paper). To explain the idea behind io-regions, consider the transition system shown in Figure 2(a), which is isomorphic to the reachability graph of the ENL-system shown in Figure 2(b). (Note that the two events there, e and f, are assumed to be co-located.) The standard region-based synthesis procedure would attempt to construct the conditions of the net in Figure 2(b), by identifying each of these with the set of the nodes of the transition system where it 'holds'. For example, the region corresponding to b_1 comprises just one state, $r = \{s_{init}\}$. Similarly, $r' = \{s\}$ is a region where b_2 holds. (Note that there are two more 'trivial' regions, $\{s_{init}, s\}$ and $\varnothing$, which are ignored by the synthesis procedure.) However, this is not enough to construct the ENL-system in Figure 2(b) since there are only two non-trivial regions and we need four, one for each of the conditions.

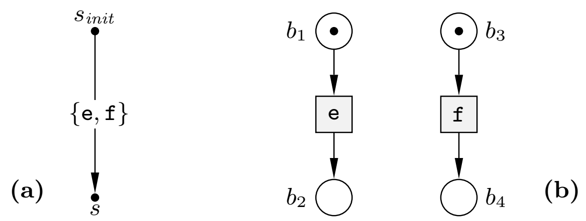

Fig. 2. A transition system with co-located transitions e and f **(a)**, and a corresponding ENL-system **(b)**.

An intuitive reason why the standard construction does not work for the transition system in Figure 2(a) is that the 'set-of-states' notion of region is not rich enough for the purposes of synthesising ENL-systems. The modification to the original notion proposed in [4] is based on having also explicit *input* and *output* events of a set of states, which point at those events which are 'responsible' for entering the region and for leaving it. More precisely, an io-regions is a triple: $\mathfrak{r} = (in, r, out)$, where r is a set of states, in is a set of events which are responsible for entering r, and out is a set of events which are responsible for leaving r. In the case of the example in Figure 2(a), one can find four non-trivial io-regions: $\mathfrak{r}_1 = (\varnothing, \{s_{init}\}, \{\mathtt{e}\})$ $\mathfrak{r}_2 = (\{\mathtt{e}\}, \{s\}, \varnothing\})$, $\mathfrak{r}_3 = (\varnothing, \{s_{init}\}, \{\mathtt{f}\})$ and $\mathfrak{r}_4 = (\{\mathtt{f}\}, \{s\}, \varnothing)$. Now one has enough regions to construct the conditions of the ENL-system in Figure 2(b), namely each $\mathfrak{r}_i$ corresponds to b_i.

In this paper, we will extend the idea of an io-region to also cope with context arcs. Briefly, we will base our synthesis solution on *context regions* (or c-regions), each such region being a tuple (r, in, out, inh, act) where the two additional components, inh and act, carry information about events which are related with r due to the presence of a context arc.

The paper is organised as follows. In the next section, we introduce formally ENCL-systems. After that we define ENCL-transition systems and later show that the reachability graphs of ENCL-systems are indeed ENCL-transition systems. We finally demonstrate how to construct an ENCL-system corresponding to a given ENCL-transition system.

2 ENCL-systems

Throughout the paper we assume that $\mathcal{E}$ is a fixed non-empty set of *events*. Each event e is assigned a *locality* $\mathfrak{L}(e)$, and it is *co-located* with another event f whenever $\mathfrak{L}(e) = \mathfrak{L}(f)$.

Definition 1 (net with context arcs). *A* net with context arcs *is a tuple* $\mathfrak{net} \stackrel{\mathrm{df}}{=} (B, E, F, I, A)$ *such that* B *and* $E \subseteq \mathcal{E}$ *are finite disjoint sets,* $F \subseteq (B \times E) \cup (E \times B)$ *and* $I, A \subseteq B \times E$.

The meaning and graphical representation of B (conditions), E (events) and F (flow relation) is as in the standard net theory. An *inhibitor* arc $(b, e) \in I$ means that e can be enabled only if b is not marked (in the diagrams, it is represented by an edge ending with a small circle), and an *activator* arc $(b, e) \in A$ means that e can be enabled only if b is marked (in the diagrams, it is represented by an edge ending with a small black circle). In diagrams, boxes representing events are shaded, with different shading being used for different localities (see Figure 1). We denote, for every $x \in B \cup E$,

$$^{\bullet}x \stackrel{\text{df}}{=} \{y \mid (y, x) \in F\} \qquad x^{\bullet} \stackrel{\text{df}}{=} \{y \mid (x, y) \in F\}$$

$$^{\blacklozenge}x \stackrel{\text{df}}{=} \{y \mid (x, y) \in I \cup I^{-1}\} \qquad ^{\blacktriangleleft}x \stackrel{\text{df}}{=} \{y \mid (x, y) \in A \cup A^{-1}\}$$

and we call the above sets the *pre-elements*, $^{\bullet}x$, *post-elements*, $x^{\bullet}$, *inh-elements*, $^{\blacklozenge}x$, and *act-elements*, $^{\blacktriangleleft}x$. Moreover, we denote

$$^{\bullet}x^{\bullet} \stackrel{\text{df}}{=} {}^{\bullet}x \cup x^{\bullet} \qquad ^{\bullet}x^{\blacktriangleleft} \stackrel{\text{df}}{=} {}^{\bullet}x \cup {}^{\blacktriangleleft}x \qquad ^{\blacklozenge}x^{\bullet} \stackrel{\text{df}}{=} x^{\bullet} \cup {}^{\blacklozenge}x \ .$$

All these notations extend in the usual way (i.e., through the set union) to sets of conditions and/or events. It is assumed that for every event $e \in E$, $e^{\bullet}$ and $^{\bullet}e$ are non-empty sets, and $^{\bullet}e$, $e^{\bullet}$, $^{\blacklozenge}e$ and $^{\blacktriangleleft}e$ are mutually disjoint sets. For the ENCL-system in Figure 1, we have $^{\bullet}b_3 = \{\mathtt{c2}\}$, $b_1^{\bullet} = \{\mathtt{p1}, \mathtt{p3}\}$ and $^{\blacklozenge}\mathtt{p3} = {}^{\blacktriangleleft}\mathtt{p1} = \{b_3\}$.

Definition 2 (ENCL-system). *An* elementary net system with context arcs and localities *(*ENCL*-system) is a tuple* $\mathfrak{encl} \stackrel{\text{df}}{=} (B, E, F, I, A, c_{init})$ *such that* $\mathfrak{net}_{\mathfrak{encl}} \stackrel{\text{df}}{=} (B, E, F, I, A)$ *is the underlying net with context arcs, and* $c_{init} \subseteq B$ *is the* initial case. *In general, any subset of* B *is a* case.

The execution semantics of $\mathfrak{encl}$ is based on steps of simultaneously executed events. We first define the set of *valid steps*:

$$\mathbb{U}_{\mathfrak{encl}} \stackrel{\text{df}}{=} \{u \subseteq E \mid u \neq \varnothing \ \wedge \ \forall e, f \in u : \ e \neq f \Rightarrow {}^{\bullet}e^{\bullet} \cap {}^{\bullet}f^{\bullet} = \varnothing\} \ .$$

For the ENCL-system in Figure 1, we have $\{\mathtt{p1}, \mathtt{c2}, \mathtt{c3}\} \in \mathbb{U}_{\mathfrak{encl}}$, but $\{\mathtt{p1}, \mathtt{c1}, \mathtt{c4}\} \notin \mathbb{U}_{\mathfrak{encl}}$ since $\mathtt{p1}^{\bullet} \cap {}^{\bullet}\mathtt{c1} \neq \varnothing$.

A step $u \in \mathbb{U}_{\mathfrak{encl}}$ is *enabled* at a case $c \subseteq B$ if $^{\bullet}u^{\blacktriangleleft} \subseteq c$ and $^{\blacklozenge}u^{\bullet} \cap c = \varnothing$, and there is no step $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{encl}}$ satisfying $\mathfrak{L}(e) \in \mathfrak{L}(u)$, $^{\bullet}e^{\blacktriangleleft} \subseteq c$ and $^{\blacklozenge}e^{\bullet} \cap c = \varnothing$.

For the ENCL-system in Figure 1, we have that $\{\mathtt{p1}, \mathtt{c4}\}$ is a step enabled at the initial case, but $\{\mathtt{p3}, \mathtt{c4}\}$ is not since b_3 belongs to c_{init} and there is an inhibitor arc between $\mathtt{p3}$ and b_3. We also note that $u = \{\mathtt{p2}, \mathtt{c1}\}$ is not enabled at the case $c = \{b_2, b, b_3, b_6\}$ because it can be extended by an event $e = \mathtt{c4}$ according to the definition of enabledness.

The above definition of enabledness is based on an *a priori* condition: the activator and inhibitor conditions of events occurring in a step obey their respective constraints *before* the step is executed. In an *a posteriori* approach (see [5]), the respective properties must also be true *after* executing the step. Yet another definition for enabling when activator arcs (or rather read arcs) are involved is given in [17].

The transition relation of $\mathfrak{net}_{\mathfrak{encl}}$, denoted by $\rightarrow_{\mathfrak{net}_{\mathfrak{encl}}}$, is then given as the set of all triples $(c, u, c') \in 2^B \times \mathbb{U}_{\mathfrak{encl}} \times 2^B$ such that u is enabled at c and $c' = (c \setminus {}^\bullet u) \cup u^\bullet$.

The *state space* of $\mathfrak{encl}$, denoted by $C_{\mathfrak{encl}}$, is the least subset of 2^B containing c_{init} such that if $c \in C_{\mathfrak{encl}}$ and $(c, u, c') \in \rightarrow_{\mathfrak{net}_{\mathfrak{encl}}}$ then $c' \in C_{\mathfrak{encl}}$. The *transition relation* of $\mathfrak{encl}$, denoted by $\rightarrow_{\mathfrak{encl}}$, is then defined as $\rightarrow_{\mathfrak{net}_{\mathfrak{encl}}}$ restricted to $C_{\mathfrak{encl}} \times \mathbb{U}_{\mathfrak{encl}} \times C_{\mathfrak{encl}}$. We will use $c \xrightarrow{u}_{\mathfrak{encl}} c'$ to denote that $(c, u, c') \in \rightarrow_{\mathfrak{encl}}$. Also, $c \xrightarrow{u}_{\mathfrak{encl}}$ if $(c, u, c') \in \rightarrow_{\mathfrak{encl}}$, for some c'. For the ENCL-system in Figure 1:

$$c_{init} \xrightarrow{\{\mathtt{p1}\}}_{\mathfrak{encl}} \{b_2, b, b_3, b_6\} \xrightarrow{\{\mathtt{p2,c1,c4}\}}_{\mathfrak{encl}} \{b_1, b_4, b_5\} .$$

Proposition 1 ([4]). *If* $c \xrightarrow{u}_{\mathfrak{encl}} c'$ *then* $c \setminus c' = {}^\bullet u$ *and* $c' \setminus c = u^\bullet$.

3 Step transition systems and context regions

In this section, we first recall the notion of a general step transition systems which, after further restrictions, will be used to provide a behavioural model for ENCL-systems, and introduce the notion of a context region.

Definition 3 (transition system, [18, 19]). *A* step transition system *is a triple* $\mathfrak{ts} \stackrel{\text{df}}{=} (S, T, s_{init})$ *where:*

TSYS1 S *is a non-empty finite set of* states.
TSYS2 $T \subseteq S \times (2^{\mathcal{E}} \setminus \{\varnothing\}) \times S$ *is a finite set of* transitions.
TSYS3 $s_{init} \in S$ *is the* initial state.

Throughout this section, the step transition system $\mathfrak{ts}$ will be fixed. We will denote by $\mathcal{E}_{\mathfrak{ts}}$ the set of all the events appearing in its transitions, i.e.,

$$\mathcal{E}_{\mathfrak{ts}} \stackrel{\text{df}}{=} \bigcup_{(s,u,s') \in T} u .$$

We will denote $s \xrightarrow{u} s'$ whenever (s, u, s') is a transition in T, and respectively call s the *source* and s' the *target* of this transition. We will also say that the step u is *enabled* at s, and denote this by $s \xrightarrow{u}$.

For every event $e \in \mathcal{E}_{\mathfrak{ts}}$, we will denote by T_e the set of all the transitions labelled by steps containing e, $T_e \stackrel{\text{df}}{=} \{(s, u, s') \in T \mid e \in u\}$, and by U_e the set of all the steps labelling these transitions, $U_e \stackrel{\text{df}}{=} \{u \mid (s, u, s') \in T_e\}$.

We now introduce a central notion of this paper which is meant to link the nodes of a transition system (global states) with the conditions in the hypothetical corresponding net (local states).

Definition 4 (context region). *A* context region *(or c-region) is a tuple*

$$\mathfrak{r} \stackrel{\text{df}}{=} (r, in, out, inh, act) \in 2^S \times 2^{\mathcal{E}_{\mathfrak{ts}}} \times 2^{\mathcal{E}_{\mathfrak{ts}}} \times 2^{\mathcal{E}_{\mathfrak{ts}}} \times 2^{\mathcal{E}_{\mathfrak{ts}}}$$

such that the following are satisfied, for every transition $s \xrightarrow{u} s'$ *of* $\mathfrak{ts}$*:*

1. $s \in r$ *and* $s' \notin r$ *imply* $|u \cap in| = 0$ *and* $|u \cap out| = 1$.
2. $s \notin r$ *and* $s' \in r$ *imply* $|u \cap in| = 1$ *and* $|u \cap out| = 0$.
3. $u \cap inh \neq \varnothing$ *implies* $s \notin r$.
4. $u \cap act \neq \varnothing$ *implies* $s \in r$.
5. $u \cap out \neq \varnothing$ *implies* $s \in r$ *and* $s' \notin r$.
6. $u \cap in \neq \varnothing$ *implies* $s \notin r$ *and* $s' \in r$.
7. $in \cap inh = \varnothing$ *and* $out \cap act = \varnothing$.

We denote $\|\mathfrak{r}\| \stackrel{\text{df}}{=} r$, ${}^{\bullet}\mathfrak{r} \stackrel{\text{df}}{=} in$, $\mathfrak{r}^{\bullet} \stackrel{\text{df}}{=} out$, ${}^{\blacklozenge}\mathfrak{r} \stackrel{\text{df}}{=} inh$ *and* ${}^{\blacktriangleleft}\mathfrak{r} \stackrel{\text{df}}{=} act$.

The step transition system shown in Figure 2(a) has the following c-regions:

$$\begin{array}{ll}
\mathfrak{r}_1 = (\varnothing,\varnothing,\varnothing,\varnothing,\varnothing) & \mathfrak{r}_2 = (\varnothing,\varnothing,\varnothing,\{\mathtt{e}\},\varnothing) \\
\mathfrak{r}_3 = (\varnothing,\varnothing,\varnothing,\{\mathtt{f}\},\varnothing) & \mathfrak{r}_4 = (\varnothing,\varnothing,\varnothing,\{\mathtt{e},\mathtt{f}\},\varnothing) \\
\mathfrak{r}_5 = (\{s_{init},s\},\varnothing,\varnothing,\varnothing,\varnothing) & \mathfrak{r}_6 = (\{s_{init},s\},\varnothing,\varnothing,\varnothing,\{\mathtt{e}\}) \\
\mathfrak{r}_7 = (\{s_{init},s\},\varnothing,\varnothing,\varnothing,\{\mathtt{f}\}) & \mathfrak{r}_8 = (\{s_{init},s\},\varnothing,\varnothing,\varnothing,\{\mathtt{e},\mathtt{f}\}) \\
\mathfrak{r}_9 = (\{s_{init}\},\varnothing,\{\mathtt{f}\},\varnothing,\varnothing\}) & \mathfrak{r}_{10} = (\{s_{init}\},\varnothing,\{\mathtt{f}\},\varnothing,\{\mathtt{e}\}) \\
\mathfrak{r}_{11} = (\{s_{init}\},\varnothing,\{\mathtt{e}\},\varnothing,\varnothing\}) & \mathfrak{r}_{12} = (\{s_{init}\},\varnothing,\{\mathtt{e}\},\varnothing,\{\mathtt{f}\}) \\
\mathfrak{r}_{13} = (\{s\},\{\mathtt{f}\},\varnothing,\varnothing,\varnothing\}) & \mathfrak{r}_{14} = (\{s\},\{\mathtt{f}\},\varnothing,\{\mathtt{e}\},\varnothing\}) \\
\mathfrak{r}_{15} = (\{s\},\{\mathtt{e}\},\varnothing,\varnothing,\varnothing\}) & \mathfrak{r}_{16} = (\{s\},\{\mathtt{e}\},\varnothing,\{\mathtt{f}\},\varnothing)\,.
\end{array}$$

In the rest of this section, we discuss and prove properties of c-regions which will subsequently be needed to solve the synthesis problem for ENCL-systems.

Trivial c-regions. A c-region $\mathfrak{r}$ is *trivial* if $\|\mathfrak{r}\| = \varnothing$ or $\|\mathfrak{r}\| = S$; otherwise it is *non-trivial*. For example, the step transition system shown in Figure 2(a) has eight trivial c-regions ($\mathfrak{r}_1,\ldots,\mathfrak{r}_8$) and eight non-trivial c-regions ($\mathfrak{r}_9,\ldots,\mathfrak{r}_{16}$). Note that only non-trivial c-regions will be used in the synthesis procedure.

Proposition 2. *If* $\mathfrak{r}$ *is a trivial c-region then* ${}^{\bullet}\mathfrak{r} = \mathfrak{r}^{\bullet} = \varnothing$.

Proof. Follows from Definition 4(5,6) and TSYS2. □

Proposition 3. *If* $\mathfrak{r}$ *is a c-region then the* complement *of* $\mathfrak{r}$, *defined as* $\overline{\mathfrak{r}} \stackrel{\text{df}}{=} (S \setminus \|\mathfrak{r}\|, \mathfrak{r}^{\bullet}, {}^{\bullet}\mathfrak{r}, {}^{\blacktriangleleft}\mathfrak{r}, {}^{\blacklozenge}\mathfrak{r})$, *is also a c-region.*

Proof. Follows directly from Definition 4. □

The set of all non-trivial c-regions will be denoted by $\mathrm{REG}_{\mathfrak{ts}}$ and, for every state $s \in S$, we will denote by REG_s the set of all the non-trivial c-regions containing s, $\mathrm{REG}_s \stackrel{\text{df}}{=} \{\mathfrak{r} \in \mathrm{REG}_{\mathfrak{ts}} \mid s \in \|\mathfrak{r}\|\}$. For the example in Figure 2(a), we have $\mathrm{REG}_{s_{init}} = \{\mathfrak{r}_9,\mathfrak{r}_{10},\mathfrak{r}_{11},\mathfrak{r}_{12}\}$ and $\overline{\mathfrak{r}_{12}} = \mathfrak{r}_{16}$.

Lattices of c-regions. We call two c-regions, $\mathfrak{r}$ and $\mathfrak{r}'$, *compatible* if it is the case that $\|\mathfrak{r}\| = \|\mathfrak{r}'\|$, ${}^\bullet\mathfrak{r} = {}^\bullet\mathfrak{r}'$ and $\mathfrak{r}^\bullet = \mathfrak{r}'^\bullet$. We denote this by $\mathfrak{r} \approx \mathfrak{r}'$. For two compatible c-regions, $\mathfrak{r}$ and $\mathfrak{r}'$, we define their *union* and *intersection*, in the following way:

$$\mathfrak{r} \cup \mathfrak{r}' \stackrel{\text{df}}{=} (\|\mathfrak{r}\|, {}^\bullet\mathfrak{r}, \mathfrak{r}^\bullet, {}^\blacklozenge\mathfrak{r} \cup {}^\blacklozenge\mathfrak{r}', {}^\blacktriangleleft\mathfrak{r} \cup {}^\blacktriangleleft\mathfrak{r}') \quad \text{and} \quad \mathfrak{r} \cap \mathfrak{r}' \stackrel{\text{df}}{=} (\|\mathfrak{r}\|, {}^\bullet\mathfrak{r}, \mathfrak{r}^\bullet, {}^\blacklozenge\mathfrak{r} \cap {}^\blacklozenge\mathfrak{r}', {}^\blacktriangleleft\mathfrak{r} \cap {}^\blacktriangleleft\mathfrak{r}') \ .$$

Moreover, we denote $\mathfrak{r} \preceq \mathfrak{r}'$ whenever ${}^\blacklozenge\mathfrak{r} \subseteq {}^\blacklozenge\mathfrak{r}'$ and ${}^\blacktriangleleft\mathfrak{r} \subseteq {}^\blacktriangleleft\mathfrak{r}'$. For the example in Figure 2(a), we have $\mathfrak{r}_1 \approx \mathfrak{r}_2 \approx \mathfrak{r}_3 \approx \mathfrak{r}_4$, $\mathfrak{r}_2 \cup \mathfrak{r}_3 = \mathfrak{r}_4$ and $\mathfrak{r}_{15} \preceq \mathfrak{r}_{16}$.

Proposition 4. *If $\mathfrak{r}$ is a c-region, and $inh \subseteq {}^\blacklozenge\mathfrak{r}$ and $act \subseteq {}^\blacktriangleleft\mathfrak{r}$ are two sets of events, then $(\|\mathfrak{r}\|, {}^\bullet\mathfrak{r}, \mathfrak{r}^\bullet, inh, act)$ is also a c-region.*

Proof. Follows directly from Definition 4. □

Proposition 5. *If $\mathfrak{r}$ and $\mathfrak{r}'$ are compatible c-regions, then $\mathfrak{r} \cup \mathfrak{r}'$ and $\mathfrak{r} \cap \mathfrak{r}'$ are also c-regions.*

Proof. The first part follows directly from Definition 4, and the second from Proposition 4. □

Given a c-region $\mathfrak{r}$, the equivalence class of c-regions compatible with $\mathfrak{r}$, denoted by $[\mathfrak{r}]_\approx$, forms a complete lattice w.r.t. the partial order $\preceq$ and the operations $\cup$ (join) and $\cap$ (meet). The *$\preceq$-minimal* and *$\preceq$-maximal* c-regions it contains are given respectively by:

$$(\|\mathfrak{r}\|, {}^\bullet\mathfrak{r}, \mathfrak{r}^\bullet, \varnothing, \varnothing) \qquad \text{and} \qquad (\|\mathfrak{r}\|, {}^\bullet\mathfrak{r}, \mathfrak{r}^\bullet, \bigcup_{\mathfrak{r}' \in [\mathfrak{r}]_\approx} {}^\blacklozenge\mathfrak{r}', \bigcup_{\mathfrak{r}' \in [\mathfrak{r}]_\approx} {}^\blacktriangleleft\mathfrak{r}') \ .$$

The step transition system in Figure 2(a) has six $\preceq$-minimal c-regions ($\mathfrak{r}_1$, $\mathfrak{r}_5$, $\mathfrak{r}_9$, $\mathfrak{r}_{11}$, $\mathfrak{r}_{13}$ and $\mathfrak{r}_{15}$) and six $\preceq$-maximal c-regions ($\mathfrak{r}_4$, $\mathfrak{r}_8$, $\mathfrak{r}_{10}$, $\mathfrak{r}_{12}$, $\mathfrak{r}_{14}$ and $\mathfrak{r}_{16}$).

We feel that the algebraic properties enjoyed by sets of compatible c-regions will be useful in the synthesis procedure aimed at constructing optimal ENCL-systems. We will come back to this issue later on.

Relating regions and events. Given an event $e \in \mathcal{E}_{\mathfrak{ts}}$, its sets of *pre-c-regions*, ${}^\circ e$, *post-c-regions*, e°, *inh-c-regions*, ${}^\Diamond e$, and *act-c-regions*, ${}^\triangleleft e$, are respectively defined as:

$$ {}^\circ e \stackrel{\text{df}}{=} \{\mathfrak{r} \in \text{Reg}_{\mathfrak{ts}} \mid e \in \mathfrak{r}^\bullet\} \qquad e^\circ \stackrel{\text{df}}{=} \{\mathfrak{r} \in \text{Reg}_{\mathfrak{ts}} \mid e \in {}^\bullet\mathfrak{r}\}$$
$$ {}^\Diamond e \stackrel{\text{df}}{=} \{\mathfrak{r} \in \text{Reg}_{\mathfrak{ts}} \mid e \in {}^\blacklozenge\mathfrak{r}\} \qquad {}^\triangleleft e \stackrel{\text{df}}{=} \{\mathfrak{r} \in \text{Reg}_{\mathfrak{ts}} \mid e \in {}^\blacktriangleleft\mathfrak{r}\} \ .$$

Moreover, ${}^\circ e^\circ \stackrel{\text{df}}{=} {}^\circ e \cup e^\circ$, ${}^\circ e^\triangleleft \stackrel{\text{df}}{=} {}^\circ e \cup {}^\triangleleft e$ and ${}^\Diamond e^\circ \stackrel{\text{df}}{=} e^\circ \cup {}^\Diamond e$. All these notations can be applied to sets of events by taking the union of sets of regions defined for the individual events. For the step transition system in Figure 2(a), we have ${}^\circ\mathtt{e} = \{\mathfrak{r}_{11}, \mathfrak{r}_{12}\}$ and ${}^\Diamond\mathtt{f} = \{\mathfrak{r}_{16}\}$.

Proposition 6. *If $s \xrightarrow{u} s'$ is a transition of $\mathfrak{ts}$, then:*

1. $\mathfrak{r} \in {}^{\circ}u$ *implies* $s \in \|\mathfrak{r}\|$ *and* $s' \notin \|\mathfrak{r}\|$.
2. $\mathfrak{r} \in u^{\circ}$ *implies* $s \notin \|\mathfrak{r}\|$ *and* $s' \in \|\mathfrak{r}\|$.
3. $\mathfrak{r} \in {}^{\diamond}u$ *implies* $s \notin \|\mathfrak{r}\|$.
4. $\mathfrak{r} \in {}^{\triangleleft}u$ *implies* $s \in \|\mathfrak{r}\|$.

Proof. Follows directly from the definitions of ${}^{\circ}u$, u°, ${}^{\diamond}u$ and ${}^{\triangleleft}u$ as well as Definition 4(3,4,5,6). □

The sets of pre-c-regions and post-c-regions of events in an executed step are mutually disjoint. Moreover, they can be 'calculated' using the c-regions associated with the source and target states.

Proposition 7. *If* $s \xrightarrow{u} s'$ *is a transition of* $\mathfrak{ts}$*, then:*

1. ${}^{\circ}e \cap {}^{\circ}f = \varnothing$ *and* $e^{\circ} \cap f^{\circ} = \varnothing$*, for all distinct* $e, f \in u$.
2. ${}^{\circ}u \cap u^{\circ} = \varnothing$.
3. ${}^{\circ}u = \textsc{Reg}_s \setminus \textsc{Reg}_{s'}$ *and* $u^{\circ} = \textsc{Reg}_{s'} \setminus \textsc{Reg}_s$.

Proof. (1) Suppose that $\mathfrak{r} \in {}^{\circ}e \cap {}^{\circ}f$, i.e., $e, f \in \mathfrak{r}^{\bullet}$. This means, by Definition 4(5), that $s \in \|\mathfrak{r}\|$ and $s' \notin \|\mathfrak{r}\|$. Thus, by Definition 4(1), $|u \cap \mathfrak{r}^{\bullet}| = 1$, a contradiction with $e, f \in u \cap \mathfrak{r}^{\bullet}$. The second part can be shown in a similar way.

(2) Suppose that $\mathfrak{r} \in {}^{\circ}u \cap u^{\circ}$. Then, by Proposition 6(1,2), $s \in \|\mathfrak{r}\|$ and $s \notin \|\mathfrak{r}\|$, a contradiction.

(3) We only show that $\textsc{Reg}_s \setminus \textsc{Reg}_{s'} = {}^{\circ}u$, as the second part can be shown in a similar way. By Proposition 6, ${}^{\circ}u \subseteq \textsc{Reg}_s$ and ${}^{\circ}u \cap \textsc{Reg}_{s'} = \varnothing$. Hence ${}^{\circ}u \subseteq \textsc{Reg}_s \setminus \textsc{Reg}_{s'}$. Suppose that $\mathfrak{r} \in \textsc{Reg}_s \setminus \textsc{Reg}_{s'}$, which implies that $s \in \|\mathfrak{r}\|$ and $s' \notin \|\mathfrak{r}\|$. Hence, by Definition 4(1) and $s \xrightarrow{u} s'$, $u \cap \mathfrak{r}^{\bullet} \neq \varnothing$. Thus $\mathfrak{r} \in {}^{\circ}u$ and so $\textsc{Reg}_s \setminus \textsc{Reg}_{s'} \subseteq {}^{\circ}u$. Consequently, $\textsc{Reg}_s \setminus \textsc{Reg}_{s'} = {}^{\circ}u$. □

The next two propositions provide a useful characterisation of inh-c-regions and act-c-regions of an event in terms of transitions involving this event. For example, if $\mathfrak{r}$ is an inh-c-region of event e, then no transition involving e lies completely within $\mathfrak{r}$. In what follows, for an event e and a c-region $\mathfrak{r}$, we denote $\mathcal{B}^e_{\mathfrak{r}} \stackrel{\mathrm{df}}{=} \{(s, u, s') \in T_e \mid s, s' \in \|\mathfrak{r}\|\}$ to be the set of all transitions involving e which are *buried* in $\mathfrak{r}$, i.e., their source and target states belong to $\|\mathfrak{r}\|$.

Proposition 8. *If* $e \in \mathcal{E}_{\mathfrak{ts}}$ *and* $\mathfrak{r} \in {}^{\diamond}e$*, then one of the following holds:*

1. $\mathcal{B}^e_{\mathfrak{r}} = \varnothing$, $\mathcal{B}^e_{\overline{\mathfrak{r}}} \neq \varnothing$ *and* $\mathfrak{r} \notin {}^{\circ}u$*, for all* $u \in U_e$.
2. $\mathcal{B}^e_{\mathfrak{r}} = \varnothing$, $e \notin {}^{\bullet}\mathfrak{r}$ *and* $\mathfrak{r} \in u^{\circ} \setminus {}^{\circ}u$*, for all* $u \in U_e$.

Proof. Suppose that $(s, u, s') \in T_e$. From $\mathfrak{r} \in {}^{\diamond}e \subseteq {}^{\diamond}u$ and Proposition 6(3), we have that $s \notin \|\mathfrak{r}\|$. Hence $\mathcal{B}^e_{\mathfrak{r}} = \varnothing$ and $\mathfrak{r} \notin {}^{\circ}u$, for all $u \in U_e$. We will now show that $\mathcal{B}^e_{\overline{\mathfrak{r}}} \neq \varnothing$, or that $e \notin {}^{\bullet}\mathfrak{r}$ and $\mathfrak{r} \in u^{\circ}$, for all $u \in U_e$.

Suppose that $\mathcal{B}^e_{\overline{\mathfrak{r}}} = \varnothing$. We first observe that $e \notin {}^{\bullet}\mathfrak{r}$ since it follows directly from $e \in {}^{\blacklozenge}\mathfrak{r}$ (as $\mathfrak{r} \in {}^{\diamond}e$) and Definition 4(7). What remains to be shown is that if $(s, u, s') \in T_e$ then $\mathfrak{r} \in u^{\circ}$. We already know that $s \notin \|\mathfrak{r}\|$. Moreover, since $\mathcal{B}^e_{\overline{\mathfrak{r}}} = \varnothing$, we have $s' \in \|\mathfrak{r}\|$. This means, by Definition 4(2), that $|u \cap {}^{\bullet}\mathfrak{r}| = 1$. Hence there is $f \in u$ such that $f \in {}^{\bullet}\mathfrak{r}$, and so $\mathfrak{r} \in f^{\circ} \subseteq u^{\circ}$. □

Proposition 9. *If $e \in \mathcal{E}_{\mathfrak{ts}}$ and $\mathfrak{r} \in {}^{\triangleleft}e$, then one of the following holds:*

1. $\mathcal{B}^e_{\overline{\mathfrak{r}}} = \varnothing$, $\mathcal{B}^e_{\mathfrak{r}} \neq \varnothing$ *and* $\mathfrak{r} \notin u^{\circ}$, *for all* $u \in U_e$.
2. $\mathcal{B}^e_{\overline{\mathfrak{r}}} = \varnothing$, $e \notin \mathfrak{r}^{\bullet}$ *and* $\mathfrak{r} \in {}^{\circ}u \setminus u^{\circ}$, *for all* $u \in U_e$.

Proof. Similar to that of Proposition 8. □

It is easy to show that a step can be executed at a state only if the inh-c-regions of the former do not comprise the latter, and the act-c-regions do.

Proposition 10. *If $s \xrightarrow{u} s'$ is a transition of $\mathfrak{ts}$, then ${}^{\Diamond}u \cap \mathrm{REG}_s = \varnothing$ and ${}^{\triangleleft}u \subseteq \mathrm{REG}_s$.*

Proof. Suppose that $\mathfrak{r} \in {}^{\Diamond}u \cap \mathrm{REG}_s \neq \varnothing$. Then from $\mathfrak{r} \in {}^{\Diamond}u$ and Proposition 6(3) we have that $s \notin \|\mathfrak{r}\|$ which contradicts $\mathfrak{r} \in \mathrm{REG}_s$. Suppose now that $\mathfrak{r} \in {}^{\triangleleft}u$. From Proposition 6(4) we have that $s \in \|\mathfrak{r}\|$, and so $\mathfrak{r} \in \mathrm{REG}_s$. □

Proposition 11. *If $e \in \mathcal{E}_{\mathfrak{ts}}$, then ${}^{\circ}e^{\circ} \cap ({}^{\Diamond}e \cup {}^{\triangleleft}e) = \varnothing$.*

Proof. Suppose that $\mathfrak{r} \in e^{\circ} \cap {}^{\Diamond}e \neq \varnothing$. Then $e \in {}^{\bullet}\mathfrak{r} \cap {}^{\blacklozenge}\mathfrak{r} \neq \varnothing$, contradicting Definition 4(7).

Suppose now that $\mathfrak{r} \in {}^{\circ}e \cap {}^{\Diamond}e \neq \varnothing$. By Proposition 8, one of the following two cases holds:
Case 1: There is $(s, u, s') \in T_e$ such that $s, s' \notin \|\mathfrak{r}\|$. By $\mathfrak{r} \in {}^{\circ}e$, we have that $\mathfrak{r} \in {}^{\circ}u$, and so from Proposition 6 it follows that $s \in \|\mathfrak{r}\|$ and $s' \notin \|\mathfrak{r}\|$, a contradiction.
Case 2: $e \notin {}^{\bullet}\mathfrak{r}$ and $\mathfrak{r} \in u^{\circ}$ for some $u \in U_e \neq \varnothing$. Then $\mathfrak{r} \notin e^{\circ}$ and there is $(s, u, s') \in T_e$ such that $s \notin \|\mathfrak{r}\|$ and $s' \in \|\mathfrak{r}\|$. On the other hand, by $\mathfrak{r} \in {}^{\circ}e \subseteq {}^{\circ}u$ and Proposition 6, we have $s \in \|\mathfrak{r}\|$ and $s' \notin \|\mathfrak{r}\|$, a contradiction.

Hence ${}^{\circ}e^{\circ} \cap {}^{\Diamond}e = \varnothing$, and ${}^{\circ}e^{\circ} \cap {}^{\triangleleft}e = \varnothing$ can be shown in a similar way. □

To characterise transition systems generated by ENCL-systems, we will need the set of all *potential steps* $\mathbb{U}_{\mathfrak{ts}}$ of $\mathfrak{ts}$, given by:

$$\mathbb{U}_{\mathfrak{ts}} \stackrel{\mathrm{df}}{=} \{u \subseteq \mathcal{E}_{\mathfrak{ts}} \mid u \neq \varnothing \ \wedge \ \forall e, f \in u : e \neq f \Rightarrow {}^{\circ}e^{\circ} \cap {}^{\circ}f^{\circ} = \varnothing\} \, .$$

Proposition 12. *If $s \xrightarrow{u} s'$ is a transition of $\mathfrak{ts}$, then $u \in \mathbb{U}_{\mathfrak{ts}}$.*

Proof. Follows from TSYS2 and Proposition 7(1,2). □

Thin transition systems. In general, a c-region $\mathfrak{r}$ cannot be identified only by its set of states $\|\mathfrak{r}\|$; in other words, ${}^{\bullet}\mathfrak{r}$, $\mathfrak{r}^{\bullet}$, ${}^{\blacklozenge}\mathfrak{r}$ and ${}^{\blacktriangleleft}\mathfrak{r}$ may not be recoverable from $\|\mathfrak{r}\|$. However, if the transition system is *thin*, i.e., for every event $e \in \mathcal{E}_{\mathfrak{ts}}$ we have that $\{e\} \in U_e$, then different c-regions with the same sets *inh* and *act* are based on different sets of states.

Proposition 13 ([4]). *If $\mathfrak{ts}$ is thin and $\mathfrak{r} \neq \mathfrak{r}'$ are c-regions such that ${}^{\blacklozenge}\mathfrak{r} = {}^{\blacklozenge}\mathfrak{r}'$ and ${}^{\blacktriangleleft}\mathfrak{r} = {}^{\blacktriangleleft}\mathfrak{r}'$, then $\|\mathfrak{r}\| \neq \|\mathfrak{r}'\|$.*

4 Transition systems of ENCL-systems

We now can present a complete characterisation of the transition systems generated by ENCL-systems.

Definition 5 (ENCL-transition system). *A step transition system* $\mathfrak{ts} = (S, T, s_{init})$ *is an* ENCL-transition system *if it satisfies the following axioms:*

AXIOM1 *For every* $s \in S \setminus \{s_{init}\}$, *there are* $(s_0, u_0, s_1), \ldots, (s_{n-1}, u_{n-1}, s_n) \in T$ *such that* $s_0 = s_{init}$ *and* $s_n = s$.

AXIOM2 *For every event* $e \in \mathcal{E}_{\mathfrak{ts}}$, *both* $^\circ e$ *and* e° *are non-empty.*

AXIOM3 *For all states* $s, s' \in S$, *if* $\text{REG}_s = \text{REG}_{s'}$ *then* $s = s'$.

AXIOM4 *If* $s \in S$ *and* $u \in \mathbb{U}_{\mathfrak{ts}}$ *are such that*

- $^\circ u^\triangleleft \subseteq \text{REG}_s$ *and* $^\Diamond u^\circ \cap \text{REG}_s = \varnothing$ *and*
- *there is no step* $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}}$ *with the event* e *satisfying* $\mathfrak{L}(e) \in \mathfrak{L}(u)$, $^\circ e^\triangleleft \subseteq \text{REG}_s$ *and* $^\Diamond e^\circ \cap \text{REG}_s = \varnothing$,

then we have $s \xrightarrow{u}$.

AXIOM5 *If* $s \xrightarrow{u}$ *then there is no step* $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}}$ *with the event* e *satisfying* $\mathfrak{L}(e) \in \mathfrak{L}(u)$, $^\circ e^\triangleleft \subseteq \text{REG}_s$ *and* $^\Diamond e^\circ \cap \text{REG}_s = \varnothing$.

In the above, AXIOM1 implies that all the states in $\mathfrak{ts}$ are reachable from the initial state. AXIOM2 will ensure that every event in a synthesised ENCL-system will have at least one input condition and at least one output condition. AXIOM3 was used for other transition systems as well, and is usually called the *state separation property* [16, 20], and it guarantees that $\mathfrak{ts}$ is deterministic. AXIOM4 is a variation of the *forward closure property* [20] or the *event/state separation property* [16]. AXIOM5 ensures that every step in a transition system is indeed a maximal step w.r.t. localities of the events it comprises.

Proposition 14. *If* $s \xrightarrow{u} s'$ *and* $s \xrightarrow{u} s''$, *then* $s' = s''$.

Proof. Follows from Proposition 7(3) and AXIOM3. □

The construction of a step transition system for a given ENCL-system is straightforward.

Definition 6 (from net system to transition system). *The* transition system generated *by an* ENCL-*system* $\mathfrak{encl}$ *is* $\mathfrak{ts}_{\mathfrak{encl}} \stackrel{\text{df}}{=} (C_{\mathfrak{encl}}, \rightarrow_{\mathfrak{encl}}, c_{init})$, *where* c_{init} *is the initial case of* $\mathfrak{encl}$.

Theorem 1. $\mathfrak{ts}_{\mathfrak{encl}}$ *is an* ENCL-*transition system.*

Proof. See the Appendix. □

5 Solving the synthesis problem

The translation from ENCL-transition systems to ENCL-systems is based on the pre-, post-, inh- and act-c-regions of the events appearing in a transition system.

Definition 7 (from transition system to net system). *The net system* associated *with an* ENCL*-transition system* $\mathfrak{ts} = (S, T, s_{init})$ *is:*

$$\mathfrak{encl}_{\mathfrak{ts}} \stackrel{\mathrm{df}}{=} (\mathrm{REG}_{\mathfrak{ts}}, \mathcal{E}_{\mathfrak{ts}}, F_{\mathfrak{ts}}, I_{\mathfrak{ts}}, A_{\mathfrak{ts}}, \mathrm{REG}_{s_{init}}) ,$$

where $F_{\mathfrak{ts}}$*,* $I_{\mathfrak{ts}}$ *and* $A_{\mathfrak{ts}}$ *are defined thus:*

$$\left.\begin{array}{l} F_{\mathfrak{ts}} \stackrel{\mathrm{df}}{=} \{(\mathfrak{r}, e) \in \mathrm{REG}_{\mathfrak{ts}} \times \mathcal{E}_{\mathfrak{ts}} \mid \mathfrak{r} \in {}^{\circ}e\} \;\cup\; \{(e, \mathfrak{r}) \in \mathcal{E}_{\mathfrak{ts}} \times \mathrm{REG}_{\mathfrak{ts}} \mid \mathfrak{r} \in e^{\circ}\} \\ I_{\mathfrak{ts}} \stackrel{\mathrm{df}}{=} \{(\mathfrak{r}, e) \in \mathrm{REG}_{\mathfrak{ts}} \times \mathcal{E}_{\mathfrak{ts}} \mid \mathfrak{r} \in {}^{\diamond}e\} \\ A_{\mathfrak{ts}} \stackrel{\mathrm{df}}{=} \{(\mathfrak{r}, e) \in \mathrm{REG}_{\mathfrak{ts}} \times \mathcal{E}_{\mathfrak{ts}} \mid \mathfrak{r} \in {}^{\triangleleft}e\} \ . \end{array}\right\} \quad (1)$$

Proposition 15. *For every* $e \in \mathcal{E}_{\mathfrak{ts}}$*, we have* ${}^{\circ}e = {}^{\bullet}e$*,* $e^{\circ} = e^{\bullet}$*,* ${}^{\diamond}e = {}^{\blacklozenge}e$ *and* ${}^{\triangleleft}e = {}^{\blacktriangleleft}e$*.*

Proof. Follows directly from the definition of $\mathfrak{encl}_{\mathfrak{ts}}$. □

Figure 3 shows the ENCL-system associated with the step transition system shown in Figure 2(a). It is, clearly, not the net system shown in Figure 2(b), as it contains twice as many conditions as well as four context arcs which were not present there. This is not unusual as the above construction produces nets which are saturated with conditions as well as context arcs. In fact, the whole construction would still work if we restricted ourselves to the $\preceq$-maximal non-trivial c-regions, similarly as it has been done in [21] for EN-systems with inhibitor arcs. But the resulting ENCL-system would still not be as that shown in Figure 2(b). In fact, the latter would be re-constructed if we took all the $\preceq$-minimal non-trivial c-regions of the step transition system shown in Figure 2(a). However, taking only the $\preceq$-minimal c-regions would not work in the general case (the ENCL-transition system shown in Figure 4(c) provides a suitable counterexample), and that it is possible to use them in this case is due to the maximally concurrent execution rule which underpins ENCL-systems. What this example implies is that in order to synthesise an optimal net (for example, from the point of view of the number of conditions and/or context arcs), it is a good idea to look at the whole spectrum of c-regions arranged in the lattices of compatible c-regions (and, in any case, never use two different c-regions, $\mathfrak{r}$ and $\mathfrak{r}'$, such that $\mathfrak{r} \preceq \mathfrak{r}'$).

Theorem 2. $\mathfrak{encl}_{\mathfrak{ts}}$ *is an* ENCL*-system.*

Proof. All one needs to observe is that, for every $e \in \mathcal{E}_{\mathfrak{ts}}$, it is the case that: ${}^{\bullet}e \neq \varnothing \neq e^{\bullet}$, which follows from AXIOM2 and Proposition 15; ${}^{\bullet}e \cap e^{\bullet} = \varnothing$, which follows from Propositions 7(2) and 15; and ${}^{\bullet}e^{\bullet} \cap ({}^{\blacklozenge}e \cup {}^{\blacktriangleleft}e) = \varnothing$, which follows from Propositions 11 and 15. □

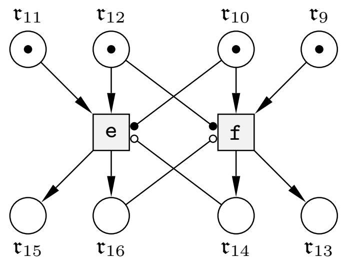

Fig. 3. ENCL-system synthesised from the ENCL-transition system in Figure 2(a).

We finally show that the ENCL-system associated with an ENCL-transition system $\mathfrak{ts}$ generates a transition system which is isomorphic to $\mathfrak{ts}$.

Proposition 16. *Let* $\mathfrak{ts} = (S, T, s_{init})$ *be an* ENCL*-transition system and*

$$\mathfrak{encl} = \mathfrak{encl}_{\mathfrak{ts}} = (\text{REG}_{\mathfrak{ts}}, \mathcal{E}_{\mathfrak{ts}}, F_{\mathfrak{ts}}, I_{\mathfrak{ts}}, A_{\mathfrak{ts}}, \text{REG}_{s_{init}}) = (B, E, F, I, A, c_{init})$$

be the ENCL*-system associated with it.*

1. $C_{\mathfrak{encl}} = \{\text{REG}_s \mid s \in S\}$.
2. $\rightarrow_{\mathfrak{encl}} = \{(\text{REG}_s, u, \text{REG}_{s'}) \mid (s, u, s') \in T\}$.

Proof. Note that from the definition of $C_{\mathfrak{encl}}$, every $c \in C_{\mathfrak{encl}}$ is reachable from c_{init} in $\mathfrak{encl}$; and that from AXIOM1, every $s \in S$ is reachable from s_{init} in $\mathfrak{ts}$.

We first show that if $c \xrightarrow{u}_{\mathfrak{encl}} c'$ and $c = \text{REG}_s$, for some $s \in S$, then there is $s' \in S$ such that $s \xrightarrow{u} s'$ and $c' = \text{REG}_{s'}$. By $c \xrightarrow{u}_{\mathfrak{encl}} c'$, $u \in \mathbb{U}_{\mathfrak{encl}}$ is a step such that ${}^{\bullet}u^{\blacktriangleleft} \subseteq c$ and ${}^{\blacklozenge}u^{\bullet} \cap c = \varnothing$, and there is no step $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{encl}}$ satisfying $\mathfrak{L}(e) \in \mathfrak{L}(u)$ and ${}^{\bullet}e^{\blacktriangleleft} \subseteq c$ and ${}^{\blacklozenge}e^{\bullet} \cap c = \varnothing$. Moreover, $c' = (c \setminus {}^{\bullet}u) \cup u^{\bullet}$.
Hence, by Proposition 15 and AXIOM4, $u \in \mathbb{U}_{\mathfrak{ts}}$ and $s \xrightarrow{u} s'$, for some $s' \in S$. Then, by Proposition 7(3), $\text{REG}_{s'} = (\text{REG}_s \setminus {}^{\circ}u) \cup u^{\circ}$. At the same time, we have $c' = (c \setminus {}^{\bullet}u) \cup u^{\bullet}$. Hence, by Proposition 15 and $c = \text{REG}_s$, we have that $c' = \text{REG}_{s'}$.
As a result, we have shown (note that $c_{init} = \text{REG}_{s_{init}} \in \{\text{REG}_s \mid s \in S\}$) that

$$\begin{array}{lcl} C_{\mathfrak{encl}} & \subseteq & \{\text{REG}_s \mid s \in S\} \\ \rightarrow_{\mathfrak{encl}} & \subseteq & \{(\text{REG}_s, u, \text{REG}_{s'}) \mid (s, u, s') \in T\} \,. \end{array}$$

We now prove the reverse inclusions. By definition, $\text{REG}_{s_{init}} \in C_{\mathfrak{encl}}$. It is enough to show that if $s \xrightarrow{u} s'$ and $\text{REG}_s \in C_{\mathfrak{encl}}$, then $\text{REG}_{s'} \in C_{\mathfrak{encl}}$ and $\text{REG}_s \xrightarrow{u}_{\mathfrak{encl}} \text{REG}_{s'}$. By AXIOM5 and Propositions 7(3), 12, 10 and 15, u is a valid step in $\mathfrak{encl}$ which is enabled at the case REG_s. So, there is a case c' such that $\text{REG}_s \xrightarrow{u}_{\mathfrak{encl}} c'$ and $c' = (\text{REG}_s \setminus {}^{\bullet}u) \cup u^{\bullet}$. From Propositions 7(3) and 15 we have that $c' = \text{REG}_{s'}$. Hence we obtain that $\text{REG}_s \xrightarrow{u}_{\mathfrak{encl}} \text{REG}_{s'}$ and so also $\text{REG}_{s'} \in C_{\mathfrak{encl}}$. ☐

Theorem 3. *Let* $\mathfrak{ts} = (S, T, s_{init})$ *be an* ENCL*-transition system and* $\mathfrak{encl} = \mathfrak{encl}_{\mathfrak{ts}}$ *be the* ENCL*-system associated with it. Then* $\mathfrak{ts}_{\mathfrak{encl}}$ *is isomorphic to* $\mathfrak{ts}$*.*

Proof. Let $\psi : S \to C_{\mathfrak{encl}}$ be a mapping given by $\psi(s) = \text{REG}_s$, for all $s \in S$ (note that, by Proposition 16(1), ψ is well-defined). We will show that ψ is an isomorphism for $\mathfrak{ts}$ and $\mathfrak{ts}_{\mathfrak{encl}}$.
Note that $\psi(s_{init}) = \text{REG}_{s_{init}}$. By Proposition 16(1), ψ is onto. Moreover, by AXIOM3, it is injective. Hence ψ is a bijection. We then observe that, by Proposition 16(2), we have $(s, u, s') \in T$ if and only if $(\psi(s), u, \psi(s')) \in \to_{\mathfrak{encl}}$. Hence ψ is an isomorphism for $\mathfrak{ts}$ and $\mathfrak{ts}_{\mathfrak{encl}}$. $\square$

Figure 4 shows two further examples of the synthesis of ENCL-systems. The first one, in Figure 4(a,b), illustrates a conflict between two events, and the synthesised ENCL-system utilises two $\preceq$-minimal c-regions, $\mathfrak{r} = (\{s_{init}\}, \varnothing, \{\mathtt{e}, \mathtt{f}\}, \varnothing, \varnothing)$ for the upper condition, and its complement $\overline{\mathfrak{r}}$ for the lower one. The second example, in Figure 4(c,d), exemplifies a situation when a correct solution has been obtained without using only $\preceq$-maximal c-regions. However, an attempt to use only $\preceq$-minimal c-regions would fail, as the resulting ENCL-system (shown in Figure 5(a)) allows one to execute the step sequence $\{\mathtt{e}\}\{\mathtt{f}\}$ which is impossible in the original transition system. Moreover, Figure 5(b) shows a correct synthesis solution based solely on $\preceq$-maximal c-regions. When compared with that in Figure 4(d) it looks less attractive since the latter uses fewer context arcs. It should already be clear that to synthesise 'optimal' ENCL-systems it will, in general, be necessary to use a mix of various kinds of c-regions, and the development of suitable algorithms is an interesting and important topic for further research.

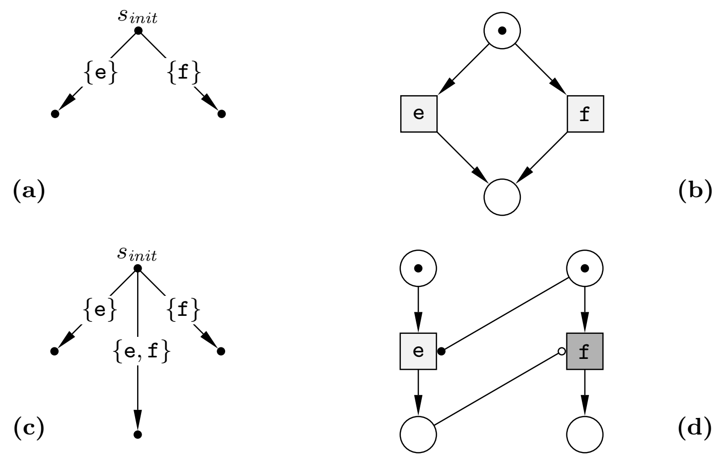

Fig. 4. A transition system with co-located events e and f **(a)**, and a corresponding ENCL-system **(b)**; and a transition system with differently located events e and f **(c)**, and a corresponding ENCL-system **(d)**.

Fig. 5. ENCL-system synthesised from the transition system in Figure 4(c) using only $\preceq$-minimal non-trivial c-regions **(a)**, and only $\preceq$-maximal non-trivial c-regions **(b)**.

6 Concluding Remarks

In this paper, we solved the synthesis problem for EN-systems with context arcs and localities, by following the standard approach in which key relationships between a Petri net and its transition system are established via the notion of a region. Moreover, in order to obtain a satisfactory solution, we augmented the standard notion of a region with some additional information, leading to the notion of a c-region. We then defined, and showed consistency of, two behaviour preserving translations between ENCL-systems and their transition systems.

Throughout this paper it has always been assumed that events' localities are known in advance. In particular, this information was present in the input to the synthesis problem. However, one might prefer to work only with step transition systems, and determine the localities of events during the synthesis procedure (perhaps choosing an 'optimal' option). This could, of course, be done by considering in turn all possibilities for the locality mapping $\mathfrak{L}$. Unfortunately, such an approach would be hardly satisfactory as there are $B_{|\mathcal{E}_{\mathfrak{ts}}|}$ different candidate mappings, where B_n is the n-th number in the fast-growing sequence of *Bell numbers*. But it is not necessary to follow this 'brute-force' approach, and two simple observations should in practice be of great help. More precisely, consider a step transition system $\mathfrak{ts} \overset{\mathrm{df}}{=} (S, T, s_{init})$. If it is generated by an ENCL-system with the locality mapping $\mathfrak{L}$, then the following hold, for every state $s \in S$:

- If $s \xrightarrow{u \uplus w}$ and $s \xrightarrow{u}$ then $\mathfrak{L}(e) \neq \mathfrak{L}(f)$, for all $e \in u$ and $f \in w$.
- If $s \xrightarrow{u}$ and there is no $w \subset u$ such that $s \xrightarrow{w}$ then $\mathfrak{L}(e) = \mathfrak{L}(f)$, for all $e, f \in u$.

Thus, for the example transition systems in Figures 2(a) and 4(c), we have respectively $\mathfrak{L}(e) = \mathfrak{L}(f)$ and $\mathfrak{L}(e) \neq \mathfrak{L}(f)$, and so the choice of localities we made was actually the only one which would work in these cases. On the other hand, for the step transition systems in Figures 4(a) and 6(a), the above rules do not provide any useful information. Indeed, in both cases we may take $\mathfrak{L}(e) = \mathfrak{L}(f)$ or $\mathfrak{L}(e) \neq \mathfrak{L}(f)$, and in each case synthesise a suitable ENCL-system, as shown in Figure 6(b) for the example in Figure 6(a). Note that these rules can be used for a quick decision that a step transition system is not a valid ENCL-transition system; for example, if we have $s_1 \xrightarrow{\{e,f\}}$ and $s_1 \xrightarrow{\{e\}}$ and $s_2 \xrightarrow{\{e,f\}}$ and $\neg s_2 \xrightarrow{\{e\}}$, for two distinct states, s_1 and s_2, of the same step transition system.

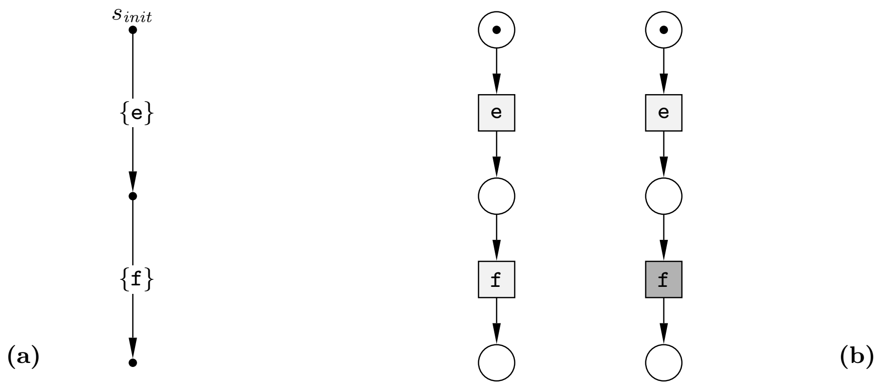

Fig. 6. A step transition system where no assumption about co-locating the events has been made **(a)**, and two corresponding ENCL-systems with different locality mappings **(b)**.

Previous work which appears to be closest to what has been proposed in this paper is due to Badouel and Darondeau [16]. It discusses the notion of a step transition system (generalising that introduced by Mukund [12]), which provides much more general a framewo rk than the basic EN-transition systems; in particular, by dropping the assumption that a transition system should exhibit the so-called *intermediate state property*:

$$s \xrightarrow{\alpha+\beta} s' \implies \exists s'' : s \xrightarrow{\alpha} s'' \xrightarrow{\beta} s' \ .$$

But the step transition systems of [16] still exhibit a *subset property*:

$$s \xrightarrow{\alpha+\beta} \implies s \xrightarrow{\alpha} \ .$$

Neither of the these properties holds for ENL-transition systems (and hence also for ENCL-transition systems). Instead, transition systems with localities enjoy their *weaker* version. More precisely, for ENL-transition systems we have:

$$s \xrightarrow{\alpha+\beta} s' \implies (s \xrightarrow{\alpha} s'' \implies s'' \xrightarrow{\beta} s' \wedge s \xrightarrow{\beta}) \ ,$$

and for ENCL-transition systems, we have:

$$s \xrightarrow{\alpha+\beta} \implies (s \xrightarrow{\alpha} \implies s \xrightarrow{\beta}) \ .$$

For example, the first of these properties implies that the transition system in Figure 4(c) cannot be generated by an ENL-system, and so the use of some context arcs is unavoidable as shown, e.g., in Figure 4(d). We feel that both properties might be useful in finding out whether (or to what extent) the theory of [16] could be adopted to work for the ENCL-transition systems as well.

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Appendix: Proof of Theorem 1

Clearly, $\mathfrak{ts}_{\mathfrak{encl}}$ is a step transition system. We need to prove that it satisfies the five axioms in Definition 5. Before doing this, we will show that, for every $b \in B$,

$$\mathfrak{r}_b \stackrel{\text{df}}{=} (\{c \in C_{\mathfrak{encl}} \mid b \in c\}, {}^\bullet b, b^\bullet, {}^\blacklozenge b, {}^\blacktriangleleft b)$$

is a (possibly trivial) c-region of $\mathfrak{ts}_{\mathfrak{encl}}$. Moreover, if $\varnothing \neq \|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$ then $\mathfrak{r}_b$ is non-trivial.

To show that Definition 4 holds for $\mathfrak{r}_b$, we assume that $c \xrightarrow{u}_{\mathfrak{encl}} c'$ in $\mathfrak{ts}_{\mathfrak{encl}}$, and proceed as follows:

Proof of Definition 4(1) for $\mathfrak{r}_b$. We need to show that $c \in \|\mathfrak{r}_b\|$ and $c' \notin \|\mathfrak{r}_b\|$ implies $|u \cap {}^\bullet b| = 0$ and $|u \cap b^\bullet| = 1$.
From $c \in \|\mathfrak{r}_b\|$ ($c' \notin \|\mathfrak{r}_b\|$) it follows that $b \in c$ (resp. $b \notin c'$). Hence $b \in c \setminus c'$. From Proposition 1 we have $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. Hence $b \in {}^\bullet u$ and, as a consequence, there exists $e \in u$ such that $b \in {}^\bullet e$, and so $e \in b^\bullet$. We therefore have $e \in u \cap b^\bullet$. Hence $|u \cap b^\bullet| \geq 1$. Suppose that there is $f \neq e$ such that $f \in u \cap b^\bullet$. Then we have $f \in u$ and $b \in {}^\bullet f$ which implies $b \in {}^\bullet f \cap {}^\bullet e$, producing a contradiction with $e, f \in u \in \mathbb{U}_{\mathfrak{encl}}$. Hence $|u \cap b^\bullet| = 1$.
From $b \notin c'$ and $c' \setminus c = u^\bullet$, we have $b \notin u^\bullet$. Let $g \in u$ ($u \neq \varnothing$ by definition). Then $b \notin g^\bullet$, and so $g \notin {}^\bullet b$. Hence $|u \cap {}^\bullet b| = 0$.

Proof of Definition 4(2) for $\mathfrak{r}_b$. Can be proved similarly as Definition 4(1).

Proof of Definition 4(3) for $\mathfrak{r}_b$. We need to show that $u \cap {}^\blacklozenge b \neq \varnothing$ implies $c \notin \|\mathfrak{r}_b\|$.
From $u \cap {}^\blacklozenge b \neq \varnothing$ we have that there is $e \in u$ such that $e \in {}^\blacklozenge b$ and so $b \in {}^\blacklozenge e$. Thus, since u is enabled at c in $\mathfrak{encl}$, $b \notin c$. Hence $c \notin \|\mathfrak{r}_b\|$.

Proof of Definition 4(4) for $\mathfrak{r}_b$. Can be proved similarly as Definition 4(3).

Proof of Definition 4(5) for $\mathfrak{r}_b$. We need to show that $u \cap b^\bullet \neq \varnothing$ implies $c \in \|\mathfrak{r}_b\|$ and $c' \notin \|\mathfrak{r}_b\|$.
From Proposition 1 we have $c \setminus c' = {}^\bullet u$ and $c' \setminus c = u^\bullet$. From $u \cap b^\bullet \neq \varnothing$ we have that there is $e \in u$ such that $e \in b^\bullet$, and so $b \in {}^\bullet e$. Consequently, $b \in {}^\bullet u = c \setminus c'$, and so $b \in c$ and $b \notin c'$. Hence $c \in \|\mathfrak{r}_b\|$ and $c' \notin \|\mathfrak{r}_b\|$.

Proof of Definition 4(6) for $\mathfrak{r}_b$. Can be proved similarly as Definition 4(5).

Proof of Definition 4(7) for $\mathfrak{r}_b$. We need to show that ${}^\bullet b \cap {}^\blacklozenge b = b^\bullet \cap {}^\blacktriangleleft b = \varnothing$. This follows directly from the fact that $\mathfrak{encl}$ is an ENCL-system, where for every event e, the sets ${}^\bullet e$, $e^\bullet$, ${}^\blacklozenge e$ and ${}^\blacktriangleleft e$ are mutually disjoint.

Clearly, if $\varnothing \neq \|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$ then $\mathfrak{r}_b$ is a non-trivial c-region, and we may now proceed with the proof proper.

Proof of AXIOM1. Follows directly from the definition of $C_{\mathfrak{encl}}$.

Proof of AXIOM2. We observe that if $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$ then $\{\mathfrak{r}_b \mid b \in {}^\bullet e\} \subseteq {}^\circ e$ and $\{\mathfrak{r}_b \mid b \in e^\bullet\} \subseteq e^\circ$ (follows from ${}^\bullet e \neq \varnothing \neq e^\bullet$, Proposition 2 and the definitions of ${}^\circ e$, e° and $\mathfrak{r}_b$). This and ${}^\bullet e \neq \varnothing \neq e^\bullet$ yields ${}^\circ e \neq \varnothing \neq e^\circ$.

Proof of AXIOM3. Suppose that $c \neq c'$ are two cases in $C_{\mathfrak{encl}}$. Without loss of generality, we may assume that there is $b \in c \setminus c'$. Hence $c \in \|\mathfrak{r}_b\|$ and $c' \notin \|\mathfrak{r}_b\|$. Thus, by the fact that $\mathfrak{r}_b$ is not trivial ($\varnothing \neq \|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$) and $\mathfrak{r}_b \in \text{REG}_c \setminus \text{REG}_{c'}$, AXIOM3 holds.

Proof of AXIOM4. Suppose that $c \in C_{\mathfrak{encl}}$ and $u \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ are such that ${}^\circ u^\triangleleft \subseteq \text{REG}_c$ and ${}^\lozenge u^\circ \cap \text{REG}_c = \varnothing$ and and there is no $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ satisfying: $\mathfrak{L}(e) \in \mathfrak{L}(u)$ and ${}^\circ e^\triangleleft \subseteq \text{REG}_c$ and ${}^\lozenge e^\circ \cap \text{REG}_c = \varnothing$. We need to show that $c \xrightarrow{u}_{\mathfrak{encl}}$.
We have already shown that for $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$, $b \in {}^\bullet e$ implies $\mathfrak{r}_b \in {}^\circ e$, and $b \in e^\bullet$ implies $\mathfrak{r}_b \in e^\circ$. From this and $u \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ we have that $u \in \mathbb{U}_{\mathfrak{encl}}$.
First we show ${}^\bullet u \subseteq c$. Let $e \in u$. Consider $b \in {}^\bullet e$. We have already shown that this implies $\mathfrak{r}_b \in {}^\circ e$. From ${}^\circ u \subseteq \text{REG}_c$, we have that $\mathfrak{r}_b \in \text{REG}_c$, and so $c \in \|\mathfrak{r}_b\|$. Consequently, $b \in c$. Hence, for all $e \in u$ we have ${}^\bullet e \subseteq c$, and so ${}^\bullet u \subseteq c$.
Now we show that $u^\bullet \cap c = \varnothing$. Let $e \in u$. Consider $b \in e^\bullet$. We have already shown that this implies $\mathfrak{r}_b \in e^\circ$. From $u^\circ \cap \text{REG}_c = \varnothing$, we have that $\mathfrak{r}_b \notin \text{REG}_c$, and so $c \notin \|\mathfrak{r}_b\|$. Consequently, $b \notin c$. Hence, for all $e \in u$ we have $e^\bullet \cap c = \varnothing$, and so $u^\bullet \cap c = \varnothing$.
Now we show that ${}^\blacklozenge u \cap c = \varnothing$. Suppose to the contrary that ${}^\blacklozenge u \cap c \neq \varnothing$. Then there is $e \in u$ such that ${}^\blacklozenge e \cap c \neq \varnothing$, and as a consequence there is $b \in {}^\blacklozenge e$ such that $b \in c$. Hence, $c \in \|\mathfrak{r}_b\|$ and so $\|\mathfrak{r}_b\| \neq \varnothing$. We now prove that $\|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$. Suppose $\|\mathfrak{r}_b\| = \{c \in C_{\mathfrak{encl}} \mid b \in c\} = C_{\mathfrak{encl}}$. Then b is a condition present in every case c of $\mathfrak{encl}$ making it impossible for any step containing e to be enabled ($b \in {}^\blacklozenge e$). This, in turn, contradicts the fact that $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$ (as an event in $u \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$) and must appear in some step labelling a transition from $\mathfrak{ts}_{\mathfrak{encl}}$. Hence $\|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$, and so $\mathfrak{r}_b$ is a non-trivial c-region. From $b \in {}^\blacklozenge e$ we have $e \in {}^\blacklozenge b = {}^\blacklozenge \mathfrak{r}_b$, which means that $\mathfrak{r}_b \in {}^\lozenge e$. Consequently, $\mathfrak{r}_b \in {}^\lozenge u$. From this and ${}^\lozenge u \cap \text{REG}_c = \varnothing$ we have $\mathfrak{r}_b \notin \text{REG}_c$, and so $c \notin \|\mathfrak{r}_b\|$. Consequently $b \notin c$, and so we obtained a contradiction. Hence ${}^\blacklozenge u \cap c = \varnothing$.
Now we show that ${}^\blacktriangleleft u \subseteq c$. Suppose to the contrary that there is $b \in {}^\blacktriangleleft u \setminus c$. From $b \in {}^\blacktriangleleft u$ we have that there is $e \in u$ such that $b \in {}^\blacktriangleleft e$. From $b \notin c$ we have that $c \notin \|\mathfrak{r}_b\|$, and so $\|\mathfrak{r}_b\| \neq C_{\mathfrak{encl}}$. We now prove that $\|\mathfrak{r}_b\| \neq \varnothing$. Assume that $\|\mathfrak{r}_b\| = \varnothing$. This implies that, for all $c \in C_{\mathfrak{encl}}$, $b \notin c$. But this would make it impossible to

execute any step containing e in $\mathfrak{encl}$. This, in turn, contradicts the fact that $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$ and so it must appear in some step labelling a transition in $\mathfrak{ts}_{\mathfrak{encl}}$. Hence $\|\mathfrak{r}_b\| \neq \varnothing$, and so the c-region $\mathfrak{r}_b$ is non-trivial. From $b \in {}^{\blacktriangleleft}e$ we have that $e \in {}^{\blacktriangleleft}b = {}^{\blacktriangleleft}\mathfrak{r}_b$. Consequently, we have that $\mathfrak{r}_b \in {}^{\triangleleft}e$, and so $\mathfrak{r}_b \in {}^{\triangleleft}u$. From this and ${}^{\triangleleft}u \subseteq \textsc{Reg}_c$ we have that $\mathfrak{r}_b \in \textsc{Reg}_c$, and so $c \in \|\mathfrak{r}_b\|$. Consequently $b \in c$, and so we obtained a contradiction. Hence ${}^{\blacktriangleleft}u \subseteq c$.
All what remains to be shown is that there is no step $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{encl}}$ satisfying: $\mathfrak{L}(e) \in \mathfrak{L}(u)$, ${}^{\bullet}e^{\blacktriangleleft} \subseteq c$ and ${}^{\blacklozenge}e^{\bullet} \cap c = \varnothing$. Suppose that this is not the case, and $u \uplus \{e_1\} \in \mathbb{U}_{\mathfrak{encl}}$ is a step satisfying these conditions. We consider two cases.
Case 1: There is no $u \uplus \{e_1\} \uplus \{f\} \in \mathbb{U}_{\mathfrak{encl}}$ such that $\mathfrak{L}(f) \in \mathfrak{L}(u \uplus \{e_1\})$, ${}^{\bullet}f^{\blacktriangleleft} \subseteq c$ and ${}^{\blacklozenge}f^{\bullet} \cap c = \varnothing$. This implies $c \xrightarrow{u \uplus \{e_1\}}_{\mathfrak{encl}}$. By Proposition 12, we have that $u \uplus \{e_1\} \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$. Moreover, $\mathfrak{L}(e_1) \in \mathfrak{L}(u)$ and, by Propositions 7(3) and 10, we have ${}^{\circ}(u \uplus \{e_1\})^{\triangleleft} \subseteq \textsc{Reg}_c$ and ${}^{\lozenge}(u \uplus \{e_1\})^{\circ} \cap \textsc{Reg}_c = \varnothing$. We therefore obtained a contradiction with our assumptions.
Case 2: We can find $u \uplus \{e_1\} \uplus \{e_2\} \in \mathbb{U}_{\mathfrak{encl}}$ such that $\mathfrak{L}(e_2) \in \mathfrak{L}(u \uplus \{e_1\})$, ${}^{\bullet}e_2{}^{\blacktriangleleft} \subseteq c$ and ${}^{\blacklozenge}e_2{}^{\bullet} \cap c = \varnothing$. Then we consider Cases 1 and 2 again, taking $u \uplus \{e_1\} \uplus \{e_2\}$ instead of $u \uplus \{e_1\}$. Since the number of events in E is finite, we will eventually end up in Case 1. This means that, eventually, we will obtain a contradiction.

Proof of Axiom5. We need to show that if $c \xrightarrow{u}_{\mathfrak{encl}}$ then there is no $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ satisfying $\mathfrak{L}(e) \in \mathfrak{L}(u)$, ${}^{\circ}e^{\triangleleft} \subseteq \textsc{Reg}_c$ and ${}^{\lozenge}e^{\circ} \cap \textsc{Reg}_c = \varnothing$.
Suppose to the contrary that there is $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ as above (†).
We have already shown that for $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$, $b \in {}^{\bullet}e$ implies $\mathfrak{r}_b \in {}^{\circ}e$, and $b \in e^{\bullet}$ implies $\mathfrak{r}_b \in e^{\circ}$. From this and $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{ts}_{\mathfrak{encl}}}$ we have $u \uplus \{e\} \in \mathbb{U}_{\mathfrak{encl}}$.
We will show that ${}^{\bullet}e \subseteq c$. Consider $b \in {}^{\bullet}e$. We have that $b \in {}^{\bullet}e$ implies $\mathfrak{r}_b \in {}^{\circ}e$. But ${}^{\circ}e \subseteq \textsc{Reg}_c$, and so $\mathfrak{r}_b \in \textsc{Reg}_c$. This means that $c \in \|\mathfrak{r}_b\|$, and consequently $b \in c$. Hence ${}^{\bullet}e \subseteq c$.
We now show that $e^{\bullet} \cap c = \varnothing$. Consider $b \in e^{\bullet}$. We have that $b \in e^{\bullet}$ implies $\mathfrak{r}_b \in e^{\circ}$. But $e^{\circ} \cap \textsc{Reg}_c = \varnothing$, and so $\mathfrak{r}_b \notin \textsc{Reg}_c$. This means that $c \notin \|\mathfrak{r}_b\|$, and consequently, $b \notin c$. Hence $e^{\bullet} \cap c = \varnothing$.
Now we show that ${}^{\blacklozenge}e \cap c = \varnothing$. Suppose to the contrary that $b \in {}^{\blacklozenge}e \cap c \neq \varnothing$. We have already shown in the proof of Axiom4 that for $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$, $b \in {}^{\blacklozenge}e \cap c$ implies $\mathfrak{r}_b \in {}^{\lozenge}e$. But ${}^{\lozenge}e \cap \textsc{Reg}_c = \varnothing$, so $\mathfrak{r}_b \notin \textsc{Reg}_c$. This means $c \notin \|\mathfrak{r}_b\|$, and so $b \notin c$, a contradiction.
Finally, we show that ${}^{\blacktriangleleft}e \subseteq c$. Suppose to the contrary that there is $b \in {}^{\blacktriangleleft}e \setminus c$. We have already shown in the proof of Axiom4 that for $e \in \mathcal{E}_{\mathfrak{ts}_{\mathfrak{encl}}}$, $b \in {}^{\blacktriangleleft}e \setminus c$ implies $\mathfrak{r}_b \in {}^{\triangleleft}e$. But ${}^{\triangleleft}e \subseteq \textsc{Reg}_c$, so $\mathfrak{r}_b \in \textsc{Reg}_c$. This means that $c \in \|\mathfrak{r}_b\|$ and, consequently $b \in c$, a contradiction.
As a result, assuming that (†) holds leads to a contradiction with $c \xrightarrow{u}_{\mathfrak{encl}}$.