

ON THE K-THEORY OF BOUNDARY C^* -ALGEBRAS OF \tilde{A}_2 GROUPS

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ABSTRACT. Let Γ be an \tilde{A}_2 subgroup of $\mathrm{PGL}_3(\mathbb{K})$, where \mathbb{K} is a local field with residue field of order q . The module of coinvariants $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})_{\Gamma}$ is shown to be finite, where $\mathbb{P}_{\mathbb{K}}^2$ is the projective plane over \mathbb{K} . If the group Γ is of Tits type and if $q \not\equiv 1 \pmod{3}$ then the exact value of the order of the class $[\mathbf{1}]_{K_0}$ in the K-theory of the (full) crossed product C^* -algebra $C(\Omega) \rtimes \Gamma$ is determined, where Ω is the Furstenberg boundary of $\mathrm{PGL}_3(\mathbb{K})$. For groups of Tits type, this verifies a conjecture of G. Robertson and T. Steger.

1. INTRODUCTION

Let \mathbb{K} be a nonarchimedean local field with residue field k of order q and uniformizer π . Denote by $\mathbb{P}_{\mathbb{K}}^2$ the set of one dimensional subspaces of the vector space \mathbb{K}^3 , i.e. the set of points in the projective plane over \mathbb{K} . Then $\mathbb{P}_{\mathbb{K}}^2$ is a compact totally disconnected space with the topology inherited from \mathbb{K} , and there is a continuous action of $G = \mathrm{PGL}_3(\mathbb{K})$ on $\mathbb{P}_{\mathbb{K}}^2$. The group G also acts on its Furstenberg boundary Ω , which is the space of maximal flags $(0) < V_1 < V_2 < \mathbb{K}^3$.

The Bruhat-Tits building Δ of G is a topologically contractible simplicial complex of dimension 2, which is a union of apartments. Each apartment in Δ is a flat subcomplex isomorphic to a tessellation of \mathbb{R}^2 by equilateral triangles: that is, an affine Coxeter complex of type \tilde{A}_2 . For this reason, Δ is referred to as an affine building of type \tilde{A}_2 .

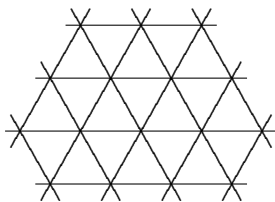


FIGURE 1. The Coxeter complex of type \tilde{A}_2

An apartment in Δ is a union of six sectors, based at a fixed vertex (Figure 2). These six sectors correspond to six points ω_i in the Furstenberg boundary Ω . They also represent the six edges of a hexagon in the spherical building at infinity Δ^∞ . This hexagon (which is a spherical Coxeter complex of type A_2) is an apartment in Δ^∞ . For this reason, Δ^∞ is referred to as a spherical building of type A_2 . More details are provided in Section 4 below.

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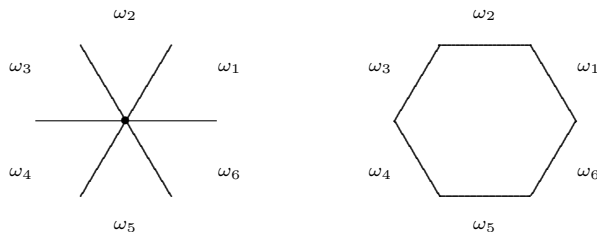


FIGURE 2. The six boundary points of an apartment

Let Γ be a lattice subgroup of G . The abelian group $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})$ of continuous integer-valued functions on $\mathbb{P}_{\mathbb{K}}^2$ is a Γ -module and the module of coinvariants $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})_{\Gamma} = H_0(\Gamma; C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z}))$ is a finitely generated group. Now suppose that Γ acts freely and transitively on the vertex set of the Bruhat-Tits building of G , i.e. Γ is an \tilde{A}_2 group [4]. Such a group is a natural analogue of a free group, which acts freely and transitively on the vertex set of a tree (which is a building of type \tilde{A}_1). We prove that $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})_{\Gamma}$ is a finite group and that the class $[\mathbf{1}]$ in $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})_{\Gamma}$ has order bounded by $q^2 - 1$.

An \tilde{A}_2 group of *Tits type* has a presentation with large automorphism group [4, II, Sections 4,5]. If Γ is of Tits type, and if $q \not\equiv 1 \pmod{3}$ it is shown that the class $[\mathbf{1}]$ in $C(\mathbb{P}_{\mathbb{K}}^2, \mathbb{Z})_{\Gamma}$ has order exactly $q - 1$. For groups of Tits type, this proves a conjecture of [8] regarding the order of the class $[\mathbf{1}]_{K_0}$ in the K-theory of boundary C^* -algebras.

Theorem 1.1. *Let Γ be an \tilde{A}_2 group of Tits type and let \mathfrak{A}_{Γ} be the (full) crossed product C^* -algebra $C(\Omega) \rtimes \Gamma$. If $q \not\equiv 1 \pmod{3}$ then the order of $[\mathbf{1}]_{K_0}$ in $K_0(\mathfrak{A}_{\Gamma})$ is precisely $q - 1$.*

It is worth noting that the full crossed product C^* -algebra $C(\Omega) \rtimes \Gamma$ coincides with the reduced crossed product, since the action of Γ on Ω is amenable [6, Section 4.2]. For $q \equiv 1 \pmod{3}$, the conjectured order of $[\mathbf{1}]_{K_0}$ is $\frac{q-1}{3}$, and this is verified for $q \leq 31$. The element $[\mathbf{1}]_{K_0}$ is significant because the C^* -algebras \mathfrak{A}_{Γ} are classified up to isomorphism by their two K -groups, together with the class $[\mathbf{1}]_{K_0}$ [7, Remark 6.5]. Many of the results are proved in a more general context, where the \tilde{A}_2 group Γ is not necessarily a lattice subgroup of $\mathrm{PGL}_3(\mathbb{K})$.

2. BACKGROUND

2.1. The Bruhat-Tits building of $\mathrm{PGL}_3(\mathbb{K})$. Given a local field \mathbb{K} , with discrete valuation $v : \mathbb{K}^{\times} \rightarrow \mathbb{Z}$, let $\mathcal{O} = \{x \in \mathbb{K} : v(x) \geq 0\}$ and let $\pi \in \mathbb{K}$ satisfy $v(\pi) = 1$. A *lattice* L is a rank-3 \mathcal{O} -submodule of \mathbb{K}^3 . In other words $L = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3$, for some basis $\{v_1, v_2, v_3\}$ of \mathbb{K}^3 . Two lattices L_1 and L_2 are *equivalent* if $L_1 = uL_2$ for some $u \in \mathbb{K}^{\times}$. The Bruhat-Tits building Δ of $\mathrm{PGL}_3(\mathbb{K})$ is a two dimensional simplicial complex whose vertices are equivalence classes of lattices in \mathbb{K}^3 . Two lattice classes $[L_0], [L_1]$ are *adjacent* if, for suitable representatives L_0, L_1 , we have $L_0 \subset L_1 \subset \pi^{-1}L_0$. A *simplex* is a set of pairwise adjacent lattice classes. The maximal simplices (chambers) are the sets $\{[L_0], [L_1], [L_2]\}$ where $L_0 \subset L_1 \subset L_2 \subset \pi^{-1}L_0$. These inclusions determine a canonical ordering of the vertices in a chamber, up to cyclic permutation.

Each vertex v of Δ has a *type* $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber of Δ has exactly one vertex of each type. If the Haar measure on \mathbb{K}^3 is normalized so that \mathcal{O}^3 has measure 1 then the type map may be defined by $\tau([L]) = \text{vol}(L) + 3\mathbb{Z}$. The cyclic ordering of the vertices of a chamber coincides with the natural ordering given by the vertex types. The building Δ is of type \tilde{A}_2 and there is a natural action of $\text{PGL}_3(\mathbb{K})$ on Δ induced by the action of $\text{GL}_3(\mathbb{K})$ on the set of lattices.

2.2. \tilde{A}_2 groups. More generally, consider any locally finite building Δ of type \tilde{A}_2 . Each vertex v of Δ is labeled with a type $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber has exactly one vertex of each type. Each edge e is directed, with initial vertex $o(e)$ of type i and final vertex $t(e)$ of type $i + 1$. An automorphism α of Δ is said to be *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices $v \in \Delta$.

Suppose that Γ is a group of type rotating automorphisms of Δ , which acts freely and transitively on the vertex set of Δ . Such a group Γ is called an \tilde{A}_2 group. The theory of \tilde{A}_2 groups has been developed in [4] and some, but not all, \tilde{A}_2 groups embed as lattice subgroups of $\text{PGL}_3(\mathbb{K})$. Any \tilde{A}_2 group can be constructed as follows [4, I, Section 3]. Let (P, L) be a projective plane of order q . There are $q^2 + q + 1$ points (elements of P) and $q^2 + q + 1$ lines (elements of L). Let $\lambda : P \rightarrow L$ be a bijection. A *triangle presentation* compatible with λ is a set \mathcal{T} of ordered triples (x, y, z) where $x, y, z \in P$, with the following properties.

- (i) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident, i.e. $y \in \lambda(x)$.
- (ii) $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$.
- (iii) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

In [4] there is displayed a complete list of triangle presentations for $q = 2$ and $q = 3$. Given a triangle presentation \mathcal{T} , one can form the group

$$(1) \quad \Gamma = \Gamma_{\mathcal{T}} = \langle P \mid xyz = 1 \text{ for } (x, y, z) \in \mathcal{T} \rangle.$$

The Cayley graph of Γ with respect to the generating set P is the 1-skeleton of a building Δ of type \tilde{A}_2 . Vertices are elements of Γ and a directed edge of the form $e = (a, ax)$ with $a \in \Gamma$ is labeled by a generator $x \in P$. Denote by E_+ the set of directed edges of Δ .

The link of a vertex a of Δ is the incidence graph of the projective plane (P, L) , where the lines in L correspond to the inverses in Γ of the generators in P . In other words, $\lambda(x) = x^{-1}$ for $x \in P$.

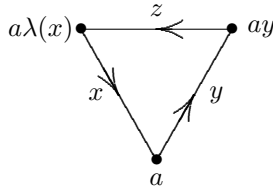


FIGURE 3. A chamber based at a vertex a

3. THE GROUP $A_{\mathcal{T}}$

The main results will be proved by defining a homomorphism from an abelian group $A_{\mathcal{T}}$ onto the module of coinvariants. Suppose that \mathcal{T} is a triangle presentation with associated \tilde{A}_2 group $\Gamma = \Gamma_{\mathcal{T}}$. Define $A_{\mathcal{T}}$ to be the abelian group generated by $P \cup \{\varepsilon\}$ subject to the relations

$$\begin{aligned} (2a) \quad & \sum_{y \notin \lambda(x)} y = x, \quad x \in P; \\ (2b) \quad & x + y + z = \varepsilon, \quad (x, y, z) \in \mathcal{T}; \\ (2c) \quad & \sum_{x \in P} x = \varepsilon. \end{aligned}$$

It follows immediately from (2a) and (2c) that, for each $x \in P$,

$$\varepsilon = \sum_{y \in P} y = \sum_{y \notin \lambda(x)} y + \sum_{y \in \lambda(x)} y = x + \sum_{y \in \lambda(x)} y.$$

If we define $\hat{x} = \sum_{y \in \lambda(x)} y$ then we obtain

$$(3) \quad x + \hat{x} = \varepsilon, \quad x \in P.$$

The group $A_{\mathcal{T}}$ has an alternative presentation with relations (2b), (2c), (3). It turns out that $A_{\mathcal{T}}$ is finite. First of all, observe that the element ε has finite order.

Lemma 3.1. *In the group $A_{\mathcal{T}}$, $(q^2 - 1)\varepsilon = 0$.*

Proof. Define a $\{0, 1\}$ -matrix M by $M(x, y) = 1 \Leftrightarrow y \notin \lambda(x)$. Then

$$\begin{aligned} \varepsilon &= \sum_{x \in P} x = \sum_{x \in P} \sum_{y \in P} M(x, y)y \\ &= \sum_{y \in P} \left(\sum_{x \in P} M(x, y) \right) y = \sum_{y \in P} q^2 y \\ &= q^2 \varepsilon \end{aligned}$$

□

Proposition 3.2. *The group $A_{\mathcal{T}}$ is finite.*

Proof. The \tilde{A}_2 group Γ has Kazhdan's property T [3, 12]. It follows that the abelianization $\Gamma_{ab} = H_1(\Gamma, \mathbb{Z})$ is finite [2, Corollary 1.3.6]. By relation (2b), $A_{\mathcal{T}}/\langle \varepsilon \rangle$ is a quotient of Γ_{ab} . Thus $A_{\mathcal{T}}/\langle \varepsilon \rangle$ is finite. Therefore so also is $A_{\mathcal{T}}$, since $\langle \varepsilon \rangle$ is finite, by Lemma 3.1. □

Question. What is the order of ε in $A_{\mathcal{T}}$?

Lemma 3.3. *The order of ε is at least $\frac{q-1}{(q-1, 3)}$.*

Proof. The map $f : P \cup \{\varepsilon\} \rightarrow \mathbb{Z}_{q^2-1}$ defined by $f(\varepsilon) = 3(q+1)$ and $f(x) = q+1$ for each $x \in P$, extends to a homomorphism $f : A_{\mathcal{T}} \rightarrow \mathbb{Z}_{q^2-1}$. For f preserves the relations (2), since

$$\sum_{y \notin \lambda(x)} f(y) = q^2(q+1) = q+1 = f(x), \quad x \in P,$$

$$f(x) + f(y) + f(z) = 3(q+1) = f(\varepsilon), \quad (x, y, z) \in \mathcal{T},$$

and

$$\sum_{x \in P} f(x) = (q^2 + q + 1)(q + 1) = 3(q + 1) = f(\varepsilon).$$

Since $f(\varepsilon) = 3(q+1)$, the order of $f(\varepsilon)$ is

$$\frac{q^2 - 1}{(q^2 - 1, 3(q + 1))} = \frac{q^2 - 1}{(q + 1)(q - 1, 3)} = \frac{q - 1}{(q - 1, 3)}.$$

Therefore ε has order at least $\frac{q-1}{(q-1,3)}$. \square

Proposition 3.4. *Suppose that there exists a subset $\mathcal{M} \subset \mathcal{T}$ such that each element in P occurs precisely 3 times in the triples belonging to \mathcal{M} . Then $(q-1)\varepsilon = 0$.*

Proof. Since $\#P = q^2 + q + 1$, we have $\#\mathcal{M} = q^2 + q + 1$. Therefore

$$\begin{aligned} 3\varepsilon &= 3 \sum_{x \in P} x = \sum_{(x,y,z) \in \mathcal{M}} (x + y + z) \\ &= \sum_{(x,y,z) \in \mathcal{M}} \varepsilon = (q^2 + q + 1)\varepsilon. \end{aligned}$$

The result follows, since $q^2\varepsilon = \varepsilon$, by Lemma 3.1 \square

Remark 3.5. Such a set \mathcal{M} exists if \mathcal{T} is a triangle presentation of any torsion free \tilde{A}_2 group with $q = 2$. Thus $\varepsilon = 0$ if $q = 2$. To see this, observe that for each $x_0 \in P$ there are 3 elements $y \in P$ such that $y \in \lambda(x_0)$. Thus there are 3 elements in \mathcal{T} of the form (x_0, y, z) , where z is uniquely determined by x_0 and y . The 7 possible $x_0 \in P$ give rise to $7 \cdot 3 = 21$ elements of \mathcal{T} . Thus $\#\mathcal{T} = 21$, and each element of P occurs $3 + 3 + 3 = 9$ times in the triples belonging to \mathcal{T} . Since \mathcal{T} contains no triple of the form (x, x, x) , the orbit of an element of \mathcal{T} under cyclic permutations contains 3 elements. Choosing one element of \mathcal{T} from each such orbit, we obtain a set \mathcal{M} containing 7 elements of \mathcal{T} , in which each generator appears 3 times.

A weak form of Proposition 3.4 is the following.

Corollary 3.6. *Suppose that there exists a subset $\mathcal{M} \subset \mathcal{T}$ such that each of the three coordinate projections from \mathcal{M} onto P is bijective. Then*

$$(q-1)\varepsilon = 0.$$

The rest of this section provides examples where Corollary 3.6 applies.

3.1. Invariant triangle presentations. Consider the Desarguesian projective plane $(P, L) = PG(2, q)$. The points and lines are the 1- and 2-dimensional subspaces, respectively, of a 3-dimensional vector space V over \mathbb{F}_q , with incidence being inclusion. The field \mathbb{F}_q is a subfield of \mathbb{F}_{q^3} and so \mathbb{F}_{q^3} is a 3-dimensional vector space over \mathbb{F}_q . We may therefore identify P with $\mathbb{F}_{q^3}^\times / \mathbb{F}_q^\times$. Now $\mathbb{F}_{q^3}^\times$ is a cyclic group, with generator ζ , say, and \mathbb{F}_q^\times is the subgroup generated by ζ^{1+q+q^2} (the unique subgroup with $q-1$ elements). Thus P is a cyclic group of order $1+q+q^2$ generated by the element $\sigma = \mathbb{F}_q^\times \zeta$. Multiplication by ζ is an \mathbb{F}_q -linear transformation and so multiplication by σ is a collineation of (P, L) . Thus σ generates a cyclic collineation group of order q^2+q+1 which acts regularly on P (and on L). A group with these properties is called a Singer group, after [10].

Remark 3.7. It is not known whether a projective plane with a Singer group is necessarily Desarguesian. Also, there are no known \tilde{A}_2 groups for which the underlying projective plane (P, L) is not Desarguesian.

Call a triangle presentation S -invariant if it is invariant under a Singer group of collineations of (P, L) . Such presentations are studied in [4, I, Section 5]. It is shown in [4, I, Theorem 5.1] that there are exactly $2^{(q+1)/3}$, $2^{q/3}$ or $3 \cdot 2^{(q-1)/3}$ distinct S -invariant triangle presentations, according as $q \equiv -1, 0$ or $1 \pmod{3}$, respectively. These presentations are constructed explicitly in [4] and all of them are also invariant under the Frobenius collineation $x \mapsto x^q$. It is not known whether all the corresponding \tilde{A}_2 groups embed as lattices in $PGL_3(\mathbb{K})$.

Remark 3.8. If $\lambda : P \rightarrow L$ is a bijection and \mathcal{T} is an S -invariant triangle presentation compatible with λ then $y \in \lambda(x) \leftrightarrow \sigma y \in \lambda(\sigma x)$. Therefore the presentation (2) of the group $\mathcal{A}_{\mathcal{T}}$ is invariant under the Singer group $\langle \sigma \rangle$. It is also invariant under the collineation $x \mapsto x^q$.

Corollary 3.9. *If the triangle presentation \mathcal{T} is S -invariant then, in the group $\mathcal{A}_{\mathcal{T}}$, $(q-1)\varepsilon = 0$.*

Proof. Fix a triple $(x_0, y_0, z_0) \in \mathcal{T}$. The orbit \mathcal{M}_0 of (x_0, y_0, z_0) under the Singer group satisfies the hypotheses of Corollary 3.6. \square

Remark 3.10. It follows from Lemma 3.3 that if \mathcal{T} is S -invariant and $q \not\equiv 1 \pmod{3}$ then the order of ε is $q-1$. Computations on the family of triangle presentations described below confirm that, up to $q = 32$, the order of ε is $\frac{q-1}{(q-1,3)}$. We conjecture that this is true for all S -invariant triangle presentations and for all q .

3.2. S -invariant triangle presentations of Tits type [4, I Section 4]. The mapping $x \mapsto x^q$ is an automorphism of \mathbb{F}_{q^3} over \mathbb{F}_q , i.e. it has fixed field \mathbb{F}_q . The trace $\text{Tr} : \mathbb{F}_{q^3} \rightarrow \mathbb{F}_q$ is the \mathbb{F}_q -linear mapping defined by $\text{Tr}(a) = a + a^q + a^{q^2}$. Now $(x_0, y_0) \mapsto \text{Tr}(x_0 y_0)$ is a regular symmetric bilinear form on \mathbb{F}_{q^3} , and so the map

$$V \mapsto V^\perp = \{y_0 \in \mathbb{F}_{q^3} : \text{Tr}(x_0 y_0) = 0 \text{ for all } x_0 \in V\}$$

is a bijection $P \rightarrow L$.

Let $x = \mathbb{F}_q^\times x_0 \in P = \mathbb{F}_{q^3}^\times / \mathbb{F}_q^\times$. Write $\text{Tr}(x) = 0$ if $\text{Tr}(x_0) = 0$ and write x^\perp for $(\mathbb{F}_q x_0)^\perp$. We identify any line, i.e., 2-dimensional subspace, with the set of

its 1-dimensional subspaces. Under these identifications, the lines are the subsets $x^\perp = \{y \in P : \text{Tr}(xy) = 0\}$ of the group P . For $x \in P$, let

$$\lambda_0(x) = \left(\frac{1}{x}\right)^\perp = \left\{y \in P : \text{Tr}\left(\frac{y}{x}\right) = 0\right\}.$$

This defines a point-line correspondence $\lambda_0 : P \rightarrow L$. The following set of triples is a triangle presentation \mathcal{T}_0 compatible with λ_0 :

$$(4) \quad \mathcal{T}_0 = \{(x, x\xi, x\xi^{q+1}) : x, \xi \in P \text{ and } \text{Tr}(\xi) = 0\}.$$

The only nontrivial thing to check is that \mathcal{T}_0 is invariant under cyclic permutations. If $(x, x\xi, x\xi^{q+1}) \in \mathcal{T}_0$ then

$$(x\xi, x\xi^{q+1}, x) = (x\xi, (x\xi)\xi^q, (x\xi)(\xi^q)^{q+1}) \in \mathcal{T}_0$$

since $\text{Tr}(\xi_0^q) = \text{Tr}(\xi_0)$ and $\xi_0^{1+q+q^2} \in \mathbb{F}_q^\times$ for each $\xi_0 \in \mathbb{F}_{q^3}^\times$.

The group $\Gamma_{\mathcal{T}}$ associated with the presentation (4) is called a group of Tits type [4, I Section 4]. The presentation (4) is clearly invariant under multiplication by any $x \in P$. That is, it is S-invariant. It is also easy to see that (4) is invariant under the Frobenius collineation $x \mapsto x^q$.

Remark 3.11. Consider the related triangle presentation

$$(5) \quad \begin{aligned} \mathcal{T}'_0 &= \left\{ \left(\frac{1}{z}, \frac{1}{y}, \frac{1}{x} \right) : (x, y, z) \in \mathcal{T}_0 \right\} \\ &= \{(x, x\xi, x\xi^{q^2+1}) : x, \xi \in P \text{ and } \text{Tr}(\xi) = 0\} \end{aligned}$$

which is compatible with λ_0 and S-invariant. For $q = 2, 3$, the two presentations $\mathcal{T}_0, \mathcal{T}'_0$ are the only S-invariant triangle presentations [4, II, p.165, Remark].

The group $\Gamma_{\mathcal{T}'_0}$ is isomorphic to $\Gamma_{\mathcal{T}_0}$, via the map $x \mapsto (\frac{1}{x})$, but the groups $A_{\mathcal{T}'_0}, A_{\mathcal{T}_0}$ are not isomorphic. In fact, writing $q = p^r$, p prime, the MAGMA computer algebra package calculates the following expressions for the groups $A_{\mathcal{T}_0}, A_{\mathcal{T}'_0}$ for $2 \leq q \leq 32$.

	$q \not\equiv 1 \pmod{3}$	$q \equiv 1 \pmod{3}$
$A_{\mathcal{T}_0}$	\mathbb{Z}_{q-1}	$\mathbb{Z}_{q-1} \oplus \mathbb{Z}_3$
$A_{\mathcal{T}'_0}$	$(\mathbb{Z}_p)^{3r} \oplus \mathbb{Z}_{q-1}$	$(\mathbb{Z}_p)^{3r} \oplus \mathbb{Z}_{q-1} \oplus \mathbb{Z}_3$

In all cases, the order of ε is precisely $\frac{q-1}{(q-1,3)}$.

Example 3.12. We verify the stated value of ε in the case $q = 4$. The set P is identified with $\mathbb{F}_{64}^\times / \mathbb{F}_4^\times$. Write $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$, where $\alpha^2 + \alpha + 1 = 0$ and $\mathbb{F}_{64} = \mathbb{F}_2(\alpha)(\delta)$, where $\delta^3 + \delta^2 + \alpha\delta + \alpha + 1 = 0$. Then $\delta^{21} = \alpha + 1$ and $\delta^{42} = \alpha$ are the roots of $x^2 + x + 1$. A short calculation shows that $(\delta^7)^4 = (\alpha + 1)\delta^7$, so that the fixed points of the Frobenius collineation $x \mapsto x^4$ on $P = \mathbb{F}_{64}^\times / \mathbb{F}_4^\times$ are $\dot{1}, \omega, \omega^2$, where $\dot{1} = \mathbb{F}_4^\times$ and $\omega = \delta^7 \mathbb{F}_4^\times$. It is easy to check that $\{\xi \in P : \text{Tr}(\xi) = 0\} = \{\omega, \omega^2, s, s^4, s^{16}\}$, where $s = \delta^3 \mathbb{F}_4^\times$. It follows from relation (3) in the group $A_{\mathcal{T}_0}$, with $x = \dot{1}$, that

$$\begin{aligned} \varepsilon &= \dot{1} + (\omega + \omega^2 + s + s^4 + s^{16}) \\ &= (\dot{1} + \omega + \omega^2) + (s + s^4 + s^{16}) \\ &= \varepsilon + (s + s^4 + s^{16}). \end{aligned}$$

In the last line, the fact that $\dot{1} + \omega + \omega^2 = \varepsilon$ follows from (2b), since $(\dot{1}, \omega, \omega^2) \in \mathcal{T}_0$ by (4). Therefore

$$\begin{aligned} \varepsilon + s^3 &= \varepsilon + (s^3 + s^4 + s) + s^{16} \\ &= \varepsilon + (s^3 + s^4 + s^8) + s^2 \\ &= \varepsilon + \varepsilon + s^2, \end{aligned}$$

using the facts that $s^8 = \delta^{24}\mathbb{F}_4^\times = s$ and $(s^3, s^4, s^8) \in \mathcal{T}_0$. Hence $s^3 = \varepsilon + s^2$. By invariance of the presentation \mathcal{T}_0 (Remark 3.8), we deduce that $s - \dot{1} = \varepsilon$ and therefore that $\varepsilon = s^{16} - \dot{1} = s^2 - \dot{1}$. Thus $s^2 = s$ and, by invariance, $s = \dot{1}$, which gives $\varepsilon = 0$. It is also easy to see that $A_{\mathcal{T}_0} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$, with generators δ and δ^2 .

3.3. More examples. Consider any triangle presentation \mathcal{T} . If ϕ is an order-3 collineation such that \mathcal{T} is fixed by the map $x \mapsto \phi(x)$, then one obtains a new triangle presentation

$$(6) \quad \mathcal{T}^\phi = \{(x, \phi(y), \phi^2(z)) : (x, y, z) \in \mathcal{T}\}$$

relative to the point–line correspondence $\phi \circ \lambda : P \rightarrow L$. The corresponding group

$$(7) \quad \Gamma_{\mathcal{T}^\phi} = \langle P \mid x\phi(y)\phi^2(z) = 1 \text{ for } (x, y, z) \in \mathcal{T} \rangle$$

is *not* in general isomorphic to $\Gamma_{\mathcal{T}}$ [4, II].

One possible choice for ϕ is the collineation $x \mapsto x^q$. Applied to the triangle presentations (4), ϕ and ϕ^2 give two more triangle presentations which are *not* in general S -invariant. If $q \equiv 1 \pmod{3}$ then another possible choice of ϕ is $x \mapsto \omega x$, where $\omega = \sigma^{(q^2+q+1)/3}$.

Corollary 3.13. *If the triangle presentation \mathcal{T} is S -invariant and if ϕ is an order-3 collineation of (P, L) such that \mathcal{T} is fixed by the map $x \mapsto \phi(x)$ then $(q-1)\varepsilon = 0$ in $A_{\mathcal{T}^\phi}$.*

Proof. Let $\mathcal{M} = \{(x, \phi(y), \phi^2(z)) : (x, y, z) \in \mathcal{M}_0\}$, where \mathcal{M}_0 is defined in the proof of Corollary 3.9. This satisfies the hypotheses of Corollary 3.6. \square

4. THE BOUNDARY ACTION

Associated with the building Δ is the building at infinity Δ^∞ . This is a spherical building of type A_2 [11, Theorem 8.24]. In the geometrical realization of Δ^∞ , a point $\xi \in \Delta^\infty$ is an equivalence class of rays (subsets of Δ isometric to $[0, \infty)$), where two rays are equivalent if they are *parallel*, i.e. at finite Hausdorff distance from each other [1, 11.8]. For each vertex x of Δ and each $\xi \in \Delta^\infty$, there is a unique ray $[x, \xi)$ with initial vertex x in the parallelism class of ξ . A *sector* is a $\frac{\pi}{3}$ -angled sector made up of chambers in some apartment. Each sector in Δ determines a 1-simplex (chamber) of Δ^∞ whose points are equivalence classes of rays in the sector. Two sectors determine the same chamber of Δ^∞ if and only if they contain a common subsector.

If $[x, \xi)$ is a sector wall then ξ is a *vertex* of Δ^∞ . If the initial edge of this sector wall is $[x, y]$, and if $\tau(y) = \tau(x) + i$ then the vertex ξ is said to be of type $i - 1$. This definition is independent of the base vertex x , since two parallel rays lie in a common apartment. Denote by \mathcal{P} the set of vertices $\xi \in \Delta^\infty$ of type 0 and denote by \mathcal{L} the set of vertices $\eta \in \Delta^\infty$ of type 1. Then $(\mathcal{P}, \mathcal{L})$ is a projective plane. A point $\xi \in \mathcal{P}$ and a line $\eta \in \mathcal{L}$ are incident if and only if they are the two vertices of

a common chamber in Δ^∞ . This is equivalent to saying that there is a sector with base vertex x whose walls are the rays $[x, \xi]$ and $[x, \eta]$.

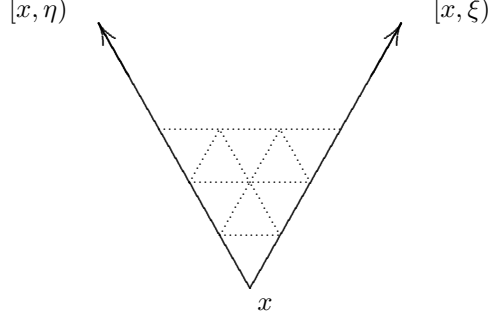


FIGURE 4. Sector walls

If $e \in E_+$, let $\Omega(e)$ denote the set of points $\xi \in \mathcal{P}$ which have representative rays with initial edge e . That is,

$$\Omega(e) = \{\xi \in \mathcal{P} : e \subset [o(e), \xi]\}.$$

The sets $\Omega(e)$, $e \in E_+$, form a basis of clopen sets for a totally disconnected compact topology on \mathcal{P} . The topological space \mathcal{P} is called the *minimal boundary* of Δ .

Remark 4.1. If Δ is the Bruhat-Tits building of $\mathrm{PGL}_3(\mathbb{K})$ then there is a natural identification of \mathcal{P} and $\mathbb{P}_{\mathbb{K}}^2$ as topological G -spaces. If $\lambda \in \mathbb{P}_{\mathbb{K}}^2$ and L_0 is a lattice, define a sequence of lattices inductively by $L_{i+1} = L_i + (\lambda \cap \pi^{-1}L_i)$. Then $L_i \subset L_{i+1} \subset \pi^{-1}L_i$ and $\tau([L_{i+1}]) = \tau([L_i]) + 1$, since L_i is maximal in L_{i+1} . The sequence of vertices $[L_0], [L_1], [L_2], \dots$ defines a ray whose parallelism class ξ_λ is an element of \mathcal{P} and the map $\lambda \mapsto \xi_\lambda$ is a bijection from $\mathbb{P}_{\mathbb{K}}^2$ onto \mathcal{P} . If $e = ([L_0], [L_1])$, where $L_0 \subset L_1 \subset \pi^{-1}L_0$ and $\tau([L_1]) = \tau([L_0]) + 1$, then $\Omega(e)$ may be identified with the set of lines $\lambda \in \mathbb{P}_{\mathbb{K}}^2$ such that $L_1 = L_0 + (\lambda \cap \pi^{-1}L_0)$.

If v is a fixed vertex of Δ , then \mathcal{P} may be expressed as a disjoint union

$$(8) \quad \mathcal{P} = \bigsqcup_{o(e)=v} \Omega(e).$$

Also, if $e \in E_+$, then $\Omega(e)$ can be expressed as a disjoint union

$$(9) \quad \Omega(e) = \bigsqcup_{\substack{o(e')=t(e) \\ \Omega(e') \subset \Omega(e)}} \Omega(e').$$

If Γ is an \tilde{A}_2 group acting on Δ , then Γ acts on \mathcal{P} , and the abelian group $C(\mathcal{P}, \mathbb{Z})$ has the structure of a Γ -module, with $(g \cdot f)(\xi) = f(g^{-1}\xi)$, $g \in \Gamma$, $\xi \in \mathcal{P}$. The module of Γ -coinvariants, $C(\mathcal{P}, \mathbb{Z})_\Gamma$, is the quotient of $C(\mathcal{P}, \mathbb{Z})$ by the submodule generated by $\{g \cdot f - f : g \in \Gamma, f \in C(\mathcal{P}, \mathbb{Z})\}$. If $f \in C(\mathcal{P}, \mathbb{Z})$ then $[f]$ denotes its class in $C(\mathcal{P}, \mathbb{Z})_\Gamma$. Also, $\mathbf{1}$ denotes the constant function defined by $\mathbf{1}(\xi) = 1$ for $\xi \in \mathcal{P}$.

If $e \in E_+$, let χ_e be the characteristic function of $\Omega(e)$. For each $g \in \Gamma$, the functions χ_e and $g \cdot \chi_e = \chi_{ge}$ project to the same element in $C(\mathcal{P}, \mathbb{Z})_\Gamma$. Therefore, for any edge $e = (a, ax)$ with $a \in \Gamma$ and $x \in \mathcal{P}$, it makes sense to denote by $[x]$ the class of χ_e in $C(\mathcal{P}, \mathbb{Z})_\Gamma$.

Suppose that $e, e' \in E_+$ with $o(e') = t(e) = v$, so that $t(e') = yv$ and $o(e) = x^{-1}v$ for (unique) $y, x \in P$. Then $\Omega(e') \subset \Omega(e)$ if and only if $y \notin \lambda(x)$. This is because $\Omega(e') \subset \Omega(e)$ if and only if the edges e and e' lie as shown in Figure 5, in some apartment.

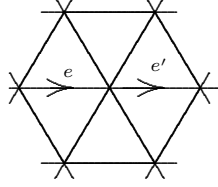


FIGURE 5

Equations (9) and (8) imply the following relations in $C(\mathcal{P}, \mathbb{Z})_\Gamma$.

$$(10a) \quad \sum_{y \notin \lambda(x)} [y] = [x], \quad x \in P;$$

$$(10b) \quad \sum_{x \in P} [x] = [\mathbf{1}].$$

These should be compared with relations (2a),(2c), respectively. Now we seek an analogue of (2b). The following fact is the key.

Lemma 4.2. [9, Lemma 9.4] *Given any chamber c and any sector S in Δ , there exists a sector $S_1 \subset S$ such that S_1 and c lie in a common apartment.*

If c is a chamber of Δ and if $\xi \in \mathcal{P}$, then ξ has a representative ray that lies relative to c in one of the three positions in Figure 6, in some apartment containing them both. This is because, by Lemma 4.2, we can choose a ray $[x, \xi)$ such that c and $[x, \xi)$ lie in a common apartment. Now choose an appropriate ray parallel to $[x, \xi)$.

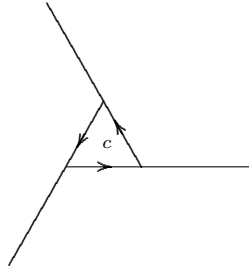


FIGURE 6. Relative positions of a chamber and representative rays

The next lemma follows immediately. Equation (11) is the desired analogue of (2b).

Lemma 4.3. *If $e_0, e_1, e_2 \in E_+$ are the edges of a chamber $c \in \Delta$ then*

$$\Omega(e_0) \sqcup \Omega(e_1) \sqcup \Omega(e_2) = \mathcal{P}.$$

Consequently, if \mathcal{T} is a triangle presentation and $(x, y, z) \in \mathcal{T}$ then

$$(11) \quad [x] + [y] + [z] = [\mathbf{1}].$$

Now we specify generators for $C(\mathcal{P}, \mathbb{Z})_\Gamma$.

Lemma 4.4. *The group $C(\mathcal{P}, \mathbb{Z})_\Gamma$ is finitely generated, with generating set $\{[x] : x \in \mathcal{P}\}$.*

Proof. Every clopen set V in \mathcal{P} may be expressed as a finite disjoint union of sets of the form $\Omega(e)$, $e \in E_+$. Any function $f \in C(\mathcal{P}, \mathbb{Z})$ is bounded, by compactness of \mathcal{P} , and so takes finitely many values $n_i \in \mathbb{Z}$. Therefore f may be expressed as a finite sum $f = \sum_j n_j \chi_{e_j}$, with $e_j \in E_+$. The result follows. \square

Proposition 4.5. *There is a homomorphism θ from $A_\mathcal{T}$ onto $C(\mathcal{P}, \mathbb{Z})_\Gamma$ defined by $\theta(x) = [x]$, for $x \in P$ and $\theta(\varepsilon) = [\mathbf{1}]$.*

Proof. Equations (10a),(10b),(11) show that θ preserves the relations (2a),(2c),(2b), respectively. Therefore θ extends to a homomorphism. Surjectivity is a consequence of Lemma 4.4. \square

5. THE MAIN RESULTS

This section collects the main consequences.

Corollary 5.1. *Let Γ be an \tilde{A}_2 group acting on an \tilde{A}_2 building Δ of order q with minimal boundary \mathcal{P} . Then $C(\mathcal{P}, \mathbb{Z})_\Gamma$ is a finite group and the class $[\mathbf{1}]$ in $C(\mathcal{P}, \mathbb{Z})_\Gamma$ has order bounded by $q^2 - 1$. If Γ has an S -invariant triangle presentation then $(q - 1)[\mathbf{1}] = 0$.*

Proof. This follows immediately from Proposition 3.2, Proposition 4.5, Lemma 3.1 and Corollary 3.9. \square

The final statement of Corollary 5.1 applies, in particular, to the groups of Tits type (Section 3.2).

5.1. K-theory. The *Furstenberg boundary* Ω of Δ is the set of chambers of Δ^∞ , endowed with a compact totally disconnected topology in which basic open sets have the form

$$\Omega(v) = \{\omega \in \Omega : [x, \omega) \text{ contains } v\}$$

where v is a vertex of Δ and $[x, \omega)$ is the representative sector for ω with base vertex x [3, Section 2]. If Δ is the Bruhat-Tits building of $G = \mathrm{PGL}_3(\mathbb{K})$ then Ω is isomorphic as a topological G -space to the space of maximal flags $(0) < V_1 < V_2 < \mathbb{K}^3$. The mapping which sends each sector to its wall of type 0 induces a natural surjection $\Omega \rightarrow \mathcal{P}$, under which \mathcal{P} has the quotient topology. Since this surjection is equivariant, there is an induced epimorphism $C(\mathcal{P}, \mathbb{Z})_\Gamma \rightarrow C(\Omega, \mathbb{Z})_\Gamma$.

The topological action of an \tilde{A}_2 group Γ on the maximal boundary is encoded in the full crossed product C^* -algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, which is studied in [6, 7, 8]. The natural embedding $C(\Omega) \rightarrow \mathfrak{A}_\Gamma$ induces a homomorphism

$$(12) \quad \psi : C(\Omega, \mathbb{Z})_\Gamma \rightarrow K_0(\mathfrak{A}_\Gamma)$$

and $\psi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$, the class of $\mathbf{1}$ in the K_0 -group of \mathfrak{A}_Γ . The article [5] estimates the order of $[\mathbf{1}]_{K_0}$ for various boundary C^* -algebras, and contains an extensive bibliography.

In [8], T. Steger and the second author performed extensive computations which determined the order of $[\mathbf{1}]_{K_0}$ for many \tilde{A}_2 groups with $q \leq 13$. The computations were done for all the \tilde{A}_2 groups in the cases $q = 2, 3$ and for several representative

groups for each of the other values of $q \leq 13$. If $q = 2$ there are precisely eight \tilde{A}_2 groups Γ , all of which embed as lattices in a linear group $\mathrm{PGL}(3, \mathbb{K})$ where $\mathbb{K} = \mathbb{F}_2((X))$ or $\mathbb{K} = \mathbb{Q}_2$. If $q = 3$ there are 89 possible \tilde{A}_2 groups, of which 65 do not embed naturally in linear groups. The experimental evidence suggested that for boundary crossed product algebras associated with \tilde{A}_2 groups it is always true that $[\mathbf{1}]_{K_0}$ has order $q - 1$ for $q \not\equiv 1 \pmod{3}$ and has order $(q - 1)/3$ for $q \equiv 1 \pmod{3}$. It is striking that the order of $[\mathbf{1}]$ appears to depend only on the parameter q . It is shown in [8] that the order of $[\mathbf{1}]_{K_0}$ is bounded above by $q^2 - 1$ and below by $\frac{q-1}{(q-1,3)}$.

Corollary 5.2. *Let \mathcal{T} be an S -invariant triangle presentation and $\Gamma = \Gamma_{\mathcal{T}}$. Then the class $[\mathbf{1}]_{K_0}$ in $K_0(\mathfrak{A}_{\Gamma})$ has order bounded by $q - 1$. If $q \not\equiv 1 \pmod{3}$ then the order of $[\mathbf{1}]_{K_0}$ is precisely $q - 1$.*

Proof. This follows directly from Corollary 5.1, since $\psi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$. \square

Theorem 1.1 is an immediate consequence of this Corollary.

Remark 5.3. It also follows from the computations in Remark 3.11 that the order of $[\mathbf{1}]_{K_0}$ is $\frac{q-1}{3}$, for all groups of Tits type with $q \equiv 1 \pmod{3}$ and $q \leq 31$.

Remark 5.4. For each \tilde{A}_2 group Γ , the algebra \mathfrak{A}_{Γ} has the structure of a higher rank Cuntz-Krieger algebra [7, theorem 7.7]. These algebras are classified up to isomorphism by their two K -groups, together with the class $[\mathbf{1}]_{K_0}$, [7, Remark 6.5]. It was proved in [8, Theorem 2.1] that

$$(13) \quad K_0(\mathfrak{A}_{\Gamma}) = K_1(\mathfrak{A}_{\Gamma}) = \mathbb{Z}^{2r} \oplus T,$$

where $r \geq 0$ and T is a finite group. The computations done in [8] give rise to some striking observations. For example, there are precisely three torsion-free \tilde{A}_2 subgroups of $\mathrm{PGL}_3(\mathbb{Q}_2)$ and these three groups are distinguished from each other by $K_0(\mathfrak{A}_{\Gamma})$.

There is also strong evidence that, for any torsion free \tilde{A}_2 group Γ , the integer r in (13) is equal to the rank of $H_2(\Gamma, \mathbb{Z})$. A typical example is provided by the group $\Gamma = \Gamma_{\mathcal{T}_0}$, associated with the triangle presentation of Tits type defined in Section 3.2, with $q = 13$. In that case,

$$K_0(\mathfrak{A}_{\Gamma}) = \mathbb{Z}^{1342} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^6 \oplus (\mathbb{Z}/13\mathbb{Z})^6.$$

As expected, the class $[\mathbf{1}]_{K_0}$ has order $4 = \frac{q-1}{3}$ and $H_2(\Gamma, \mathbb{Z}) = \mathbb{Z}^{671}$.

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