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A critical reliability evaluation of fibre reinforced composite materials based on probabilistic micro and macro-mechanical analysis

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Abstract

In probabilistic composite mechanics, uncertainty modelling may be introduced at a constituent (micro-scale), ply (meso-scale) or component (macro-scale) level. Each of these approaches has particular advantages/limitations and appropriate fusing and benchmarking is desirable in order to improve confidence in probabilistic performance estimates of composite structures. In the present study, random variable based micro and macro-scale reliability analyses are critically compared through a limit state formulation based on the analytical stress tensor components of a rectangular simply supported orthotropic FRP composite plate and the Tsai-Hill failure criterion. The study aims to promote cross-fertilisation of alternative uncertainty modelling approaches in a multi-scale analysis framework. Propagation of uncertainty from micro to macro-scale, and the corresponding influence of changes in random variability on the reliability estimates is quantified. The importance of benchmarking experimentally-based probability distributions of mechanical properties through micro-scale modelling is illustrated, and the confidence that can be placed on reliability estimates is quantified through a series of numerical examples.
1. Introduction

Composite materials are widely used in diverse engineering industries due to their unique characteristics. In many deterministic studies that have attempted to quantify the mechanical behaviour of composite materials, considerable differences are observed between theoretical predictions using micro-scale properties and experimental results at component level, e.g. [1]. This could be attributed to the complex processes and uncertainties involved in the manufacture, assembly and the associated quality control procedures. Significant uncertainty sources include [2-6]: variations in volume fractions of fibre and matrix, voids in the matrix and between the fibres and matrix, imperfect bonding between constituents, cracks, fibre damage, random and/or contiguously packed fibres; misaligned fibres, temperature effects, non-uniform curing of the matrix material, residual stresses etc. The uncertainty in these factors propagates to a larger scale and is reflected in the variability of stiffness and strength descriptors characterising overall structural performance. As a result, current deterministic approaches to structural design are associated with high safety factors.

Over the years, a range of stochastic analysis methods have been developed to account for the uncertainties at different scales. As highlighted in a recent review [6] researchers have modelled uncertainty starting at a micro-scale (Class A: constituent level, fibre/matrix), meso-scale (Class AB: ply level) or macro-scale (Class B: coupon/component level). In Class A, uncertainty is modelled either through a large number of random variables at the fibre/matrix level (Class A1), e.g. [3], or by
considering representative elements of composite microstructure, e.g. [7], typically obtained through an image processing technique and linked to a suitable micro-mechanical model (Class A2). In Class B studies, randomness in material and geometric properties is captured through experimental results at coupon level (with characteristic dimensions of tens or hundreds of millimetres) which feed into mechanics-based structural models, e.g. [2]. Meso-scale modelling (Class AB), e.g. [8], could be thought of as an alternative starting point in Class A1 methods or at the failure assessment stage in Class B methods.

Class A1 is popular when the corresponding probability models and associated computer codes are available whereas Class A2 is typically suitable when the design requirement aims to control a particular micro-structural damage mechanism (e.g. matrix cracking, local yielding etc.), as it links local constitutive modelling with the random microstructure. On the other hand, Class B is useful for investigating the global behaviour of composites such as displacements, average stresses, strains, etc. and/or when the performance requirements do not rely on intense micro-mechanical modelling. Class AB may be appropriate when the ply characteristics influence significantly the properties of the composite.

As a contribution to the cross-fertilisation between different approaches in a multi-scale stochastic framework, the present paper evaluates reliability estimates using probabilistic micro and macro-level analysis. The scope of this study is limited to the consideration of Class A1 and Class B modelling approaches, which are the most widely used in assessing the global response of composite structures.

2. Formulation of reliability limit states
In order to quantify uncertainty propagation effects, and to compare reliability estimates based on micro- and macro-mechanical material modelling, a suitable limit state function needs to be considered. Herein, it is formulated so that the alternative probabilistic approaches may be validated for a set of closed form stress tensor expressions.

2.1. Composite material failure criteria

The general problem of a plate or shell is considered, made from a material which can be characterised by a set of continuum elastic constitutive relations, and for which a number of standard failure criteria can be used to predict the onset of failure in response to some prescribed load. Lin [9] computed the failure probabilities of laminated composite plates, subjected to transverse loads using four different failure criteria, namely, the Tsai-Wu, Tsai-Hill, maximum stress and Hoffman criteria. Comparing with a range of experimental results, it was observed that the Tsai-Wu criterion yielded a failure load that was closest to the corresponding experimental value whereas Tsai-Hill was slightly less accurate. Both were significantly more accurate than the maximum stress and Hoffman criteria. The results of Lin [9] are consistent with the conclusions of Hinton et al. [10] and Daniel [11], in that the Tsai-Wu criterion has been shown to be, generally, more accurate than Tsai-Hill. However, a major limitation of the Tsai-Wu formulation lies in the difficulty of obtaining experimentally based values for the interaction coefficients. Moreover, major enhancements to existing micro-mechanics models [1, 12–15] would be required in order to be able to use Tsai-Wu in such an approach. In the present work, the Tsai-Hill failure criterion is used, bearing in mind the above comments regarding its accuracy.
In the case of a rectangular plate (Fig. 1), made of orthotropic material, the Tsai-Hill failure criterion takes the following form:

\[
\left( \frac{\sigma_{xx}}{X} \right)^2 + \left( \frac{\sigma_{yy}}{Y} \right)^2 + \left( \frac{\sigma_{zz}}{Z} \right)^2 - \left( \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) \sigma_{xx} \sigma_{yy} - \\
\left( \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \right) \sigma_{yy} \sigma_{zz} - \left( \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \right) \sigma_{zz} \sigma_{xx} + \\
\left( \frac{\sigma_{xy}}{S} \right)^2 + \left( \frac{\sigma_{xz}}{R} \right)^2 + \left( \frac{\sigma_{yz}}{T} \right)^2 \geq 1, \tag{1}
\]

\[X = \begin{cases} 
X_T, & \sigma_{xx} > 0, \\
X_C, & \sigma_{xx} < 0,
\end{cases} \quad 
Y = \begin{cases} 
Y_T, & \sigma_{yy} > 0, \\
Y_C, & \sigma_{yy} < 0,
\end{cases} \quad 
Z = \begin{cases} 
Z_T, & \sigma_{zz} > 0, \\
Z_C, & \sigma_{zz} < 0.
\end{cases}
\]

\(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\) are the components of the Cartesian coordinate form of the stress tensor. \(X_T, Y_T\) and \(Z_T\) are the tensile strength components and \(X_C, Y_C,\) and \(Z_C\) are the respective compressive strength components. \(S, R\) and \(T\) are the shear strength components in the \(xy, xz\) and \(yz\)-planes respectively.

### 2.2. Macro-mechanical limit state formulation

In order to test the probabilistic methodology independently of a FE code, benchmarks that rely on analytical solutions for the components of the stress tensor appearing in Eq. (1) are formulated. Reddy [16] has derived such solutions for a range of standard cases. Here, two loading cases are considered for a simply supported linear elastic orthotropic plate under:

(i) a uniformly distributed load (UDL) over the entire top surface and

(ii) a line load (LL) acting along the \(y=b/2\) centreline.

For these load cases, the components of the stress tensor are given by [16]:

\[
\sigma_{xx} = \frac{2\pi^2}{b^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q_{mn}}{d_{mn}} \left( s^2 m^2 Q_{x1} + n^2 Q_{x2} \right) \sin \left[ \frac{m\pi x}{a} \right] \sin \left[ \frac{n\pi y}{b} \right] \tag{2}
\]
\[ \sigma_{yy} = \frac{z\pi^2}{b^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q_{mn}}{d_{mn}} \left( s^2 m^2 Q_{12} + n^2 Q_{22} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \] (3)

\[ \sigma_{zz} = -\frac{1}{4} \left[ 1 + \left( \frac{2z}{h} \right)^2 \right] \sum_{n=1}^{\infty} q_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \] (4)

\[ \sigma_{xz} = \frac{3b}{2\pi h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \sum_{n=1}^{\infty} q_{mn} \frac{m^3 s^3 Q_{11} + mn^2 s (Q_{12} + 2Q_{66})}{s^4 m^4 Q_{11} + 2s^2 m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{22}} \] (5)

\[ \sigma_{yz} = \frac{3b}{2\pi h} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right] \sum_{n=1}^{\infty} q_{mn} \frac{n^3 Q_{22} + mn^2 s (Q_{12} + 2Q_{66})}{s^4 m^4 Q_{11} + 2s^2 m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{22}} \] (6)

\[ \sigma_{xy} = -\frac{2z\pi^2 G_{12}}{b^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{q_{mn} \sin mn}{d_{mn}} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \] (7)

where

\[ d_{mn} = \frac{\pi^4 h^3}{12b^4} \left[ s^4 m^4 Q_{11} + 2s^2 m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{22} \right] \] (8)

\[ s = \frac{b}{a}, \quad Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_{22}}{1-\nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12} \] (9)

and \( E_{11} \) and \( E_{22} \) are the Young’s moduli in the \( x \) and \( y \) directions respectively, \( G_{12} \) is the in-plane shear modulus and \( \nu_{12} \) and \( \nu_{21} \) are the in-plane Poisson ratios.

In the case of a uniformly loaded plate,

\[ q_{mn} = \frac{16q}{\pi^2 mn}, \quad (m,n = 1, 3, 5, \ldots), \quad \text{where} \quad q = q(x,y) = p_0 \] (10)

whereas, in the case of a line loaded plate,

\[ q_{mn} = \frac{8q}{\pi bm} \sin \left( \frac{n\pi}{2} \right), \quad (m = 1, 3, 5, \ldots), \quad (n = 1, 2, 3, \ldots) \] (11)

where \( q = q(x,y) = q_0 \delta(y - b/2) \). Here \( q_0 \) is a constant force per unit length term which is applied along the line \( y = b/2 \).
By substituting $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$, $\sigma_{xy}$, $\sigma_{xz}$ and $\sigma_{yz}$ from Eqs. (2)-(7) respectively, into Eq. (1), the Tsai-Hill criterion may be evaluated at any point within the plate volume.

Provided that the material is weaker in tension than in compression, the left-hand-side of Eq. (1) is found to be largest at the point located on the bottom middle of the plate (i.e. at $x = a/2$, $y = b/2$, $z = h/2$). Substituting these $x$, $y$ and $z$ co-ordinates into Eqs. (2)-(11) the shear stress components vanish,

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$  \hspace{1cm} (12)

and the other components of stress at that point are such that,

$$\sigma_{xx} \left( x = a/2, y = b/2, z = h/2 \right) > 0$$  
$$\sigma_{yy} \left( x = a/2, y = b/2, z = h/2 \right) > 0$$  
$$\sigma_{zz} \left( x = a/2, y = b/2, z = h/2 \right) > 0$$  \hspace{1cm} (13)

Assuming that for a uni-directional composite with the fibres parallel to the $x$ direction,

$$Y_T = Z_T, \quad Y_C = Z_C.$$  \hspace{1cm} (14)

on the basis that $Y$ and $Z$ represent matrix dominated strengths, and combining Eqs. (1), (12) and (13) the following limit state function (LSF) is obtained,

$$g = 1 - \left( \frac{\sigma_{yy} - \sigma_{zz}}{Y_T} \right)^2 + \frac{\sigma_{xx} \left( \sigma_{xx} + \sigma_{yy} - \sigma_{zz} \right) - \sigma_{yy} \sigma_{zz}}{X_T^2}$$  \hspace{1cm} (15)

For the purposes of a reliability analysis, if $g$ is less than or equal to zero then the plate is deemed to have failed.

This expression can be used to calculate the probability of failure for any specified distributions of all the random input variables. At a macro-level, the random input variables may include the components of composite stiffness and strength, as well as load and geometric parameters. Here, the load parameters, $p_0$ and $q_0$ defined in Eqs. (10) and (11) are treated as deterministic and only the plate thickness ($h$) is considered as a
geometric random variable. These simplifications do not compromise the objective of the work carried out here which focuses on material uncertainty by comparing reliability estimates starting from either micro or macro-mechanical versions of the limit state defined in Eq. (15).

Inspecting Eqs. (2)-(9) and (12)-(15), it can be seen that the macro-mechanical LSF of Eq. (15) does not involve the Poisson’s ratio component $\nu_{21}$, defined in Eq. (9). This is because the plate material is taken to be orthotropic (as required by Reddy’s theory [16]), and hence the following standard elasticity relationship can be applied,

$$\frac{\nu_{21}}{E_{22}} = \frac{\nu_{12}}{E_{11}}, \quad \text{i.e. } \nu_{21} = \nu_{21}(E_{11}, E_{22}, \nu_{12})$$

(16)

Thus, in summary, the macro-mechanical LSF has seven input random variables: the plate thickness ($h$), the four stiffness components ($E_{11}$, $E_{22}$, $\nu_{12}$ and $G_{12}$) characterising an orthotropic linear-elastic FRP material, and two independent strength components ($X_T$ which is the strength in the fibre direction and $Y_T$ which is the strength in a direction perpendicular to the fibres).

2.3. Micro-mechanical limit state formulation

In a micro-mechanics approach, the composite properties are expressed as functions of constituent material properties and assembly variables. A number of deterministic micromechanics studies [1, 10, 12-15] have quantified stiffness components for a variety of composite material types, achieving reasonable agreement with experiments. However, the quantification of composite strength using a micromechanics approach has proved to be more challenging. The elasticity approach of Huang [12], which is an extension of earlier theoretical work by Aboudi [17], is one of the relatively recent studies that has made extensive predictions over a range of strength components;
As stated previously, the LSF is applied at the bottom middle of the plate. It
therefore follows from Eqs. (1), (13) and (15) that tensile FRP strength components are
the only required properties in this case, and hence Huang’s model [12] will suffice.
This model is applicable to unidirectional fibre reinforced composites, which satisfy the
following conditions:

(i) the fibre material is transversely isotropic in an elastic region but becomes
isotropic in a plastic region, and

(ii) the matrix material is isotropically elastic-plastic

The resulting unidirectional composite is generally considered as a transversely
isotropic material, having three material principal axes which coincide with those of the
fibres. For this case, the equations derived by Huang [12] are summarised below:

\[ E_{11} = V_f E_{11}^f + V_m E_m \]  \hspace{1cm} (17)

\[ E_{22} = \frac{(V_f + V_m a_{11})(V_f + V_m a_{22})}{(V_f + V_m a_{11})(V_f S_{22}^f + a_{22} V_m S_{22}^m) + V_f V_m (S_{12}^m - S_{12}^f) a_{12}} \]  \hspace{1cm} (18)

\[ \nu_{12} = V_f \nu_{12}^f + V_m \nu_m \]  \hspace{1cm} (19)

\[ G_{12} = \frac{(G_{12}^f + G^m) + V_f (G_{12}^f - G^m)}{(G_{12}^f + G^m) - V_f (G_{12}^f - G^m)} \]  \hspace{1cm} (20)

where \( a_{11}, a_{22}, a_{12}, S_{11}^f, \ldots, S_{12}^m \) are functions of fibre and matrix properties (given in
[12]), \( E_{11}^f \) and \( E_{22}^f \) are the longitudinal and transverse fibre Young’s moduli, \( G_{12}^f \) and \( \nu_{12}^f \)
are the shear modulus and Poisson’s ratio of the fibre respectively, and \( E_m \) and \( \nu_m \) are
the corresponding properties of the matrix material. In addition to the constituent
material stiffness components, the assembly variables $V_f$ and $V_m$ (fibre and matrix volume fractions) need to be considered. Compared to Huang [12], who assumed a zero void volume fraction, a minor extension is made here by allowing the void volume fraction to be non-zero (as assumed by Chamis [18]) and hence,

\[ V_m + V_f + V_v = 1 \]  

(21)

where $V_v$ is the void volume fraction. The tensile strength components are given by,

\begin{align*}
X_T &= \min \left\{ \frac{\sigma_u - (\alpha_{\rho_1} - \alpha_{\rho_1}^f) \sigma_{11}^0}{\alpha_{\rho_1}^f}, \frac{\sigma_m - (\alpha_{\rho_2} - \alpha_{\rho_2}^m) \sigma_{11}^0}{\alpha_{\rho_2}^m} \right\} \\
Y_T &= \min \left\{ \frac{\sigma_u - (\alpha_{\rho_2} - \alpha_{\rho_2}^f) \sigma_{22}^0}{\alpha_{\rho_2}^f}, \frac{\sigma_m - (\alpha_{\rho_2} - \alpha_{\rho_2}^m) \sigma_{22}^0}{\alpha_{\rho_2}^m} \right\}
\end{align*}

(22)

(23)

where,

\[ \sigma_{ij}^0 = \min \left\{ \frac{\sigma_u^f}{\alpha_{\rho_1}^f}, \frac{\sigma_u^m}{\alpha_{\rho_2}^m} \right\}, \]

and $\sigma_u^f$ and $\sigma_u^m$ are the ultimate failure strengths of the fibre and matrix materials and the $\alpha$ coefficients, which are functions of fibre and matrix properties can be found in [12]. Further developments in micro-mechanical modelling would be required to produce predictive equations for compressive strength components that have comparable confidence levels, should the latter be needed as part of a limit state formulation.

A novel aspect of the present work is the extension of the deterministic micro-mechanics model of Huang [12] to a probabilistic framework. This is accomplished using a well accepted methodology [3, 18-22] which has been implemented in micro-mechanics models. Thus, the micro-mechanical LSF is derived by substituting $E_{11}, E_{22},$
$G_{12}, \nu_{12}, X_T$ and $Y_T$, from Eqs. (17) – (23) into Eq. (15). This increases the number of random input variables from 7 to 13, as shown below,

$$g = g(h, E_{11}(...), E_{22}(...), \nu_{12}(...), G_{12}(...), X_T(...), Y_T(...)) =$$

$$g\left(h, E_{11}^f, E_{22}^f, \nu_{12}^f, G_{12}^f, E^m, \nu^m, E_T^m, \sigma_y^m, \sigma_u^m, V_f, V_t\right).$$

$h$ is the thickness random variable; $E_{11}^f, E_{22}^f, \nu_{12}^f, G_{12}^f, E^m, \nu^m, E_T^m, \sigma_y^m, \sigma_u^m$ and $\sigma_u^m$ are constituent material properties and $V_f$ and $V_t$ are assembly variables, all treated as random.

3. Material statistics

3.1. Micro level material statistics

It is generally acknowledged [6, 17, 23] that obtaining the statistics for micro-scale modelling is a challenging task. This is because of the complexity of experiments, instrumentation constraints, the number of variables and the large samples of nominally identical tests required for statistical significance. Probability models for a Graphite-fibre/epoxy-matrix FRP material have been presented by NASA researchers [3, 18, 21, 23] and are shown in Table 1. These are being used in the present study to examine the reliability of a plate with dimensions $a=1.7m$, $b=1.5m$ and a normally distributed thickness $h$ (random variable $X_1$) with mean and a standard deviation of 4.7 mm and 0.5 mm respectively.

3.2. Macro level material statistics

As highlighted above, complexities and limitations associated with micro-mechanical modelling may limit the confidence in such an approach, thus pointing towards an alternative macro-mechanical approach, for which random variability needs to be
captured at a coupon/component level. Many studies reported in the literature present ‘empirical’ random variable models for coupon/component properties, combining experimental evidence with engineering judgment. Common probability laws, such as Normal (N), Log Normal (LN), Gamma (G), Weibull (W) or Extreme Value Type 1 Largest/Smallest (ET1L/ET1S) models have been proposed for various properties [e.g., 2, 9, 24, 25]. Generally, a set of nominally identical experiments are conducted and distribution selection is carried out using standard probability paper plots and hypothesis testing methods such as chi-square, Kolmogorov-Smirnov (K-S) or Anderson-Darling tests. The majority of the reported probability distribution models for composite material properties are in line with the guidelines of Military Handbook MIL-HDBK-17-1F [26]. Some researchers have used statistical analysis of test results in conjunction with engineering judgement, e.g. [27].

Alternatively, uncertainty can be propagated from micro to macro-scale. It is, thus, possible to derive the macro-level statistics by simulating micro-level variables based on Huang’s micro-mechanical model [12]. In the present study, probability distributions are derived for four stiffness components \(E_{11}, E_{22}, \nu_{12},\) and \(G_{12}\) and two strength components \(X_T\) and \(Y_T\). For each micro-level variable, \(10^5\) independent Monte Carlo simulations (MCS) are performed and the corresponding statistical properties of macro-level random variables are obtained (Table 2). It is of interest to observe the correlations that exist between these derived variables at macro-level (the significance of this correlation on reliability estimates is discussed in the following sections). This is not normally captured when different experiment types are performed at coupon level.

At this point, it is also meaningful to compare the statistical properties of derived macro-level random variables with experimentally based values for similar composites.
The values quoted by Jeong and Shenoi [2] and Lin [9] are presented in Table 2 alongside the derived parameters. As can be seen, the values for stiffness variables are in relatively good agreement, while the corresponding values of strength variables exhibit wider discrepancies. This is in line with deterministic comparisons between derived and experimental properties [1, 10, 12-15] but in the present study, these differences are quantified both in terms of the mean values and the associated dispersion. It is also worth noting the significantly higher dispersion observed in physical experiments for strength properties ($X_T$ and $Y_T$) compared to the corresponding values derived by propagating uncertainties from micro to macro-level.

The histograms of the four derived macro-scale material property random variables ($E_{11}$, $E_{22}$, $X_T$ and $Y_T$) are shown in Fig. 2 with best-fit probability densities for Weibull, normal and lognormal models superimposed. The distribution parameters for the best-fit distributions, along with the goodness-of-fit statistics, are given in Table 3. In addition to the Anderson-Darling hypothesis test (ADH), the traditionally used Chi-Square (CH) and Kolmogorov Smirnov (KSH) have also been performed. A zero indicates that the data are sampled from the corresponding distribution whereas unity indicates rejection of the assumed distribution. On the basis of these results, and following the guidelines of Military Handbook [26], it is appropriate to model $Y_T$ with a Weibull law and the rest of the variables with a normal law. Insofar as the plate thickness is concerned, it is assumed to have the same characteristics as in the micro-scale study to enable effective comparison. Thus, the lamina thicknesses are assumed to be perfectly positively correlated so that the effective statistical properties for the laminate remain same, i.e., $E[h_T]= n E[h]$ and $\text{Var}[h_T]= n^2 \text{Var}[h]$, where $n$ is the total number of laminae.

4. Reliability analysis
As discussed earlier, the reliability of composite structures could be estimated starting at either micro, meso or macro-scales. Thus, using an appropriate structural reliability method [28], such as Monte Carlo Simulation (MCS) or a first/second order reliability method (FORM/SORM), the probability of attaining a specified performance criterion is evaluated through the definition of a limit state function which distinguishes between acceptable and unacceptable performance. For the case study presented herein, starting from Eqs. (15) and (24), the reliability, $R$ may be defined as

$$R = P[g(.) > 0] = 1 - P[g(.) \leq 0] = 1 - P_f$$  \hspace{1cm} (25)

where $P_f$ is the corresponding probability of failure, i.e., $R$ and $P_f$ are complementary events.

4.1. **Estimation of reliability**

The failure probability for both uniformly distributed and line loaded (UDL/LL) cases is evaluated for a range of load parameter values ($p_0$ from Eq. (10) for UDL and $q_0$ from Eq. (11) for LL) using MCS and FORM/SORM. With respect to Eqs. (2) to (7), the Fourier series expressions for the components of the stress tensor were found to converge with, $m = n = 30$ in the case of UDL and, $m = n = 300$ in the case of LL; the same number of terms are used in the reliability computations. For the MCS, a sample size of $10^5$ was established in order to reduce the variance of the estimates to a tolerable level, which at a $10^{-3}$ level was taken to be a coefficient of variation of 10% on the failure probability estimate. FORM is undertaken using a standard Rackwitz-Fiessler algorithm [29], whereas SORM is based on the improved Breitung (Hohenbichler / Rackwitz) algorithm with curvature fitting [30].

4.2 **Comparison of micro and macro mechanical based estimates**
The probability of failure is evaluated starting from the micro-level variables given in Table 1 or the derived macro level statistics (including appropriate correlations) given in Table 2. Results for both UDL and LL cases are shown in Fig. 3. On first glance, the probability estimates appear to be practically identical for both load cases and at all load levels. However, this is quantified through the set of lines included in the UDL figure (right hand y-axis), which shows that for low failure probabilities there are indeed some differences between the estimates, depending on the starting point (micro vs. macro) and the reliability method used (MCS vs. SORM). In the context of computed reliability, these differences are small, since, as can be seen, the probabilities are within a factor of two at the $10^{-3}$ level. The overall good agreement between the two set of results is significant for complex or implicit limit state functions where it would be possible to carry out the probabilistic analysis in stages (Stage 1: Simulation and probabilistic characterisation of stiffness and strength parameters; Stage 2: reliability analysis using derived macro-level property distributions). It was also found that, for the load cases and limit states examined herein, the difference between FORM and SORM estimates is very small (results for both FORM and SORM are only shown for the LL case but identical trends were found for the UDL case).

Finally, the significance of statistical independence and distribution fitting were explored and the results are shown in Fig. 4. Insofar as independence is concerned, the effect is within a factor of 2 at a failure probability level of $10^{-3}$. Treating all variables as normally distributed (instead of modelling $Y_T$ as a Weibull variable) has practically no effect, except for the very low end of the failure probability range considered.

4.3 Comparison of derived and experimentally based estimates
As highlighted in Table 2, experimental tests can be used to obtain directly macro-level statistics; in this section, the failure probability based on such models is compared with the estimates presented in the preceding section. Fig. 5 shows the experimentally-based probability distributions for $E_{11}$, $E_{22}$, $X_T$ and $Y_T$ based on the studies by Jeong and Shenoi [2] and Lin [9] on one hand, and the derived distributions starting from micro-level statistics on the other. It is worth noting that all the experimentally-based distributions were taken to be normal whereas for the derived distributions this is the case for $E_{11}$, $E_{22}$, $X_T$ but $Y_T$ is Weibull distributed. Substantial differences are evident, particularly with respect to strength distributions, as might have been anticipated based on the results presented in Table 2. Of course, what is important is to quantify the effect of such differences on the estimated reliabilities, which is depicted in Fig. 6 for both UDL and LL load cases.

In both cases, the estimates associated with Lin’s statistics [9] are very different from the estimates based on micro/macro modelling. Those based on the Jeong and Shenoi statistics [2] come in between the two for the UDL case and are very close to the micro/macro estimates for the LL case. Specifically, for the UDL case, there is a factor of 50 difference in estimated failure probabilities at low levels ($10^{-4}$) between micro/macro and Lin [9]; this factor is less than 10 when comparing between micro/macro and Jeong and Shenoi [2]. The latter is perhaps within the acceptable variations for notional failure probability estimates, given the physical and statistical uncertainties involved in these variables. Overall, the reliability estimates based on micro-mechanics based probabilistic modelling (or macro-based on derived distributions) tend to be lower than those associated with coupon/component based modelling. This might be attributed to the wider range of factors that can influence the
dispersion observed in the latter; such factors can be real (e.g. misalignment, cracking, imperfect bonding) but could also be related to the repeatability and robustness associated with physical tests. In this respect, it would be useful to consider and quantify modelling and measuring uncertainties, which play an important role in both micro-mechanics based and coupon/component based probabilistic modelling.

5. Conclusions

In order to critically compare the probabilistic performance estimates based on micro and macro-scale modelling, closed form stress tensor components of a simply supported orthotropic rectangular FRP composite plate are considered. Both macro and micro-level limit states are formulated based on the Tsai-Hill failure criterion. For the micro-scale case, Huang’s model is cast in a probabilistic framework, and macro-material statistics are derived using Monte Carlo simulation. These derived macro-level statistics are compared with experimentally derived probabilistic models for the same composite material type. Notwithstanding the limitations of available experimental data, it is noted that probabilistic models for stiffness properties are in close agreement but, in contrast, the corresponding statistics for strength components vary substantially. Moreover, the statistical dependence that is observed in micro/macro derived properties is typically ignored when experimentally-based models are used.

Failure probabilities are evaluated for two load cases of a laterally loaded composite plate using a range of reliability methods, such as MCS, FORM and SORM. In general, failure probabilities for a micro-level analysis and a macro-level analysis based on derived statistics are found to be in very good agreement. It is, thus, possible to use derived macro-level models in reliability analysis, which could facilitate the development of a transparent multi-scale modelling framework to be established and
validated at appropriate intermediate milestones. The effect of commonly used assumptions pertaining to statistical dependence and normality are examined for the particular case study, though the observations made should not be generalised.

A further comparative study is conducted using derived micro/macro and experimentally based macro-level probability distributions. It is found that variations in the statistics of macro-level stiffness and, particularly, strength properties may result in significant differences in failure probability estimates. In this respect, it would be helpful if probabilistic modelling parameters are presented together with appropriately estimated confidence levels. Although experimental validation of probability models for micro-level parameters remains a challenging task, a reliability analysis starting at micro-scale level offers significant insight into the propagation of uncertainty, and can help benchmark corresponding macro-level analyses which are often based on scarce experimental data to derive ‘empirical’ distributions for macro-level stiffness and strength properties. On the other hand, the characterisation and propagation of uncertainty sources starting at micro-level would benefit from further research, particularly in the light of modern testing capabilities and emerging NDE technologies.

Acknowledgements

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References


Fig. 1. Geometric set-up for a simply supported, orthotropic plate

Fig. 2. Probability densities of derived macro-scale random variables ($E_{11}, E_{22}, X_T$ and $Y_T$)
Fig. 3. Micro and macro-scale probabilistic failure estimates for UDL and LL cases
Fig. 4. Micro and macro-scale probabilistic failure estimates for UDL and LL cases.
Fig. 5. Comparison of derived and experimental stiffness and strength distributions
Fig. 6. Comparison of failure probabilities for derived and experimentally-based property distributions.
Table 1. Random variability of micro-mechanical LSF input variables

<table>
<thead>
<tr>
<th>Random variable</th>
<th>Probability distribution model</th>
<th>Mean</th>
<th>Standard deviation</th>
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<tbody>
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<td>$X_3 = E_{22}^f$</td>
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$^\S$ only deterministic value available, random variability assumed.
Table 2. Random variability of macro-mechanical LSF input variables

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* Correlation is significant at the 0.01 level (2-tailed).

Table 3. Goodness of fits for derived macro-level probability models

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<th>Random variable</th>
<th>2-parameter Weibull distribution</th>
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* Slower loading rate.