Small-scale dynamo action in rotating compressible convection

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We study dynamo action in a convective layer of electrically-conducting, compressible fluid, rotating about the vertical axis. At the upper and lower bounding surfaces, perfectly-conducting boundary conditions are adopted for the magnetic field. Two different levels of thermal stratification are considered. If the magnetic diffusivity is sufficiently small, the convection acts as a small-scale dynamo. Using a definition for the magnetic Reynolds number $R_M$ that is based upon the horizontal integral scale and the horizontally-averaged velocity at the mid-layer of the domain, we find that rotation tends to reduce the critical value of $R_M$ above which dynamo action is observed. Increasing the level of thermal stratification within the layer does not significantly alter the critical value of $R_M$ in the rotating calculations, but it does lead to a reduction in this critical value in the non-rotating cases. At the highest computationally-accessible values of the magnetic Reynolds number, the saturation levels of the dynamo are similar in all cases, with the mean magnetic energy density somewhere between 4 and 9% of the mean kinetic energy density. To gain further insights into the differences between rotating and non-rotating convection, we quantify the stretching properties of each flow by measuring Lyapunov exponents. Away from the boundaries, the rate of stretching due to the flow is much less dependent upon depth in the rotating cases than it is in the corresponding non-rotating calculations. It is also shown that the effects of rotation significantly reduce the magnetic energy dissipation in the lower part of the layer. We also investigate certain aspects of the saturation mechanism of the dynamo.

1. Introduction

In a hydromagnetic dynamo, motions in an electrically-conducting fluid lead to the amplification of a seed magnetic field. Dynamo action can only occur if the inductive effects of the fluid motions outweigh the dissipative effects of magnetic diffusion. Provided that the magnetic energy density of the seed field is very much less than the kinetic energy density of the flow, the early stages of evolution of the dynamo process are effectively kinematic, which implies that the magnetic energy in the system grows exponentially with time (although the magnetic energy will tend to fluctuate about this exponential trajectory if the velocity field is time-dependent). Eventually, however, the nonlinear feedback of the magnetic field upon the flow (via the Lorentz force) becomes dynamically significant. This halts the growth of the dynamo-generated magnetic field, leading to a saturated nonlinear dynamo. There are many examples of natural dynamos. For example, it is generally accepted that the 11 year solar magnetic cycle (see, for example, Stix 2004), is driven by an oscillatory dynamo. Similar dynamo mechanisms drive cyclic magnetic activity on other stars (Baliunas et al. 1996). Dynamo action is also responsible for

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sustaining the Earth’s magnetic field (see, for example, Roberts & Glatzmaier 2000). It is also possible to drive hydromagnetic dynamos in laboratories, as demonstrated by recent liquid metal experiments (such as the French VKS experiment, see for example, Monchaux et al. 2009).

Convection plays a crucial role in many natural dynamos, particularly in the context of solar, stellar and planetary dynamos. Although it is possible to study global dynamo models numerically (Brun et al. 2004), most of the theoretical work on this problem has been based upon local models of convection, where a fluid layer is heated from below and cooled from above. In the context of Boussinesq convection, this classical setup has been studied numerically by Meneguzzi & Pouquet (1989) and Cattaneo (1999). In highly-conducting fluids, Boussinesq convection can act as an efficient dynamo, producing small-scale, intermittent magnetic fields. In the Boussinesq approximation, the effects of compressibility are neglected. More recent studies have focused upon dynamo action in local models of convection in fully compressible fluids (Vögler & Schüssler 2007; Brummell et al. 2010; Bushby et al. 2010). It is now well-established that these compressible models can also drive small-scale dynamos.

Previous calculations have clearly demonstrated that rotation is not a necessary ingredient for small-scale dynamo action in convectively-driven flows. However, rotation is present in most natural dynamos, and this additional physical effect can profoundly influence the behaviour of a convective layer. For example, when the rotation vector is aligned with the vertical axis, rapid rotation not only inhibits convection, but it also reduces the preferred horizontal scale of the convective instability (Chandrasekhar 1961). Even far from onset, when compressible convection is fully turbulent, rotation can also play an important role: helical convective plumes tend to become aligned with the rotation vector. This is particularly apparent when the axis of rotation is tilted away from the vertical (Brummell et al. 1996, 1998b). Given these hydrodynamical considerations, we might reasonably expect the dynamo properties of rotating convection to differ from the equivalent non-rotating case. There have been numerous studies of this problem in Boussinesq convection (Childress & Soward 1972; St. Pierre 1993; Jones & Roberts 2000; Stellmach & Hansen 2004; Cattaneo & Hughes 2006), but the compressible case is less well understood. Existing studies have generally adopted complex models, with multiple polytropic layers, inclined rotation vectors, or additional physical effects such as an imposed velocity shear (see, for example, Brandenburg et al. 1996; Käpylä et al. 2008, 2009). In fact, the simpler problem of dynamo action in a single layer of turbulent compressible convection, rotating about the vertical axis, is still not fully understood. Therefore, the aim of this paper is to study the ways in which rotation and compressibility influence the dynamo properties of convection in a simple polytropic layer.

The paper is organised as follows. The governing equations, boundary conditions and parameters, together with the numerical methods are discussed in the next section. Considerations about the dimensionless numbers of interest, and the Lyapunov exponents of hydrodynamic convection are presented in Section 3. In Section 4, we discuss results from a set of dynamo calculations. Finally, in Section 5, we present our conclusions.

2. Model and method

2.1. Model and governing equations

We consider the evolution of a plane-parallel layer of compressible fluid, bounded above and below by two impenetrable, stress-free walls, a distance $d$ apart. The upper and lower boundaries are held at fixed temperatures, $T_0$ and $T_0 + \Delta T$ respectively. Taking $\Delta T > 0$
implies that this layer is heated from below. The geometry of this layer is defined by a Cartesian grid, with \( x \) and \( y \) corresponding to the horizontal coordinates. The \( z \)-axis points vertically downwards, parallel to the constant gravitational acceleration \( g = ge_z \).

The layer is rotating about the \( z \)-axis, with a constant angular velocity \( \Omega = \Omega \hat{z} \). The horizontal size of the fluid domain is defined by the aspect ratio \( \lambda \) so that the fluid occupies the domain \( 0 < z < d \) and \( 0 < x, y < \lambda d \). The physical properties of the fluid, namely the specific heats \( c_p \) and \( c_v \), the dynamic viscosity \( \mu \), the thermal conductivity \( K \), the magnetic permeability \( \mu_0 \) and the magnetic diffusivity \( \eta \), are assumed to be constant.

The model is identical to that used by Matthews et al. (1995), except for the addition of rotation.

It is convenient to introduce dimensionless variables, so we adopt the scalings described in Bushby et al. (2008). Lengths are scaled with the depth of the layer \( d \). The temperature \( T \) and the density \( \rho \) are scaled with their values at the upper surface, \( T_0 \) and \( \rho_0 \) respectively. The velocity \( u \) is scaled with the isothermal sound speed \( \sqrt{R^* T_0} \) at the top of the layer, where \( R^* = c_p - c_v \) is the gas constant. We adopt the same scaling for the Alfvén speed, which implies that the magnetic field \( B \) is scaled with \( \sqrt{\mu_0 \rho_0 R^* T_0} \). Finally, we scale time by an acoustic time scale \( d/\sqrt{R^* T_0} \).

We now express the governing equations in terms of these dimensionless variables. The equation for conservation of mass is given by

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) .
\]  

(2.1)

Similarly, the dimensionless momentum equation can be written in the following form,

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \kappa \sigma T a_0^{1/2} \hat{z} \times \mathbf{u} \right) = -\nabla \left( P + \frac{B^2}{2} \right) + \theta (m + 1) \rho \hat{z} + \nabla \cdot (\mathbf{B} \mathbf{u} \rho \kappa \mathbf{u} + \kappa \sigma \mathbf{T}) ,
\]

(2.2)

where \( P \) is the pressure, given by the equation of state \( P = \rho T \), and \( \mathbf{T} \) is the stress tensor defined by

\[
\tau_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} .
\]  

(2.3)

Several non-dimensional parameters appear in Equation (2.2). The parameter \( \theta = \Delta T / T_0 \) is the dimensionless temperature difference across the layer, whilst \( m = gd/R_c \Delta T - 1 \) corresponds to the polytropic index. The dimensionless thermal diffusivity is given by \( \kappa = K/d\rho_0 c_p(R_c T_0)^{1/2} \) and \( \sigma = \mu c_p / K \) is the Prandtl number. Finally, \( T a_0 \) is the standard Taylor number, \( T a_0 = 4\rho_0^2 \Omega^2 d^4/\mu^2 \), evaluated at the upper boundary. The induction equation for the magnetic field is

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \zeta_0 \kappa \nabla \times \mathbf{B}) ,
\]

(2.4)

where \( \zeta_0 = \eta c_p \rho_0 / K \) is the ratio of magnetic to thermal diffusivity at the top of the layer. The magnetic field is solenoidal so that

\[
\nabla \cdot \mathbf{B} = 0 .
\]  

(2.5)

Finally, the heat equation is

\[
\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T - (\gamma - 1) T \nabla \cdot \mathbf{u} + \frac{\kappa \gamma}{\rho} \nabla^2 T + \frac{\kappa (\gamma - 1)}{\rho} \left( \sigma T^2 / 2 + \zeta_0 |\nabla \times \mathbf{B}|^2 \right) ,
\]

(2.6)

where \( \gamma = c_p / c_v \).

To complete the specification of the model, some boundary conditions must be imposed.
In the horizontal directions, all variables are assumed to be periodic. As has already been described, the upper and lower boundaries are assumed to be impermeable and stress-free, which implies that $u_{x,z} = u_{y,z} = u_z = 0$ at $z = 0$ (the upper boundary) and $z = 1$ (the lower boundary). Having non-dimensionalised the system, the thermal boundary conditions at these surfaces correspond to fixing $T = 1$ at $z = 0$ and $T = 1 + \theta$ at $z = 1$.

For the magnetic field boundary conditions, two choices have typically been made in previous studies. One can consider the upper and lower boundaries to be perfect electrical conductors or one can adopt a vertical field boundary condition. We choose appropriate conditions for perfectly-conducting boundaries, which implies that $B_z = B_{x,z} = B_{y,z} = 0$ at $z = 0$ and $z = 1$. This is partly to facilitate comparison with the Boussinesq study of Cattaneo & Hughes (2006), but this is not the only motivation for adopting boundary conditions that enforce $B_z = 0$ at the upper surface. Strong concentrations of vertical magnetic flux at the upper surface tend to become partially evacuated (see, for example, Bushby et al. 2008), which dramatically increases the local Alfvén speed (as well as significantly reducing the local thermal diffusion time scale). This can impose very strong constraints upon the time-step in any explicit numerical scheme. Hence, there are also numerical reasons for adopting perfectly-conducting boundary conditions. In this context, it is worth noting that it has been shown in the Boussinesq case (Thelen & Cattaneo 2000) that the dynamo efficiency of convection is largely insensitive to the detailed choice of boundary conditions for the magnetic field (see also the compressible calculations of Käpylä et al. 2010). Although not reported here, we have carried out a small number of simulations with vertical field boundary conditions, and the results are qualitatively similar in that case.

### 2.2. Model parameters and initial conditions

With the given boundary conditions, it is straightforward to show that these governing equations have a simple equilibrium solution, corresponding to a hydrostatic, polytropic layer:

$$T = 1 + \theta z, \quad \rho = (1 + \theta z)^m, \quad \mathbf{u} = \mathbf{0}, \quad \mathbf{B} = \mathbf{0}. \quad (2.7)$$

This polytropic layer (coupled with a small thermal perturbation) is an appropriate initial condition for simulations of hydrodynamic convection. All the hydrodynamic flows that are considered in this paper are obtained by evolving the governing equations from this basic initial condition. Once a statistically-steady state has been reached, the dynamo properties of these flows can be investigated by inserting a weak (seed) magnetic field into the domain.

There are many non-dimensional parameters in this system. Since it is not viable to complete a systematic survey of the whole of parameter space, we only vary a subset of the available parameters. Throughout this paper, the polytropic index is fixed at $m = 1$, whilst the ratio of specific heats is given by $\gamma = 5/3$ (the appropriate value for a monatomic gas). These choices ensure that the initial polytropic layer is unstable to convective perturbations. We also fix the Prandtl number to be $\sigma = 1$. In order to study the effects of varying the stratification, we adopt two different values of $\theta$. The case of $\theta = 3$ corresponds to a weakly-stratified layer, whilst $\theta = 10$ represents a highly-stratified case in which the temperature and density both vary by an order of magnitude across the layer.

The main aim of this study is to address the effects of rotation and compressibility upon dynamo action in a convective layer. So, for each value of $\theta$, we consider a rotating and a non-rotating case (which implies four different cases overall). Note that some care is needed when comparing simulations of rotating convection at different values of $\theta$. 

The Taylor number that appears in the governing equations, $Ta_0$, corresponds to the Taylor number at the top of the domain. Given the differing levels of stratification, it makes more sense to specify the same mid-layer Taylor number for each of the rotating cases. Since the Taylor number is proportional to $\rho^2$, the mid-layer Taylor number (in the unperturbed polytrope) is defined by $Ta = Ta_0(1 + \theta/2)^2m$. Two different values of $Ta$ are considered, $Ta = 0$ (the non-rotating cases) and $Ta = 10^5$. Further discussion regarding the depth-dependence of the Taylor number is given in the next section.

Another parameter that needs to be specified is the dimensionless thermal diffusivity, $\kappa$. However, rather than prescribing values for $\kappa$, we define the mid-layer Rayleigh number

$$Ra = (m + 1 - m\gamma)(1 + \theta/2)^{2m-1} \frac{(m + 1)\theta^2}{\kappa^2\gamma\sigma}, \quad (2.8)$$

which is inversely proportional to $\kappa^2$ and measures the destabilising effect of the superadiabatic temperature gradient relative to the stabilising effect of (viscous and thermal) diffusive processes. As described in the Introduction, rotation tends to stabilise convection, whilst larger values of $\theta$ also have a stabilising effect (Gough et al. 1976). Therefore it does not make sense to keep $Ra$ constant in all cases. Instead, we vary the Rayleigh number from one calculation to the other, ranging from $Ra = 3 \times 10^5$ up to $Ra = 6 \times 10^5$.

The aim was to reach similar values of the Reynolds number in all cases. We shall discuss another possible definition for the Reynolds number in the next section, but for now we define a global Reynolds number, based upon the rms velocity ($U_{rms}$), the kinematic viscosity at the mid-layer ($\kappa\sigma/\rho_{mid}$, where $\rho_{mid}$ is the mean density at the mid-layer of the domain) and the depth of the layer (which equals unity in these dimensionless units). Hence this global Reynolds number is given by

$$Re = \frac{\rho_{mid}U_{rms}}{\kappa\sigma}. \quad (2.9)$$

The choices for $Ra$ that we have used imply that $Re$ is approximately 150 for each of the four cases. A summary of our choice of parameters for each case is given in Table 1.

### Table 1.

<table>
<thead>
<tr>
<th>Run</th>
<th>$Ra$</th>
<th>$Ta$</th>
<th>$\theta$</th>
<th>$\kappa$</th>
<th>$Re$</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
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<td>3</td>
<td>0.0055</td>
<td>153</td>
</tr>
<tr>
<td>R2</td>
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<td>$10^5$</td>
<td>3</td>
<td>0.0044</td>
<td>157</td>
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<tr>
<td>R3</td>
<td>$5 \times 10^5$</td>
<td>0</td>
<td>10</td>
<td>0.0242</td>
<td>149</td>
</tr>
<tr>
<td>R4</td>
<td>$6 \times 10^5$</td>
<td>$10^5$</td>
<td>10</td>
<td>0.02</td>
<td>152</td>
</tr>
</tbody>
</table>

All the parameters that have been discussed so far relate to the hydrodynamic properties of the convection. If all other non-dimensional parameters in the system are fixed, the early evolution of any weak magnetic field in the domain depends solely upon value of $\zeta_0$, which is a parameter that can be set as the field is introduced. This parameter is proportional to the magnetic diffusivity, $\eta$, so we require low values of $\zeta_0$ for dynamo action. Equivalently, we could say that dynamo action is only expected in the high magnetic Reynolds number regime. As for the Reynolds number, we shall discuss an alternative definition for the magnetic Reynolds number in Section 3.1. However, for the moment,
we make an analogous global definition:

\[ R_M = \frac{U_{\text{rms}}}{\kappa \zeta_0}. \]  

(2.10)

We choose a range of values for \( \zeta_0 \) which give values of \( R_M \) that vary between approximately 30 and 480 \((0.12 \leq \zeta_0 \leq 2.0 \text{ for } \theta = 3 \text{ and } 0.05 \leq \zeta_0 \leq 0.8 \text{ for } \theta = 10)\). This range of values is restricted by numerical constraints: smaller values of \( \zeta_0 \) (higher magnetic Reynolds numbers) would require a higher level of numerical resolution, which would greatly increase the computational expense.

### 2.3. Numerical method

Solving the equations of compressible convection in the presence of a magnetic field is more demanding numerically than equivalent Boussinesq calculations, so it is important to use a well optimised code. The given set of equations is solved using a modified version of the mixed pseudo-spectral/finite difference code originally described by Matthews et al. (1995). Due to periodicity in the horizontal direction, horizontal derivatives are computed in Fourier space using fast Fourier transforms. In the vertical direction, a fourth-order finite differences scheme is used, using an upwind stencil for the advective terms. The time-stepping is performed by an explicit third-order Adams-Bashforth technique, with a variable time-step. For all calculations presented here, the aspect ratio is \( \lambda = 4 \). The resolution is 256 grid-points in each horizontal directions and 120 grid-points in the vertical direction. A poloidal-toroidal decomposition is used for the magnetic field in order to ensure that the field remains solenoidal. As explained later, we also aim to calculate Lagrangian statistics. Trajectories of fluid particles are therefore computed using the following equation:

\[ \frac{\partial \mathbf{x}_p}{\partial t} = \mathbf{u}(\mathbf{x}_p, t), \]  

(2.11)

where \( \mathbf{x}_p \) is the position of the particle. The velocity at the particle position is interpolated from the grid values using a sixth-order Lagrangian interpolation scheme. The boundaries are treated with a decentred scheme and we carefully check that all the particles remain in the fluid domain. The time-stepping used to solve Equation (2.11) is the same as for the other evolution equations in the system.

### 3. Hydrodynamical considerations

As we have already described, fully-developed hydrodynamic convection is used as a starting point for all dynamo calculations. In this section, we consider the properties of the hydrodynamical flows that are obtained in each of four cases that are listed in Table 1.

#### 3.1. Dimensionless numbers

In a stratified layer, most quantities of interest will be a function of depth. This is true not only for the thermodynamic quantities in the flow, but also for some of the parameters in the system. In the previous section, we defined the Taylor number and the Reynolds number using mid-layer values for the density, in addition to using the layer depth for the characteristic length scale. These are certainly valid definitions for these quantities, but further insight can be gained by considering the depth-dependence of these parameters, particularly when comparing calculations with different levels of stratification. This is a point that we consider in detail in this subsection.

For each value of \( z \), it is possible to define local dimensionless numbers by considering
horizontally-averaged quantities. Under such circumstances, it is more sensible to define these parameters in terms of a horizontal length-scale rather than using the depth of the layer (which equals unity in this dimensionless system). We choose here the horizontal integral length scale, defined by

$$l_0(z) = \frac{1}{\langle u_i(x,t)u_i(x,t) \rangle_z} \int_0^\lambda \langle u_i(x,t)u_i(x+re_x,t) \rangle_z \, dr$$ \hspace{1cm} (3.12)

(where we have assumed the flow is horizontally-isotropic). Here, and in the following, the brackets $<.>_z$ mean a statistical average over horizontal coordinates and time at a given depth $z$, whereas the brackets $<.>$ (with no subscript) mean a statistical average over time and all spatial coordinates. We note that this horizontal scale (unlike the vertical dimension of the domain) will vary not only with depth, but also from one flow to another.

Figure 1 shows the variation with depth of both the integral scale, $l_0(z)$, and the local rms velocity, $U_{\text{rms}}(z) = <|u(x,t)|^2>^{1/2}$, for the four cases. Each of these horizontally-averaged quantities is averaged over more than 100 acoustic crossing times. In all four cases, the trends are similar. Near the surface, the integral scale $l_0(z)$ decreases with depth as the flow becomes more turbulent, and therefore less correlated. As the flow reaches the lower part of the layer, the integral length scale increases again, presumably due to the presence of boundaries. Comparing the two non-rotating cases, one observes that the effect of increasing $\theta$ is to decrease the integral scale of the flow. However, whatever the value of $\theta$ is, the length scales are very similar in the two rotating cases, and are always smaller than the corresponding integral scales in the non-rotating calculations. This mirrors the result from linear theory that rotation tends to decrease the size of the convective cells (Chandrasekhar 1961). The rms velocity is comparatively independent of depth, except close to the lower boundary where (like the integral scale) it increases.
Figure 2. Dimensionless numbers versus depth. From left to right: Reynolds number $Re(z)$, magnetic Reynolds number $R_M(z)$ (the global value of $R_M$ is 30 in each of the four cases shown here), Rossby number $Ro(z)$ and Taylor number $Ta(z)$.

Note that $U_{\text{rms}}(z)$ is larger in both the highly-stratified cases. At fixed stratification, the rms velocity is smaller in the rotating cases.

We can now use these velocity and length scales to define local dimensionless parameters for this system. Unlike their corresponding global values (based upon the depth of the layer and the depth-averaged rms velocity), these are always functions of $z$. We now define the local Reynolds number to be

$$Re(z) = \frac{\langle \rho(x, t) \rangle_z U_{\text{rms}}(z) \ell_0(z)}{\kappa \sigma}.$$  

(3.13)

Similarly, we can define the local Rossby number to be

$$Ro(z) = \frac{U_{\text{rms}}(z)}{2\Omega \ell_0(z)},$$

(3.14)

whilst

$$Ta(z) = 4 \frac{\langle \rho(x, t) \rangle_z^2 \Omega^2 \ell_0(z)}{\kappa^2 \sigma^2},$$

(3.15)

gives a local definition for the Taylor number. Although we are focusing upon the hydrodynamic properties of convection in this section, this is also a convenient place to define the local magnetic Reynolds number,

$$R_M(z) = \frac{U_{\text{rms}}(z) \ell_0(z)}{\kappa \zeta_0}.$$  

(3.16)

In the dynamo calculations in the next section, we shall often refer to the mid-layer value of the local magnetic Reynolds number, i.e. $R_M(0.5)$.

The depth-dependence of the local Reynolds number, the local magnetic Reynolds number (for four cases corresponding to a global value of $R_M = 30$), the local Rossby number and the local Taylor number is shown in Figure 2. First of all, it is clear that the local Reynolds number is an increasing function of depth, which is a clear indication that the flow is becoming more turbulent as $z$ increases. This variation with depth is partly
due to the fact that the dynamic viscosity $\mu$ is held constant in all our simulations. An important observation is that the local Reynolds number (and the local magnetic Reynolds number) is always smaller in the rotating cases. This is despite the fact that the global Reynolds numbers and magnetic Reynolds numbers (as defined in the previous subsection) are the same in all cases. This reduction in the local Reynolds number is due to the fact that rotation leads to motions with a smaller characteristic horizontal length scale (as well as a lower rms velocity). So, if these local definitions for the Reynolds numbers give a clearer indication of the level of turbulence in the flow, the rotating calculations are actually slightly less turbulent than suggested by the values of the global Reynolds numbers. We shall return to this point in Section 4. Returning to Figure 2, we see that the Rossby number is roughly constant across the layer, but is smaller in the high $\theta$ case. Although we could conclude that this implies that the effects of the Coriolis force with respect to inertia are more important in the highly stratified case, the key point is that $Ro(z)$ is always much less than unity, which implies that the flows are both rotationally dominated. Finally, the local Taylor number is largely independent of depth ($\theta$) in the middle of the domain, but increases near the boundaries. Note that this behaviour is very different from the global Taylor number, which actually increases with depth. This apparent discrepancy is due to the fact that the local Taylor number is strongly influenced by variations in the integral scale (varying as $l_0^4(z)$), which tend to outweigh any local increases in the mean density. The Taylor number (in particular) highlights some of the difficulties that must be faced when defining appropriate dimensionless numbers in highly-stratified fluids.

3.2. Lyapunov exponents

From the point of view of dynamo action, it is interesting to quantify the amount of stretching in the flow. This can be achieved by measuring the corresponding Lyapunov exponents. Several previous studies have explored the relationship between dynamo action and Lyapunov exponents (see, for example, Finn & Ott 1988; Brandenburg et al. 1995; Brummell et al. 1998a; Tanner & Hughes 2003), and we would expect regions with large Lyapunov exponents to be playing a dominant role in the dynamo process.

To estimate the short-time Lyapunov exponent, we release $5 \times 10^4$ fluid particles randomly within the convective layer. For each particle, we also release a second particle at a distance $d_0 = 10^{-4}$ apart from the first particle, in a random direction. This pair of particles is then followed as it moves through the fluid. The short-time Lyapunov exponent $\lambda_c$ (Eckhardt & Yao 1993) is then calculated using the following expression:

$$\lambda_c = \frac{1}{t} \log \frac{d(t)}{d_0},$$

(3.17)

where $d(t)$ is the distance between the two particles at time $t$. The Lyapunov exponent is computed after a fixed time (approximately equal to one crossing time) and the distance between the particles is then renormalised to $d_0$. The results appear to be insensitive to the choice of the initial separation distance $d_0$. Furthermore, they do not appear to depend upon the details of the method that is used to reinitialise the particle positions. All initial separations will quickly align with the expanding manifold, so we would expect to see very little dependence upon the initial condition for sufficiently large time, $t$.

To study the depth dependence of the stretching within the flow, we consider the horizontal average of the largest Lyapunov exponent at each value of $z$. In addition, we time-average the resulting Lyapunov exponent over approximately 20 convective turnover times. Figure 3(a) shows the mean maximum Lyapunov exponent versus depth for each of the four cases. In all cases, the stretching increases with depth (as the flow becomes
more and more turbulent), with the Lyapunov exponents taking their maximum values at the bottom of the convective domain. Unsurprisingly, this is reminiscent of the depth-dependence of the local Reynolds number. This depth-dependence has important implications for the dynamo problem: we would expect the magnetic field to be mostly generated in the lower part of the layer, where the rate of stretching is maximal. Compared to the equivalent non-rotating cases, in rotating convection we see higher values of the Lyapunov exponent at the top of the layer and lower values at the bottom of the layer. In other words, the maximum Lyapunov exponent is less depth-dependent in the rotating cases than it is in the non-rotating calculations. From a dynamo perspective, this suggests that we should expect the magnetic fields to be more homogeneously distributed across the layer in the rotating cases than they are in the non-rotating simulations. Figure 3(b) shows a plot in which the Lyapunov exponents are scaled by the local convective turnover time $l_0/U_{\text{rms}}(z)$. We use a similar scaling for the dynamo growth rates in the next section, where the implications of this plot will be discussed in more detail. Here we simply note two key features of this scaled plot. Firstly, this scaling causes the four curves to collapse onto a single curve in the upper part of the layer. Secondly, we note
that the scaled Lyapunov exponents suggest that there is less stretching in the lower part of the domain in the rotating cases.

On the right-hand side of Figure 3, we show a snapshot of the temperature, enstrophy and the short time Lyapunov exponent (STLE) in a horizontal plane at two different depths ($z = 0.2$ and $z = 0.8$) for both non-rotating (top) and rotating (bottom) convection in the weakly-stratified ($\theta = 3$) case. In the temperature plot, brighter contours correspond to warmer regions of fluid. As expected from our considerations of the horizontal integral scale, it is clear that the horizontal scale of convection is smaller in the rotating case than it is in non-rotating convection. This reduction in horizontal scale is also apparent in the enstrophy plot where bright regions, corresponding to areas of high (squared) vorticity, outline the edges of the convective cells. The STLE map is obtained by releasing $10^5$ particle pairs at a given depth. The particles are followed for approximately one crossing time. The STLE is then computed using Equation (3.17) and plotted at the initial particle pair position. Comparing the STLE map with the enstrophy distribution, it is clear that there is a correlation between zones of strong stretching and regions of high vorticity. Given this correlation, it is natural to conclude that the higher (unscaled) Lyapunov exponent in the upper part of the rotating simulations, as observed in Figure 3(a), is mostly due to the larger number of convective cells that are present. The scaled plot that is shown in Figure 3(b) tends to support this conclusion: the scaling that has been used here takes into account this “filling factor” effect, which explains why all the scaled Lyapunov exponent curves collapse onto a single curve near the surface in that case.

4. **Dynamo simulations**

In this section, we investigate the dynamo properties of the four convective flows that are being considered in this paper. Each dynamo calculation is initialised by introducing a seed magnetic field into statistically-steady hydrodynamic convection. The initial spatial structure of this magnetic field is given by $B(x, y, z) = (A \cos(k_1 y), A \cos(k_1 x), 0)$, where $k_1 = 2\pi/\lambda$ and $A$ is the peak amplitude of the initial perturbation. This is almost the simplest possible magnetic field configuration that is consistent with the imposed boundary conditions, whilst also ensuring that there is no net (horizontal or vertical) magnetic flux across the computational domain. Test calculations suggest that the evolution of the dynamo is largely insensitive to the precise choice of initial conditions.

4.1. **The kinematic regime**

Initially, we consider the kinematic dynamo regime in which the seed magnetic field is assumed to be weak. This implies that the magnetic field tends to grow (or decay) exponentially at a rate that is determined by the value of magnetic Reynolds number. Given that the actual growth of the magnetic energy fluctuates in time, long time-averages are often needed in order to accurately measure growth rates. This kinematic phase of evolution can be indefinitely prolonged by switching off the Lorentz force terms in the momentum equation and the ohmic heating terms in the temperature equation. This is the procedure that is adopted in this subsection. For each velocity field, we carry out 6 kinematic calculations at different values of $\zeta_0$. This parameter is chosen so that the global magnetic Reynolds number ranges from approximately 30 to 480 in each case. This implies that the magnetic Prandtl number (the ratio of the global magnetic Reynolds number to the global Reynolds number) varies from approximately 0.2 to 3.2.

The main aim of these kinematic calculations is to determine the ways in which rotation and stratification influence the rate of growth (or decay) of the seed magnetic field. The
Figure 4. Growth rates of the magnetic energy versus the magnetic Reynolds number for $\theta = 3$ (left) and $\theta = 10$ (right). In the upper plots, the global magnetic Reynolds number has been used. In the lower plots, the same growth rates have been plotted against the mid-layer value of the local magnetic Reynolds number, $R_M(0.5)$. The rotating results are plotted with triangles whereas the non-rotating results correspond to the circles. The growth rates are normalised by the mid-layer turnover-time $l_0(0.5)/U_{\text{rms}}(0.5)$. The error bars correspond to the maximum and minimum value for the growth rate when considering different time intervals. The logarithmic and square-root scalings are plotted for reference.

The upper part of Figure 4 shows two plots (for $\theta = 3$ and $\theta = 10$) of the kinematic growth rate of the magnetic energy, $\lambda$, versus the global magnetic Reynolds number. In all cases, $\lambda$ has been scaled by the mid-layer turnover time of the turbulence, $l_0(0.5)/U_{\text{rms}}(0.5)$. This rescaling has been carried out to facilitate comparison between the four cases, as well as the Lyapunov exponent calculations from the previous section. Results from this kinematic study highlight several key points. Firstly, it is clear that the critical value (for the onset of dynamo action) of the global magnetic Reynolds number is rather similar in all four cases. It is only in the non-rotating $\theta = 3$ case that there is a suggestion of a slightly higher critical global magnetic Reynolds number (although the error bars in the growth rate close to criticality are large enough to suggest that this difference may
not be significant). Adopting this definition for the magnetic Reynolds number, we see that the scaled growth rates in the highly stratified case seem to be rather insensitive to the presence of rotation. It is only in the weakly stratified case, at small values of the global magnetic Reynolds number, that significant differences are seen between the rotating and the non-rotating cases. So it is tempting to conclude from these results that the dynamo properties of compressible convection (particularly at higher values of the global magnetic Reynolds number) are largely insensitive to the presence of rotation, as well as variations in the level of stratification.

However, some caution is needed when interpreting these results. As we discussed in the previous section, a global definition for the magnetic Reynolds number takes no account of the horizontal scale of convection (a quantity that varies significantly between the four cases that are being considered). So we would argue that it is more sensible to consider representative values of the local magnetic Reynolds number for the purposes of this comparison. The lower part of Figure 4 also shows the growth rate curves for the four different cases, but this time $\lambda$ has been plotted against the mid-layer value of the local magnetic Reynolds number, $R_M(0.5)$. It is now possible to discern clear differences between the four sets of calculations. We first discuss the effect of varying $\theta$ without rotation. For $\theta = 3$, the critical value for $R_M(0.5)$ is about 420, whereas for $\theta = 10$, the critical value for $R_M(0.5)$ is about 290. Therefore, an increase in the level of stratification reduces the critical value of the local magnetic Reynolds number. Let us now consider the two rotating cases. A striking feature of the plots in the lower part of Figure 4 is how similar the growth rate curves of the two rotating cases are. For both values of $\theta$, the critical value of $R_M(0.5)$ is about 220. Therefore, independently of the level of stratification, rotation tends to reduce the critical value for the local magnetic Reynolds number.

Whatever definition is adopted for the magnetic Reynolds number, it is clear that the growth rate curves exhibit certain characteristic features. The dotted lines on Figure 4, which are shown on these plots for indication, correspond to a logarithmic scaling (see, for example, Rogachevskii & Kleeroin 1997) and a square-root scaling (Schekochihin et al. 2004) of the growth rate with the magnetic Reynolds number. A logarithmic scaling seems to be valid (at least for all magnetic Reynolds numbers considered here) for the non-rotating $\theta = 3$ case. This situation is less clear in the highly-stratified non-rotating case, although an $R_M^{1/2}$ scaling may be more appropriate here. The other possibility is that there are two different regimes in this case, with a logarithmic scaling holding only for larger values of the magnetic Reynolds number. The scalings are again not completely convincing in the rotating cases, but the data seem to be compatible with an $R_M^{1/2}$ scaling, irrespective of the value of $\theta$. In both plots in the lower part of Figure 4 it is clear that the rotating cases always have higher growth rates than the equivalent non-rotating calculations. The largest difference in growth rates between the rotating and non-rotating calculations can be seen at low magnetic Reynolds number in the $\theta = 3$ case.

The observed variations in the growth rate curves suggest that rotation is beneficial for dynamo action. However, as discussed in the previous section, it could be argued that the rotating calculations are (in some sense) less turbulent that their non-rotating counterparts: although the global Reynolds numbers are similar in all cases, the local Reynolds number is significantly smaller in the rotating calculations. So we have to consider the possibility that it is actually the difference in the local Reynolds numbers that is giving the impression of enhanced dynamo action in the rotating cases. In order to address this issue, we performed an additional set of kinematic simulations for a new
Results from an additional set of non-rotating simulations at lower local Reynolds number. (a) The local Reynolds numbers versus depth for simulations $R_1$, $R_{1b}$ and $R_2$. (b) Growth rates of the magnetic energy versus the mid-layer value of the local magnetic Reynolds number, $R_M(0.5)$, for all the $\theta = 3$ simulations. The rotating results, from simulation $R_2$, are plotted with triangles whereas the non-rotating results correspond to the circles (filled for $R_1$, and empty for $R_2$). In all cases, the growth rates are normalised by the mid-layer turnover-time $t_0(0.5)/U_{rms}(0.5)$.

Before concluding this section on kinematic growth rates, it is worth commenting on the magnitudes of the growth rates in the dynamo regime. Obviously the range of magnetic Reynolds numbers is restricted by numerical constraints. However, it is worth noting that our positive growth rates (when properly normalised) are comparable to the values reported by Käpylä et al. (2008), who also considered simulations of compressible convection (albeit with overshoot and shear), and those found in the forced homogeneous turbulence calculations of Haugen et al. (2004). It is also of interest to compare the peak kinematic growth rates to the scaled Lyapunov exponents that are shown in Figure 3(b). In all cases, it is clear that the growth rates are significantly smaller than the corre-
Figure 6. The time evolution of the normalised mean kinetic and mean magnetic energy densities for (a) \( \theta = 3 \) and (b) \( \theta = 10 \). Both quantities are normalised by the mean kinetic energy density during the saturated phase (i.e. from \( t = 200 \) to \( t = 500 \) in the \( \theta = 3 \) cases, and from \( t = 100 \) to \( t = 250 \) in the \( \theta = 10 \) cases). The magnetic energy has been multiplied by 5.

Corresponding Lyapunov exponents. This is unsurprising given that the magnetic Reynolds numbers are comparatively modest in these numerical simulations, and we would expect to see higher growth rates at higher values of \( R_M \). Nevertheless our results would appear to be consistent with the findings of Brandenburg et al. (1995) who, for a related dynamo calculation, found that the Lyapunov exponents were systematically larger than the observed kinematic growth rates.

4.2. Magnetic fields in the nonlinear regime

We now consider the fully nonlinear set of governing equations (Equations (2.1) to (2.6)) including the back-reaction of the magnetic field on the velocity field. For each of the four cases under consideration, we choose the values of \( \zeta_0 \) corresponding to the largest values of the magnetic Reynolds number, in order to maximise the growth rate of the dynamo, thus minimising the duration of the kinematic phase. In the following, the time \( t = 0 \) corresponds to the insertion time of the small magnetic perturbation. The initial ratio between the total magnetic energy in the seed field and the total kinetic energy in the flow is roughly \( 10^{-3} \) in all cases. This implies that the initial field is weak enough to be kinematic, without unnecessarily extending the kinematic phase in these nonlinear calculations. Each nonlinear calculation has been evolved over a significant fraction of the ohmic decay time (based upon the magnetic diffusivity and the horizontal integral scale).

Figure 6 shows the evolution of the mean kinetic and mean magnetic energy densities for the four cases. All of these dynamos are highly time-dependent, but there are clear patterns of behaviour. During the early stages of evolution, the magnetic perturbation grows. However, the kinematic phase of the dynamo is extremely brief in these calculations, as the seed field rapidly becomes dynamically significant. After the short kinematic phase, the dynamo undergoes a more extended period of nonlinear growth, eventually settling down to a time-dependent saturated state. Conservation of energy implies that
the kinetic energy decreases during the nonlinear phase of the dynamo, reaching a final level that is clearly lower than the kinetic energy of the initial hydrodynamic state. Across all simulations, the mean magnetic energy density, $\langle |B|^2 \rangle /2$, saturates at a level that is somewhere between 4 and 9% of the mean kinetic energy density, $\langle \rho |u|^2 \rangle /2$. Although not shown here, a corresponding nonlinear calculation for the $R1b$ case (described in the previous section) saturates at a similar level. These nonlinear results suggest that the saturation level of the dynamo is comparatively insensitive to the level of stratification within the domain. There is weak evidence to suggest that the rotating cases are saturating at a slightly higher level (particularly for $\theta = 3$), despite the fact that the mid-layer values of the local magnetic Reynolds number are actually smaller in the rotating calculations than they are in the corresponding non-rotating cases. If we were comparing nonlinear calculations at similar values of the (mid-layer) local magnetic Reynolds number, as opposed to similar values of the global $R_M$, we would expect the rotating cases to saturate at a higher level. However, we did not investigate this parameter regime here given that this would require higher spatial resolution in the rotating cases.

Looking again at Figure 6, it is worth noting that the time-dependence of the magnetic energy in the non-rotating $\theta = 3$ case is strongly intermittent. This is clearly seen at $t \approx 270$, where a burst of magnetic energy corresponds to a drop in the kinetic energy. These fluctuations in the magnetic energy make it more difficult to determine the level at which the dynamo saturates in this case. As discussed in the previous subsection, this non-rotating $\theta = 3$ case may have a slightly higher critical global magnetic Reynolds number than the other three cases. If the dynamo is indeed closer to criticality than the other cases, we would expect it to be more sensitive to time-dependent variations in the flow. This would explain the observed behaviour.

A more detailed description of the magnetic field in the saturated phase can be obtained by considering the horizontal and vertical components of the magnetic energy density. When comparing different cases, it is useful to normalise the magnetic energy densities by the local equipartition energy defined by $B_{eq}(z) = \langle \rho |u|^2 \rangle_z$. The normalised horizontal magnetic energy density $\langle B_x^2 \rangle_z$ and the normalised vertical magnetic energy density $\langle B_y^2 \rangle_z$ are shown in Figures 7(a) and (b). The unnormalised horizontal magnetic energy density is shown in Figure 7(c), whilst the depth dependence of the equipartition energy $B_{eq}(z)$ is shown in Figure 7(d). In the middle of the layer, the horizontal magnetic energy density is comparable to the vertical magnetic energy density, whereas it clearly dominates close to the boundaries, as expected from the chosen boundary conditions. It is clear that the (unnormalised) horizontal magnetic energy density is stronger at the lower boundary than it is at the upper boundary, although these are more comparable when they are scaled in terms of the equipartition field strength. Note that all of these magnetic energy densities are sub-equipartition. Although there are some quantitative differences in these curves, the variation with depth of each of these quantities is broadly similar in most cases. The main difference to note is that the vertical component of the magnetic energy density tends to be larger in the rotating cases.

Figure 8 shows some examples of contours of the horizontal component of the magnetic field, $B_x$, during the saturated phase. In all cases, $\theta = 3$, and two different depths are considered ($z = 0.2$ and $z = 0.8$). The upper row corresponds to the non-rotating case, $R1$, whereas the lower row corresponds to the rotating case, $R2$. It is useful to compare this figure with the corresponding temperature and enstrophy plots in Figure 3. Near the upper boundary (i.e. at $z = 0.2$), there is a clear correlation between the convective downflows and the strongest magnetic fields. The characteristic scale of the magnetic field is smaller in the rotating case and one observes that the helical structures of the
fluid motion (see Figure 3) affect the magnetic field. Close to the lower boundary (i.e. at \( z = 0.8 \)), both cases are similar even if the characteristic scale of the magnetic structures is still smaller in the rotating case. Note also that the same colour table is used for both upper and lower layers. The magnetic field is therefore stronger and more intermittent near the lower boundary in both cases, as already illustrated by Figure 7(c).

The magnetic field geometry plays a crucial role in determining the rate at which magnetic energy is dissipated via ohmic diffusion. This is clearly an important process to consider when comparing the efficiency of convectively-driven dynamos. In this model, the rate of magnetic energy dissipation (per unit volume) due to ohmic diffusion is given by \( \zeta_0 \kappa |\mathbf{j}|^2 \), where \( \mathbf{j} = \nabla \times \mathbf{B} \) is the current density. This expression for the ohmic dissipation can be derived by taking the scalar product of the induction equation (2.4) with \( \mathbf{B} \), and then integrating over the domain. Note that the rate of dissipation is also proportional to the ohmic heating term in Equation (2.6). The rate of ohmic dissipation in the nonlinear phase, averaged over time and horizontal coordinates, is plotted in Figure 9(a) for \( \theta = 3 \).
and Figure 9(b) for $\theta = 10$. The dissipation is much larger in the $\theta = 10$ case, since both the kinetic and the magnetic energy densities are greater by roughly one order of magnitude. For each value of $\theta$, the magnitude of the dissipation term near the top of the layer is roughly the same, whether or not the layer is rotating. On the other hand, rotation clearly reduces the dissipation of magnetic energy near the lower boundary. Put another way, the dissipation term is more weakly dependent upon depth in the rotating cases than it is in the corresponding non-rotating calculations. A similar reduction of the magnetic dissipation by rotation has already been observed in rotating homogeneous magnetohydrodynamic turbulence by Favier et al. (2011). It is also worth noting that this reduction in magnetic dissipation is not a nonlinear effect: it can also be observed in the kinematic phase (although it is more difficult to quantify such a reduction since it becomes necessary to normalise the dissipation term by an exponentially growing field in order to take an appropriate time-average). The regions of strongest ohmic dissipation in the non-rotating cases coincide with the regions of strongest shear. We have already seen, in Figures 3(a) and 3(b), that the stretching is more homogeneously distributed across the layer in the rotating calculations. As a result, the dynamo-generated magnetic field organises itself in such a way that the rate of dissipation in these cases is lower than it is in the equivalent non-rotating calculations. So even though the scaled Lyapunov
Figure 9. The horizontally-averaged ohmic dissipation, $\zeta_0 \langle \dot{j}^2 \rangle_z$, for (a) $\theta = 3$ and (b) $\theta = 10$.

exponents, as shown in Figure 3(b), suggest that there is generally less stretching in the rotating cases, this reduction in dissipation explains why rotating convection can still act as an efficient dynamo (at least at moderate magnetic Reynolds numbers).

4.3. Saturation mechanisms

In the previous subsection, we discussed nonlinear results, without really saying anything about the saturation mechanism for these dynamo calculations. Due to the complexity of these dynamo models, it is difficult to say anything definitive about this. However, some insights into the saturation process can be gained by comparing certain aspects of the kinematic and nonlinear phases of the dynamo.

Firstly, we consider some of the global properties of the dynamo. We define the magnetic energy spectrum in the following way:

$$E_M(k_{\perp}) = \frac{1}{2} \sum_z \sum_{k_{\perp}} \hat{B}(k_x, k_y, z) \cdot \hat{B}^*(k_x, k_y, z)$$

(4.18)

where $k_{\perp}^2 = k_x^2 + k_y^2$ is the horizontal wave number, $\hat{B}(k_x, k_y, z)$ is the two-dimensional Fourier transform of $B(x, y, z)$ and the star denotes the complex conjugation. Similarly, the kinetic energy spectrum is defined as follows:

$$E_K(k_{\perp}) = \frac{1}{4} \sum_z \sum_{k_{\perp}} \hat{u}(k_x, k_y, z) \cdot \hat{\rho u}^*(k_x, k_y, z) + \hat{\rho u}^*(k_x, k_y, z) \cdot \hat{\rho u}(k_x, k_y, z).$$

(4.19)

In Figure 10, we show the energy spectra corresponding to the $\theta = 3$ calculations ($R1$ and $R2$), in both the kinematic and nonlinear regimes. The kinetic energy spectrum is time-averaged, whereas the magnetic energy spectrum is normalised by the time-average of the magnetic energy for the saturated dynamo before also being time-averaged. The results are qualitatively similar for the highly-stratified case. Comparing the kinetic energy spectra, we see that there is less kinetic energy at large scales ($i.e.$ for $1 < k_{\perp} < 3$) in the rotating case. This is connected to the rotationally-induced reduction in the horizontal scale of motion. Whether or not rotation is present, the magnetic energy spectra
always peak at small scales, which is consistent with the fact that we do not observe a large-scale dynamo in these simulations. Comparing the kinematic and nonlinear phases, we see that the kinetic energy spectra are only weakly perturbed by the magnetic fields. So there is no evidence here to suggest that saturation is accompanied by a significant variation in the kinetic energy spectrum. More interestingly, there is a small (but perhaps significant) alteration of the magnetic energy spectra as the dynamo saturates. There is a clear reduction of magnetic energy at small scales and a corresponding increase at large scales. This is observed in both the rotating and the non-rotating cases. Although not shown here, the same trend is observed in the θ = 10 calculations. We note that this is consistent with the increase of the Taylor microscale of the magnetic field reported by Brandenburg et al. (1996).

When considering potential saturation mechanisms, one of the most interesting things to consider is whether or not the stretching properties of the flow are modified by the magnetic fields. Local variations in the stretching would probably not lead to significant variations in the kinetic energy spectrum. However, we would expect to see changes in the Lyapunov exponents if the stretching properties of the flow are modified in the nonlinear dynamo regime. Using the flows from our nonlinear calculations, we evaluate the Lyapunov exponents using exactly the same procedure as for the kinematic phase (as described in Section 3.2). The particle pairs are followed from \( t = 300 \) to \( t = 500 \) in the \( \theta = 3 \) cases, and from \( t = 150 \) to \( t = 250 \) in the \( \theta = 10 \) cases. The (horizontally-averaged) maximum Lyapunov exponents for the four cases are shown in Figure 11 by the solid lines. The dotted lines show the corresponding Lyapunov exponents from the kinematic calculations. In the non-rotating cases, regardless of the value of \( \theta \), the Lyapunov exponents are slightly lower in the nonlinear phase everywhere except near the upper and lower boundaries. In the rotating cases, the Lyapunov exponent is slightly reduced almost everywhere, even close to the lower boundary. So there is some indication of a
suppression of stretching due the presence of magnetic fields. Furthermore, given the intermittent nature of the magnetic field distribution, we might expect the local reduction in stretching (in regions of strong magnetic fields) to be greater than that suggested by these horizontally-averaged quantities. However, this does not mean that the saturation of the dynamo can be explained simply by a reduction in stretching. Cattaneo & Tobias (2009) have shown that a dynamo-saturated velocity field in Boussinesq convection can still act as a kinematic dynamo (provided that the new seed field is not aligned with the original magnetic field). Although this has not been tested here, the depth-dependence of the Lyapunov exponents in the nonlinear regime is qualitatively similar to that of the original hydrodynamic flows, so we would speculate that these dynamos would exhibit similar behaviour. Certainly we do not see the drastic reduction in the Lyapunov exponents that Cattaneo et al. (1996) observed in their model of nonlinear dynamo action in a much simpler one-scale flow. The saturation process in convectively-driven dynamos appears to be more subtle than this.

We also consider the possibility that the correlation between $\mathbf{u}$ and $\mathbf{B}$ plays some role in the saturation process. The alignment between these two vectors, and the influence that this has upon some of the key nonlinearities in the dynamo system, has been extensively studied in recent years (see, for example, Servidio et al. 2008). Figure 12(a) shows the probability density function of the cosine of the angle between $\mathbf{u}$ and $\mathbf{B}$:

$$\cos(\mathbf{u}, \mathbf{B}) = \frac{\mathbf{u} \cdot \mathbf{B}}{|\mathbf{u}||\mathbf{B}|}.$$  \hspace{1cm} (4.20)

The pdfs have been obtained by averaging over space (between $z = 0.2$ and $z = 0.8$ only, so as to focus upon the interior of the domain) and time (using around 50 snapshots in each case). Note that it is not necessary to carry out any normalisation to compare pdfs in the kinematic and nonlinear regimes. Given that we would expect any preferential alignment to be rather localised, we separate regions of strong magnetic fields from weaker field regions. This is achieved by plotting two pdfs, one for those mesh points for which
Figure 12. Alignment properties for the simulation R2 (the results are qualitatively similar for our other simulations). (a) Probability density function of the cosine of the angle between \(\mathbf{u}\) and \(\mathbf{B}\). We separate contributions from points where \(|\mathbf{B}| > 3B_{\text{rms}}\) (solid lines) and \(|\mathbf{B}| < 3B_{\text{rms}}\) (dotted lines). (b) Same as (a) but for the angle between \(\mathbf{B}\) and \(\mathbf{e}_3\), where \(\mathbf{e}_3\) is the eigenvector associated to the largest eigenvalue of the rate of strain tensor.

\(|\mathbf{B}| > 3B_{\text{rms}}\), where \(B_{\text{rms}}\) is the rms magnetic field, the other for those mesh points that fall below this threshold field strength. Despite the fact that this flow is compressible, inhomogeneous and anisotropic, we see remarkable alignment between \(\mathbf{u}\) and \(\mathbf{B}\). Perhaps unsurprisingly, the alignment is always slightly more pronounced for strong magnetic fields than for the weaker fields. However, the strong alignment between the two fields is observed in both the kinematic and the nonlinear phases, and is therefore a property of the induction equation. Hence the saturation of these convectively-driven dynamos cannot be explained by a modified alignment between \(\mathbf{u}\) and \(\mathbf{B}\).

Following Brandenburg et al. (1996), we also consider the alignment between \(\mathbf{B}\) and the eigenvectors of the rate of strain tensor \(\mathbf{S}\). The symmetric matrix \(S_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})\) is computed at each mesh point and the three corresponding eigenvalues are ordered from the smallest to the largest. The smallest eigenvalue is always negative and its eigenvector corresponds to the direction of compression. The magnetic field is found to be mostly perpendicular to this direction (as already reported by Brandenburg et al. 1996), in both the kinematic and the nonlinear phases. The largest eigenvalue is always positive and its eigenvector, denoted here by \(\mathbf{e}_3\), corresponds to the direction of maximum stretching. The probability density function of \(\cos(\mathbf{B}, \mathbf{e}_3)\), is shown in Figure 12(b). Note that we again separate the contributions from the weak field and strong field regions. During the kinematic phase, the strongest magnetic fields are mostly aligned with the direction of maximum stretching. Note that there is also some indication of preferential alignment for the weaker field regions, but this is much less pronounced. In the saturated phase, the pdf corresponding to the weak field regions shows a very modest reduction in alignment. However, there is a dramatic reduction in the alignment between \(\mathbf{B}\) and \(\mathbf{e}_3\) in the strong field regions. Note that the magnetic fields for which \(|\mathbf{B}| > 3B_{\text{rms}}\) only represent 2% of the total number of points, so that the loss of alignment between the direction of maximum stretching and the magnetic field is only observed very locally. This modification of the alignment between \(\mathbf{B}\) and \(\mathbf{e}_3\) is observed in all our simulations, with or without rotation,
and for both thermal stratifications. This indicates that this process is rather robust. Given that this modified alignment will reduce the efficiency of the dynamo in regions of strong magnetic fields, we conclude that this effect plays a role in the saturation of convectively-driven dynamos.

5. Conclusions

In this paper, we have investigated small-scale dynamo action in rotating compressible convection. Regardless of the level of stratification within the domain, both rotating and non-rotating convective flows can sustain a small-scale dynamo if the magnetic diffusivity is small enough. Using a mid-layer value for the magnetic Reynolds number (based upon the integral scale of the turbulence rather than the layer depth), rotation seems to reduce the value of the magnetic Reynolds number above which dynamo action is observed. Increasing the thermal stratification also reduces the critical value of the local magnetic Reynolds number in the non-rotating case. At high values of the magnetic Reynolds number, the growth rate of the magnetic energy of the dynamo appears to have a logarithmic dependence upon $R_M$ in the weakly-stratified, non-rotating simulation. It is more difficult to fit a scaling law in the other cases, but an $R_M^{1/3}$ scaling may be more appropriate here. At a given value of the mid-layer (local) magnetic Reynolds number, the kinematic growth rates are always larger in rotating convection than they are in the corresponding non-rotating cases. This dependence upon rotation cannot be attributed solely to the fact that the local Reynolds number is smaller in the rotating cases. So we conclude that the Coriolis force plays a key role in determining the kinematic dynamo properties of rotating convection. At the highest computationally accessible values of the magnetic Reynolds number, we find similar levels of saturation in all of our nonlinear calculations (with the magnetic energy saturating at about 4 – 9% of the final kinetic energy). At first sight, this result is slightly surprising given that the Lyapunov exponents suggest that there is less stretching in the rotating cases (particularly near the lower boundary). However, this is compensated by the fact that magnetic dissipation seems to be much less efficient in the rotating calculations. It is difficult to say anything definitive about the saturation mechanism for these convectively-driven dynamos. However, a comparison between the kinematic dynamo regimes and the nonlinear saturated states show a slight reduction in the Lyapunov exponents in the nonlinear regime (due to a local reduction of the stretching properties of the flow). Furthermore, there is some evidence to suggest that a reduction in alignment between the strongest magnetic fields and the direction of maximum stretching also plays a role in the saturation process.

The next natural step of this study is to consider the ability of rotating compressible convection to produce a magnetic field on much larger scales than the scales that are associated with the convective motions. The question of how hydromagnetic dynamos are able to generate large-scale magnetic fields is undoubtedly one of the most challenging issues in modern dynamo theory. In the standard formulation of mean-field dynamo theory (Moffatt 1978), this regenerative process relies upon the presence of helical motions, which are invariably produced when a flow is influenced by rotation. Hence, rotating convection should be able to drive a large-scale dynamo. However, despite predictions from mean-field dynamo theory, Cattaneo & Hughes (2006) failed to find evidence for a large-scale dynamo in their Boussinesq model. Interestingly, Käpylä et al. (2009) did find large-scale magnetic fields in their compressible model. They argue that the absence of large-scale fields in the Boussinesq model of Cattaneo and Hughes can be attributed to a rotation rate that is too slow. This is a possibility, however the effects of compressibility may also be playing a role. The flow is (locally) strongly helical in the Boussinesq
calculations of Cattaneo & Hughes (2006), but the mid-layer symmetry of their setup implies that the mean helicity distribution is antisymmetric about the mid-plane. It may be important to break this symmetry in order to generate large-scale fields. In work that is currently in progress, we are investigating this issue by carrying out simulations of rapidly-rotating convection in a wide compressible layer. The size of the layer is important, because it is necessary to have a clear separation in scales between the small-scale fields and any large scale magnetic fields that may be generated. The computational domain that was considered in the present study was too small to allow for this separation in scales, which may explain why no large scale fields were observed here (despite the fact that the helicity of the flow is asymmetric about the mid-plane in these stratified rotating calculations). Of course, other physical ingredients could also be included into this system once the basic ingredients of rotation and stratification are properly understood. For example, the role of the penetrative layer in the model of Käpylä et al. (2009) is also unclear. This may be promoting the formation of large-scale fields in some way. It may also be of interest to consider the effect of including a shear flow (see, for example, Hughes & Proctor 2009) in this compressible model.

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