Causal Structures for General Concurrent Behaviours

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Abstract

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In this paper, we introduce and investigate generalised mutex order structures which can capture the invariant causal relationships in any concurrent history consisting of step sequence executions. Each such structure comprises two relations, viz. interleaving/mutex and weak causality. As our main result we prove that each generalised mutex order structure is the intersection of step sequence executions which are consistent with it.
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Non-interleaving semantics of concurrent systems is often expressed using posets, where causally related events are ordered and concurrent events are unordered. Each causal poset describes a unique concurrent history which is a set of executions, expressed as sequences or step sequences, consistent with it. Moreover, such a poset captures all precedence-based invariant relationships between the events in the executions belonging to the concurrent history. Causal poset semantics underpins efficient verification techniques based on unfoldings of safe Petri nets and concurrent automata models. However, when one considers extensions of these standard models, such as nets with inhibitor arcs, concurrent histories become too intricate to be described solely in terms of causal posets.

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Suggested keywords

CONCURRENT HISTORY
CAUSAL POSET
WEAK CAUSAL ORDER
MUTEX RELATION
INTERLEAVING
STEP SEQUENCE
CAUSALITY SEMANTICS
Causal Structures for General Concurrent Behaviours

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Abstract. Non-interleaving semantics of concurrent systems is often expressed using posets, where causally related events are ordered and concurrent events are unordered. Each causal poset describes a unique concurrent history which is a set of executions, expressed as sequences or step sequences, consistent with it. Moreover, such a poset captures all precedence-based invariant relationships between the events in the executions belonging to the concurrent history. Causal poset semantics underpins efficient verification techniques based on unfoldings of safe Petri nets and concurrent automata models. However, when one considers extensions of these standard models, such as nets with inhibitor arcs, concurrent histories become too intricate to be described solely in terms of causal posets.

In this paper, we introduce and investigate generalised mutex order structures which can capture the invariant causal relationships in any concurrent history consisting of step sequence executions. Each such structure comprises two relations, viz. interleaving/mutex and weak causality. As our main result we prove that each generalised mutex order structure is the intersection of step sequence executions which are consistent with it.

Keywords: concurrent history, causal poset, weak causal order, mutex relation, interleaving, step sequence, causality semantics.

1 Introduction

In order to design and validate complex concurrent systems, it is essential to understand the fundamental relationships between events occurring in their executions. However, looking at sequential descriptions of executions in the form of sequences or step sequences is insufficient when it comes to providing faithful information about causality and independence between events. To address this drawback, one may resort to using partially ordered sets of events to provide an explicit representation of causality in the executions of a concurrent system. In particular, the order in which independent events are observed is accidental and
those executions which only differ in the order of occurrences of independent events may be regarded as belonging to the same concurrent history. Each such concurrent history is underpinned by a causal poset, and the resulting semantics [1, 21, 23] can, in particular, support efficient verification techniques [3].

In general, concurrent behaviours can be investigated at the level of individual executions as well as at the level of order structures, such as causal posets, capturing the essential invariant dependencies between events. A key link between these two levels comes from the notion of a concurrent history [10] which is an invariant closed set \( \Delta \) of executions. The latter means that \( \Delta \) can be derived from an order structure built from simple invariant relationships on events \( X \) occurring in \( \Delta \), including causality (\( e \prec_\Delta f \) if \( e \) precedes \( f \) in the executions of \( \Delta \)), weak causality (\( e \sqcap_\Delta f \) if \( e \) precedes or is simultaneous with \( f \) in the executions of \( \Delta \)) and interleaving/mutex (\( e \rightleftharpoons_\Delta f \) if \( e \) is never simultaneous with \( f \) in the executions of \( \Delta \)). In the case of safe Petri nets with sequential executions, \( \prec_\Delta \) is the only invariant we need (as then, e.g., \( \prec_\Delta = \sqcap_\Delta \) and \( \rightleftharpoons_\Delta = \prec_\Delta \cup \prec_\Delta^{-1} \)). In particular, \( \Delta \) is the set of all sequential executions corresponding to the linearisations of \( \prec_\Delta \). The soundness of such an approach is validated by Szpirański’s Theorem [26] which states that each poset is equal to the intersection of its linearisations.

In this paper, executions are observed as step sequences, i.e., sequences of finite sets (steps) of simultaneously executed events. As an example, consider the safe Petri net depicted in Figure 1(a) which generates three step sequences involving \( a, c \) and \( d \), viz. \( \sigma = \{a\}\{c,d\} \), \( \sigma' = \{a\}\{c\}\{d\} \) and \( \sigma'' = \{a\}\{d\}\{c\} \). They can be seen as forming a single concurrent history \( \Delta = \{\sigma,\sigma',\sigma''\} \) under-

![Fig. 1.](image_url)
pinned by a causal poset $\prec_{\Delta}$ satisfying $a \prec_{\Delta} c$ and $a \prec_{\Delta} d$. Moreover, such a $\Delta$ adheres to the following true concurrency paradigm:

Given two events ($c$ and $d$ in $\Delta$), they can be observed as simultaneous
(in $\sigma$) $\iff$ they can be observed in both orders ($c$ before $d$ in $\sigma'$, and $d$
before $c$ in $\sigma''$).

Concurrent histories adhering to TRUECON are underpinned by causal partial orders, in the sense that each such history comprises all step sequence executions consistent with a unique causal poset on events involved in the history.

Following such an approach, [10] identified eight fundamental concurrency paradigms, $\pi_1-\pi_8$, with $\pi_8$ being precisely the above ‘true concurrency’ paradigm. Another paradigm is $\pi_3$ characterised by (TRUECON) with $\iff$ replaced by $\preceq$. Paradigm $\pi_3$ has a natural system model interpretation provided by safe Petri nets with inhibitor arcs. Figure 1(b) depicts such a net generating two step sequences involving $a$, $c$ and $d$, viz. $\sigma = \{a\} \{c,d\}$ and $\sigma' = \{a\} \{c\} \{d\}$. They form a concurrent history $\Delta' = \{\sigma, \sigma'\}$ adhering to paradigm $\pi_3$, but not to paradigm $\pi_8$ as there is no step sequence in $\Delta'$ in which $c$ is observed after $d$ even though $c$ and $d$ are observed in $\sigma$ as simultaneous.

As a result, histories adhering to $\pi_3$ are not underpinned by causal partial orders, but rather by causality structures $(X, \prec, \sqcap)$ as proposed in [11] — called stratified order structures (SO-structures) — based on causal posets and, in addition, weak causal posets. Again, a version of Szpilrajn’s Theorem can be shown to hold for SO-structures and the concurrent histories they generate. Stratified order structures were independently introduced in [4] (as ‘prosets’) and in [8] (as ‘composets’). Their comprehensive theory was developed in [12], and recently investigations include [7, 17, 20]. Moreover, they have been successfully applied to model inhibitor and priority systems, asynchronous races, and synthesis problems in, e.g., [13, 19, 22, 24]. In this paper, we will show that SO-structures can be represented in a one-to-one manner by mutex order structures, or MO-structures, $(X, =\!, \sqcap)$ based on interleaving/mutex and weak causality. The former, symmetric, relation singles out events that never occur simultaneously. Hence strict event precedence (causality) can be captured as a combination of mutex and weak causality.

This paper focuses on paradigm $\pi_1$ which is the least restrictive of the eight general paradigms of concurrency investigated in [10], i.e., there are no constraints like those imposed by $\pi_8$ and $\pi_3$. It simply admits all concurrent histories comprising step sequence executions. In such a case, as shown in [10], one again only needs to use two invariants, mutex and weak causality.

Figure 1(c) depicts a safe Petri net with mutex arcs (see [14–16]) generating two step sequences involving $a$, $c$ and $d$, viz. $\sigma' = \{a\} \{c\} \{d\}$ and $\sigma'' = \{a\} \{d\} \{c\}$. We first observe that in $\Delta'' = \{\sigma', \sigma''\}$ the executions of $c$ and $d$ interleave, and are both preceded by $a$; in other words, $c =_{\Delta''} d$, $a \sqsubseteq_{\Delta''} c$, $a \sqsubseteq_{\Delta''} d$ and $c =_{\Delta''} a =_{\Delta''} d$. That $\Delta''$ is a concurrent history then follows from the observation that $\Delta''$ contains all step sequences involving $a$, $c$ and $d$ which obey these invariant (common) relationships. As a result, $\Delta''$ adheres to paradigm $\pi_1$. 
(which has no constraints), but it does not conform to paradigms $\pi_8$ nor $\pi_3$ as there is no step sequence in $\Delta''$ in which $c$ and $d$ occur simultaneously.

As another motivating example, consider the following three atomic operations under the assumption that simultaneous reading of variables is allowed:

$$
\begin{align*}
a & : x \leftarrow x + 1 \\
b & : x \leftarrow x + 2 \\
c & : y \leftarrow y + 1.
\end{align*}
$$

It is reasonable to consider them all as 'concurrent' as any order of their executions yields exactly the same result. However, while simultaneous execution of $a$ and $c$, or $b$ and $c$ is harmless, the simultaneous execution of $a$ and $b$ is not, even though both orders $ab$ and $ba$ are clearly valid. Hence, in this case, $a \equiv b$ and $\sqsubset = \varnothing$, and the induced concurrent history comprises all sequential executions of $a$, $b$ and $c$, together with $\{a, c\}\{b\}$, $\{b\}\{a, c\}$, $\{b, c\}\{a\}$ and $\{a\}\{b, c\}$. The idea that concurrency models with simultaneity should include the case that the executions 'a followed by b' and 'b followed by a' are equivalent yet $\{a, b\}$ is not allowed, was presumably first postulated in [18] on which the above example is based.

To summarise, a nonempty set $\Delta$ of step sequence executions over the same set of events $X$ is a concurrent history in $\pi_1$ iff $\Delta$ consists of all step sequences $\sigma$ over $X$ such that: $e \equiv_\Delta f$ implies that $e$ and $f$ are not simultaneous in $\sigma$, and $e \sqsubset_\Delta f$ implies that $e$ preceded or was simultaneous with $f$ in $\sigma$. The aim of this paper is to fully characterise all the order structures $(\pi_1, \equiv_\Delta, \sqsubset_\Delta)$ underpinning concurrent histories $\Delta$ adhering to paradigm $\pi_1$. An earlier attempt to describe structures of this kind was proposed in [3]; however, the resulting generalised stratified order structures (or GSO-structures) do not always capture all the implied invariant relationships involving the mutex relation. A comprehensive treatment of GSO-structures can be found in [6].

In this paper, we will show that general concurrent histories (i.e., conforming to paradigm $\pi_1$) are underpinned by generalised mutex order structures (or GMO-structures). We also develop a notion of GMO-closure which is the GMO-structure counterpart of the well-known construction of the transitive closure of an acyclic relation. The main result is an extended version of Szpilrajn’s Theorem which is formulated and proven to hold for GMO-structures and step sequence executions.

The paper is organised in the following way. In the next section, we recall key notions and notations used throughout the paper. In Section 3, we introduce MO-structures and establish their relationship with stratified order structures. Then, Section 4 introduces GMO-structures and proves their main properties, including GMO-closure and the GMO-structure version of Szpilrajn’s Theorem. Section 5 presents concluding remarks.

## 2 Preliminary definitions

Throughout the paper we use the standard notions of set theory and formal language theory. In particular, $\{\}$ denotes disjoint set union. The identity relation on a set $X$ is defined as $\text{Id}_X = \{(a, a) \mid x \in X\}$, the index $X$ may be omitted if it is clear from the context.
Composing relations. The composition of two binary relations, $R$ and $Q$, over a set $X$ is given by $R \circ Q = \{(a, b) \mid \exists x \in X : aRx \land xQb\}$. Moreover, if $P \subseteq X \times X$, then we define (see Figure 2):

$$R \circ_P Q = \{(a, b) \mid \exists (x, y) \in P: aRxQb \land aRyQb\}.$$ 

![Fig. 2. A visualisation of $\circ_P$ composition.](image)

Note that $\circ = \circ_{Id}$, and the associativity of relation composition holds for the extended notion. We will also denote $a_1 \ldots a_k R b_1 \ldots b_m$ whenever $a_i R b_j$, for all $i, j$. For example, $aRbcQd$ means that $aRbQd$ and $aRcQd$.

Given a relation $R \subseteq X \times X$, $R^0 = Id$ and $R^n = R^{n-1} \circ R$, for all $n \geq 1$. Then: (i) the reflexive closure of $R$ is defined by $R \cup Id$; (ii) the transitive closure by $R^+ = \bigcup_{i \geq 1} R^i$; (iii) the reflexive transitive closure by $R^* = Id \cup R^+$; and (iv) the irreflexive transitive closure by $R^\perp = R^+ \setminus Id = R^* \setminus Id$. Moreover, the inverse of $R$ is given by $R^{-1} = \{(a, b) \mid \langle b, a \rangle \in R\}$, and the symmetric closure by $R^{sym} = R \cup R^{-1}$.

**Order relations.** A relation $R \subseteq X \times X$ is: (i) symmetric if $R = R^{-1}$; (ii) antisymmetric if $R \cap R^{-1} \subseteq Id$; (iii) reflexive if $Id \subseteq R$; (iv) irreflexive if $Id \cap R = \emptyset$; (v) transitive if $R \circ R \subseteq R \cup Id$; and (vi) total if $R \cup R^{-1} = X \times X$.

A relation $R \subseteq X \times X$ is: (i) an equivalence relation if it is symmetric, transitive and reflexive; (ii) a pre-order if it is transitive and irreflexive; (iii) a partial order if it is an antisymmetric pre-order; and (iv) a total order if it is a partial order and $R \cup Id$ is total; (v) a stratified order if it is a partial order such that $X \times X \setminus R^{sym}$ – its incomparability relation – is an equivalence relation. Note that stratified orders are uniquely represented as (step) sequences of equivalence classes of the incomparability relation. In the context of concurrent behaviours, we will identify step sequences with stratified orders, similarly as sequences are often identified with total orders (see [7, 12] for formal details).

Every irreflexive relation $R \subseteq X \times X$ induces a least pre-order containing $R$ defined by $R^\perp$. Following E. Schröder [23], we define the largest equivalence relation contained in $R^\ast$ as:

$$R^\ominus = R^\ast \cap (R^\ast)^{-1} = (R^\perp \cap (R^\perp)^{-1}) \cup Id.$$ (1)
For a stratified order $R \subseteq X \times X$ we define two relations, $\sqsubseteq_R$ and $\Rightarrow_R$, such that, for all distinct $a, b \in X$:

\begin{align*}
  a \sqsubseteq_R b & \iff \neg(bRa) \quad (2) \\
  a \Rightarrow_R b & \iff \neg(a \sqsubseteq_R b) \iff aRb \lor bRa. \quad (3)
\end{align*}

Intuitively, if $R$ represents a stratified order execution, $aRb$ means ‘$a$ occurred earlier than $b$’. In such a case $a \sqsubseteq_R b$ means ‘$a$ occurred not later than $b$’, $a \Rightarrow_R b$ means ‘$a$ did not occur simultaneously with $b$’, and $a \sqsubseteq^\ast_R b$ means ‘$a$ occurred simultaneously with $b$’.

Relational structures. A tuple $S = (X, R_1, R_2, \ldots, R_n)$, where $n \geq 1$ and each $R_i \subseteq X \times X$ is a binary relation on $X$, is an $(n$-ary) relational structure. By the domain of a relational structure $S$ we mean the set $X$. An extension of $S$ is any relational structure $S' = (X, R'_1, R'_2, \ldots, R'_n)$ satisfying $R_i \subseteq R'_i$, for every $1 \leq i \leq n$. We denote this by $S \subseteq S'$. The intersection of a nonempty family $R = \{(X, R_1, \ldots, R_n) \mid i \in I\}$ of relational structures with the same domain and arity is given by:

$$\bigcap R = (X, \bigcap_{i \in I} R_i^1, \ldots, \bigcap_{i \in I} R_i^n).$$

In what follows, we consider only relational structures that contain two relations, while the set $X$ is finite.

A relational structure $S = (X, Q, R)$ is: (i) separable if $Q \cap R^\ast = \emptyset$, $Q$ is symmetric and $R$ is irreflexive; and (ii) saturated in a family of relational structures $\mathcal{X}$ if it belongs to $\mathcal{X}$ and for every extension $S' \in \mathcal{X}$ of $S$, we have $S = S'$. It is easily seen that an intersection of separable relational structures is also separable. Intuitively, if $Q$ represents ‘mutex’ and $R$ ‘weak precedence’, then separability means that simultaneous events cannot be in the mutex relation.

A stratified order structure (or so-structure) [4, 8, 12] is defined as a relational structure $\text{sos} = (X, \prec, \sqsubseteq)$, where $\prec$ and $\sqsubseteq$ are binary relations on $X$ such that, for all $a, b, c \in X$:

\begin{align*}
  S1 : & \quad a \not\preceq a \\
  S2 : & \quad a \prec b \implies a \sqsubseteq b \\
  S3 : & \quad a \sqsubseteq b \sqcap c \land a \neq c \implies a \sqsubseteq c \\
  S4 : & \quad a \sqsubseteq b \prec c \lor a \prec b \sqcap c \implies a \prec c.
\end{align*}

Figure 3 illustrates the ‘transitivity’ axioms $S2 - S4$.

A generalized stratified order structure [5, 6] (or gso-structure) is a relational structure $\text{gso}s = (X, \leftrightharpoons, \sqsubseteq)$ such that $\sqsubseteq$ is irreflexive, $\leftrightharpoons$ is irreflexive and symmetric, and $(X, \leftrightharpoons \cap \sqsubseteq, \sqsubseteq)$ is an so-structure.
Properties. For every binary relation $R \subseteq X \times X$ and all $a, b \in X$, we have:

\begin{align*}
(R \cup \langle a, b \rangle)^* &= R^* \cup \{(c, d) \mid cR^* a \land bR^* d\} \ . \\
\neg (bR^* a) &\implies (R \cup \langle a, b \rangle)^\circ = R^\circ \\
R^\circ &= (R^\circ)^{-1} \subseteq R^* \\
(R^\circ)^{\downarrow} &= R^\downarrow \\
(R^\circ)^* &= R^* \\
(R^\circ)^{\uparrow} &= R^\uparrow \\
(R^\downarrow)^{\circ} &= R^\circ \\
(R^\circ) \circ R^\circ &= R^\circ \circ R^* = R^* \circ R^* = R^* \\
\end{align*}

If $R \subseteq X \times X$ is a stratified order, then $\equiv_R$ is irreflexive and symmetric, while $\sqsubseteq_R$ is a pre-order such that:

\begin{align*}
\sqsubseteq_R &= \sqsubseteq_R \setminus \Id = \sqsubseteq_R \\
\sqsubseteq_R \setminus \Id &= \sqsubseteq_R \cap \sqsubseteq_R^{-1} .
\end{align*}

Moreover, $\equiv_R \cap \sqsubseteq_R = R$ and, for all distinct $a, b \in X$, we have:

\begin{align*}
\neg (a \equiv_R b) &\iff a \sqsubseteq_R b \land b \sqsubseteq_R a \\
\neg (a \sqsubseteq_R b) &\implies b \sqsubseteq_R a \\
aRb &\iff a \equiv_R b \land a \sqsubseteq_R b .
\end{align*}

and exactly one of the following holds:

\begin{align*}
a \equiv b &\iff a \sqsubseteq b \not\sqsubseteq a \\
a \equiv b &\iff a \not\sqsubseteq b \sqsubseteq a \\
a \not\equiv b &\iff a \not\sqsubseteq b \sqsubseteq a .
\end{align*}

Intuitively, (13) means that ‘$a$ occurred earlier than $b$’ iff ‘$a$ and $b$ were not simultaneous’ and ‘$a$ occurred not later than $b$’.
3 Separable order structures

In this section we take another look at the stratified order structures, substantially different from that of, e.g., [7,12,17,20]. We provide for them a new representation which we found more suitable for further generalisation. The new representation replaces causal orders by mutex relations between events. While so-structures may allow for more compact representation (strict precedence involves fewer pairs of events than mutex), the new order structures are easier to generalise to cater for general interleaving/mutex requirements and their properties.

In the rest of this paper, we will be concerned with order structures of the form $S = (X, =, \sqsubseteq)$. Intuitively, $X$ is a set of events involved in some history of a concurrent system, $=$ is a 'mutex' (or 'interleaving') relation which identifies pairs of events which cannot occur simultaneously, and $\sqsubseteq$ is a 'weak precedence' relation between events. The latter means, in particular, that if $a \sqsubseteq b \sqsubseteq a$ then $a$ and $b$ must occur simultaneously in any execution belonging to the history represented by $S$; in other words, $S$ must be separable (i.e., $= \cap \sqsubseteq^* = \emptyset$).

3.1 Mutex order structures

The definition of the first class of order structures based on mutex and weak precedence relations is motivated by the observation that the 'precedence' (or 'causality') relation is nothing but 'mutex'+'weak precedence', c.f. (13). Therefore, the axioms defining stratified order structures can easily be rendered in terms of the latter relations.

**Definition 1 (mutex order structure).** A mutex order structure (mo-structure) is a relational structure $mos = (X, =, \sqsubseteq)$, where $=$ and $\sqsubseteq$ are binary relations on $X$ such that, for all $a, b, c \in X$:

$M1 : a = b \implies b = a$
$M2 : a \not\sqsubseteq a$
$M3 : a = b \implies a \sqsubseteq b \lor b \sqsubseteq a$
$M4 : a \sqsubseteq b \sqsubseteq c \land a \neq c \implies a \sqsubseteq c$
$M5 : a \sqsubseteq b \sqsubseteq c \land (a = b \lor b = c) \implies a \sqsubseteq c$.

Axioms $M3 - M5$ are illustrated in Figure 4. We first show that all such structures are separable.

**Proposition 1.** Each mo-structure is separable.

**Proof.** Let $mos = (X, =, \sqsubseteq)$ be an mo-structure. Suppose that $a \sqsubseteq b \sqsubseteq a$ and $a = b$. Then, by $M5$, we have $a = a$, contradicting (15). Hence $a \sqsubseteq b \sqsubseteq a$ implies $a \neq b$. Moreover, $=$ is symmetric by $M1$, and $\sqsubseteq$ is irreflexive by $M2$. \qed

We then prove a number of properties of mo-structures.
Proposition 2. Let $mos = (X, \equiv, \sqsubseteq)$ be an MO-structure and $a, b, c, d \in X$. Then

\begin{align*}
    a \neq a & \quad (15) \\
    a \sqsubseteq b \sqsubseteq a \land a \equiv c & \implies b \equiv c \quad (16) \\
    a \sqsubseteq c \sqsubseteq b \land a \sqsubseteq d \sqsubseteq b \land c \equiv d & \implies a \equiv b \quad (17)
\end{align*}

Proof. To show (15), suppose that $a \equiv a$. Then, by $M3$, we have $a \sqsubseteq a$, contradicting $M2$. Hence $a \neq a$.

To show (16), suppose that $a \sqsubseteq b \sqsubseteq a$ and $a \equiv c$. From $a \equiv c$ and $M3$ it follows that $a \sqsubseteq c$ or $c \sqsubseteq a$. If $a \sqsubseteq c$ then $b \sqsubseteq a \sqsubseteq c$ and $a \equiv c$, and so, by $M5$, we obtain $b \equiv c$. While if $c \sqsubseteq a$ then $c \sqsubseteq a \sqsubseteq b$ and $a \equiv c$, and so, by $M5$, we obtain $c \equiv b$. Together with $M1$ this yields $b \equiv c$.

To show (17), suppose that $a \sqsubseteq c d \sqsubseteq b$ and $c \equiv d$. From $c \equiv d$ and $M3$ it follows that $c \sqsubseteq d$ or $d \sqsubseteq c$. Without loss of generality, we can assume that $c \sqsubseteq d$. Then $a \sqsubseteq c \sqsubseteq d$ and $c \equiv d$, and so, by $M5$, we obtain $a \equiv d$. Moreover, we have $a \sqsubseteq d \sqsubseteq b$, and so, also by $M5$, we obtain $a \equiv b$.

Theorem 1. The mappings $mo2so$ and $so2mo$ are inverse bijections.
Proof. Suppose that \( \text{mos} = (X, \equiv, \sqsubset) \) is an \( \text{mo} \)-structure. We will show that \( (X, \equiv \cap \sqsubset, \sqsubset) \) is an \( \text{so} \)-structure. Note that \( S1 = M2 \) and \( S3 = M4 \) because we use the same relation \( \sqsubset \) in both cases. Hence it remains to be shown that \( S2 \) and \( S4 \) hold.

To show \( S2 \), we note that \( \prec \) is defined as \( \equiv \cap \sqsubset \). Hence, if \( a \prec b \) then \( a \sqsubset b \).

To show \( S4 \), suppose that \( a = c \). Then \( a \sqsubset b \sqsubset a \) and \( a \equiv b \lor b \equiv a \). Thus, by \( M5 \), we obtain \( a \equiv a \), and, by \( M3 \), we obtain \( a \sqsubset a \), which contradicts \( M2 \). Hence \( a \neq c \).

Let \( a \prec b \sqsubset c \). Then \( a \sqsubset b \sqsubset c \) and \( a \equiv b \). By \( M5 \), we obtain \( a \equiv c \). Since, by \( M4 \), we also have \( a \sqsubset c \), it follows that \( a \prec c \).

Let \( a \sqsubset b \prec c \). Then \( a \sqsubset b \sqsubset c \) and \( a \equiv c \). By \( M5 \), we obtain \( a \equiv c \). Since, by \( M4 \), we also have \( a \sqsubset c \), it follows that \( a \prec c \).

Hence \( \text{mo}2\text{so}(\text{mos}) \) is an \( \text{so} \)-structure. Suppose that \( \text{mos}' = (X', \equiv', \sqsubset') \) is an \( \text{mo} \)-structure such that \( \text{mo}2\text{so}(\text{mos}) = \text{mo}2\text{so}(\text{mos}') \). Then, clearly, \( X = X' \) and \( \sqsubset = \sqsubset' \).

Let \( a \equiv b \). Then, by \( M3 \) and without loss of generality, \( a \sqsubset b \). Hence \( \langle a, b \rangle \in \equiv \cap \sqsubset \), and so \( \langle a, b \rangle \in \equiv' \cap \sqsubset' \). Thus \( a \equiv' b \). Hence \( \text{mos} = \text{mos}' \), demonstrating that \( \text{mo}2\text{so} \) is injective.

Suppose now that \( \text{sos} = (X, \prec, \sqsubset) \) is an \( \text{so} \)-structure. We will show that \( (X, \prec^{\text{sym}}, \sqsubset) \) is an \( \text{mo} \)-structure. Note that \( M2 = S1 \) and \( M4 = S3 \) because we use the same relation \( \sqsubset \) in both cases. Hence it remains to be shown that \( M1 \), \( M3 \) and \( M5 \) hold.

To show \( M1 \), we observe that it follows from the fact that \( \equiv \) is defined as \( \prec^{\text{sym}} \).

To show \( M3 \), we observe that since \( \equiv \) is defined as \( \prec^{\text{sym}} \), then \( a \equiv b \) implies \( a \prec b \lor b \prec a \). Hence, by \( S2 \), we obtain \( a \sqsubset b \lor b \sqsubset a \).

To show \( M5 \), suppose that \( a \sqsubset b \sqsubset c \) and \( a \equiv b \lor b \equiv c \). Then:

\[ a \sqsubset b \sqsubset c \land a \equiv b \lor a \sqsubset b \sqsubset c \land b \equiv c . \]

Hence, since \( \equiv \) is defined as \( \prec^{\text{sym}} \), we obtain:

\[ a \sqsubset b \sqsubset c \land a \prec b \lor b \prec a \lor b \prec c \lor c \prec b . \]

Since \( a \sqsubset b \) implies \( \neg(b \prec a) \) (see [11]), we can exclude \( b \prec a \) and \( c \prec b \), and so we have that \( a \sqsubset b \prec c \) or \( a \prec b \sqsubset c \). Hence, using \( S4 \), we get \( a \prec c \). Once more using the definition of \( \equiv \), we obtain \( a \equiv c \).

Hence \( \text{so}2\text{mo}(\text{sos}) \) is an \( \text{mo} \)-structure. Suppose that \( \text{sos}' = (X', \prec', \sqsubset') \) is an \( \text{so} \)-structure such that \( \text{so}2\text{mo}(\text{sos}) = \text{so}2\text{mo}(\text{sos}') \). Then, clearly, \( X = X' \) and \( \sqsubset = \sqsubset' \).

Let \( a \prec b \). Then, by \( S2 \), \( a \sqsubset b \) and so \( a \sqsubset' b \). Moreover, \( \langle a, b \rangle \in \prec^{\text{sym}} = (\prec')^{\text{sym}} \).

If \( \langle a, b \rangle \in (\prec')^{-1} \) then, by \( S4 \), \( a \prec' a \) in this way, we obtained a contradiction with \( S1 \) and \( S2 \). Thus \( a \prec' b \). Hence \( \text{mos} = \text{mos}' \), demonstrating that \( \text{so}2\text{mo} \) is injective.

Finally, to show that \( \text{so}2\text{mo} \) and \( \text{mo}2\text{so} \) are inverses of each other, we first observe that, for all distinct \( a, b \in X \):

\[ \langle a, b \rangle \in (\equiv \cap \sqsubset)^{\text{sym}} \iff a \equiv b \land (a \sqsubset b \lor b \sqsubset a) \iff M5 \ a \equiv b , \]
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\[ \text{so2mo} \circ \text{mo2so}(X, \preceq, \sqsubset) = \text{so2mo}(X, \preceq \cap \sqsubset, \sqsubset) \]
\[ = (X, (\preceq \cap \sqsubset)^{\text{sym}}, \sqsubset) = (X, \prec, \sqsubset). \]

Moreover, we have:
\[ \langle a, b \rangle \in \prec^{\text{sym}} \cap \sqsubset \iff a \prec b \lor b \prec a \sqsubset b \iff a \prec b, \]
since by S4, \( b \prec a \sqsubset b \) implies \( b \prec b \), contradicting S1 and S2. Hence
\[ \text{mo2so} \circ \text{so2so}(X, \prec, \sqsubset) = \text{mo2so}(X, \prec^{\text{sym}}, \sqsubset) \]
\[ = (X, \prec^{\text{sym}} \cap \sqsubset, \sqsubset) = (X, \prec, \sqsubset). \]

\[ \square \]

3.2 Layered order structures

In general, order structures like \( \text{mo} \)-structures are not saturated, and may capture histories comprising several executions (like a single causal partial order may have numerous total order linearisations). However, there is also a class of order structures which correspond in a one-to-one way to step sequences.

**Definition 2.** Let \( R \subseteq X \times X \) be a stratified order. Then \( \text{los} = (X, \preceq_R, \sqsubset_R) \) is the layered order structure (or \( \text{lo}-\text{structure} \)) induced by \( R \).

For a separable relational structure \( \text{sr} = (X, \preceq, \sqsubset) \), we will denote by \( \text{sr2los}(\text{sr}) \) the set of all \( \text{lo}-\text{structures} \) \( \text{los} \) extending \( \text{sr} \), i.e., \( \text{sr} \subseteq \text{los} \). With this notation, a nonempty set \( \text{LOS} \) of \( \text{lo}-\text{structures} \) is a concurrent history adhering to paradigm \( \pi_1 \) if
\[ \text{LOS} = \text{sr2los}(\bigcap \text{LOS}). \]

**Proposition 3.** Every layered order structure is separable and saturated in the set of all separable order structures.

**Proof.** Let \( \text{los} = (X, \preceq, \sqsubset) \) be a layered order structure, and \( a \sqsubseteq \diamond b \). Then, by (10), we get \( a = b \) or \( a \sqsubset b \sqsubset a \). Thus, by (14) and the irreflexivity of \( = \), we obtain that \( a \neq b \). Hence \( \equiv \cap \sqsubseteq \diamond = \emptyset \) and, by the irreflexivity of \( \sqsubseteq \) and symmetry of \( = \), we obtain that \( \text{los} \) is separable.

Let \( S = (X, Q, R) \) be a separable relational structure extending \( \text{los} \). Suppose that \( aQb \) and \( a \neq b \). Then, by (14), we get \( a \sqsubset b \) and \( b \sqsubset a \). Hence \( aRb \) and \( bRa \), and so \( aR^\circ b \). As a result, we obtain \( \langle a, b \rangle \in Q \cap R^\circ \) which contradicts the separability of \( S \). This means that \( Q \) is equal to \( = \).

Suppose now that \( aRb \) and \( a \nsubseteq b \). Then, by (14), we get \( b \equiv a \) and \( b \sqsubset a \). Hence \( bQa \) and \( bRa \) which, together with \( aRb \), gives \( aR^\circ b \). As a result, we obtain \( \langle a, b \rangle \in Q \cap R^\circ \) which contradicts the separability of \( S \). This means that \( R \) is equal to \( \sqsubset \), completing the proof.

\[ \square \]

**Proposition 4.** Every \( \text{lo}-\text{structure} \) is an \( \text{mo}-\text{structure} \).
Proof. Let $los = (X, \equiv, \sqsubseteq)$ be an lo-structure. We will show that it satisfies axioms $M1 - M5$.

To show $M1$, we observe that since $\equiv$ is symmetric, we have $a \equiv b \implies b \equiv a$.

To show $M2$, we observe that by Definition 2, $\sqsubseteq$ is irreflexive. Hence $a \not\sqsubseteq a$.

To show $M3$, suppose that $a \sqsubseteq b$. Then, by $(14)$, we get that $a \sqsubseteq b \land b \not\sqsubseteq a$ or $a \not\sqsubseteq b \land b \sqsubseteq a$. Hence $a \sqsubseteq b$ or $b \sqsubseteq a$.

To show $M4$, suppose that $a \sqsubseteq b \sqsubseteq c$ and $a \neq c$. Then $a \sqsubseteq c$. Hence, by transitivity of $\sqsubseteq$, we obtain $a \sqsubseteq c$.

To show $M5$, let $a \sqsubseteq b \sqsubseteq c$ and $a = b$. Suppose that $a = c$. Then $a \sqsubseteq b$ and $b \sqsubseteq a$ and $a = b$, contradicting $(14)$. Hence $a \neq c$. Then $b \sqsubseteq c \sqsubseteq a$, and so $b \sqsubseteq a$ and $a \sqsubseteq b$ and $a = b$, contradicting $(14)$. Hence $c \not\sqsubseteq a$. By $(14)$, we obtain $a = c$, which completes the proof. \[\square\]

An mo-structure $mos$ is linked with lo-structures (step sequences) through the set $sr2los(mos)$ of all lo-structures extending $mos$. Similarly, for every so-structure $sos$ we can define $so2los(sos) = sr2los(so2mo(sos))$. It can then be seen ([9,12]) that $so2los(sos)$ is a nonempty set and (in the notation used in this paper):

$$sos = \bigcap_{so} \mo2so(so2los(sos)).$$  \hfill (18)

That result corresponds to Szpiroja's Theorem that any partial order is the intersection of its linearisations (c.f. [6,12]). This result extends to mo-structures and we obtain

**Theorem 2.** For every mo-structure $mos$,

$$sr2los(mos) \neq \emptyset \quad \text{and} \quad mos = \bigcap_{sr} sr2los(mos).$$

Proof. From (18) and Theorem 1, it follows that $sr2los(mos) \neq \emptyset$ as well as:

$$mos = so2mo(mo2so(mos))$$

$$= so2mo(\bigcap \mo2so(so2los(mo2so(mos))))$$

$$= so2mo(\bigcap \mo2so(sr2los(so2mo(mo2so(mos)))))$$

$$= so2mo(\bigcap \mo2so(sr2los(mos)))$$

$$= so2mo(mo2so(\bigcap sr2los(mos)))$$

$$= \bigcap sr2los(mos),$$

where the domains of mappings are suitably extended. \[\square\]

We can therefore conclude that the saturated extensions of an mo-structure $mos$ form a concurrent history represented by $mos$. It is then important to ask which concurrent histories can be derived in this way; in other words, which concurrent histories can be represented by mo-structures.
Consider now a nonempty set \( \text{LOS} = \{(X, \equiv_i, \subseteq_i) \mid i \in I\} \) of \( \text{LO}\)-structures forming a concurrent history, and their intersection \( S = \bigcap \text{LOS} = (X, \equiv, \subseteq) \). Since every \( \text{LO}\)-structure is also a \( \text{MO}\)-structure, we immediately obtain that \( S \) is an order structure satisfying axioms \( M1, M2, M4 \) and \( M5 \). However, \( M3 \) in general does not hold although it holds for histories satisfying paradigm \( \pi_3 \) (see Section 1), meaning that, for all distinct \( a, b \in X \),

\[
\exists i \in I : (a, b) \in \equiv_i \cap \subseteq_i \quad \implies \quad \exists j \in I : (b, a) \in \equiv_j \cap \subseteq_j .
\]

In other words, histories where the possibility of executing two events in both orders implies the possibility of executing them simultaneously. One can then easily see that \( \bigcap \text{LOS} \) satisfies \( M3 \) iff \( S \) satisfies paradigm \( \pi_3 \). As a result, \( \text{MO}\)-structures provide a complete characterisation of histories satisfying paradigm \( \pi_3 \).

One might now wonder what happens if we do not assume any special properties of a concurrent history. As we will show in the rest of the paper, the situation can be rescued by taking Proposition 2 and immediately observing that it always holds for \( S = \bigcap \text{LOS} \), and so it can supply axioms for order structures underpinning the general concurrent histories.

### 4 Generalised order structures

In this section, we provide a complete characterisation of general concurrent histories where executions are represented by layered order structures; in other words, histories comprising step sequence executions. We achieve this by retaining all those \( \text{MO}\)-structure axioms which hold in general, and then replacing the only dropped axiom \( M3 \) by Proposition 2.

**Definition 3 (generalised mutex order structure).** A generalised mutex order structure (or \( \text{GMO}\)-structure) is a relational structure \( \text{gmos} = (X, =, \sqsubset) \), where \( = \) and \( \sqsubset \) are binary relations on \( X \) such that, for all \( a, b, c, d \in X \):

\[
\begin{align*}
G1 : \quad a = b & \implies b = a \quad & M1 \\
G2 : \quad a \not= a & \quad \land \quad a \not= a \quad & \text{M2} & \text{&} & (15) \\
G3 : \quad a \sqsubset b \sqsubset c & \quad \land \quad a \not= c \quad & \implies \quad a \sqsubset c \\
G4 : \quad a \sqsubset b \sqsubset c & \quad \land \quad (a = b \lor b = c) \quad & \implies \quad a = c \\
G5 : \quad a \sqsubset b \sqsubset a & \quad \land \quad a = c \quad & \implies \quad b = c \quad & \text{M5} \\
G6 : \quad a \sqsubset c \sqsubset b & \quad \land \quad a \sqsubset d \sqsubset b & \quad \land \quad c = d \quad & \implies \quad a = b \\
\end{align*}
\]

**Proposition 5.** The set of axioms in Definition 3 is minimal.

**Proof.** The following are relational structures that satisfy all but one axioms and are not \( \text{GMO}\)-structures (see also Figure 6):

- Not \( G1 \): \( X = \{a, b\} \) and \( a = b \).
Fig. 5. A visualisation of axioms $G5$ and $G6$.

- Not $G2$: $X = \{a\}$ and $a \equiv a$.
- Not $G3$: $X = \{a, b, c\}$ and $a \sqsubseteq b \sqsubseteq c$.
- Not $G4$: $X = \{a, b, c\}$ and $a \sqsubseteq b \sqsubseteq c$ and $a \sqsubseteq c$ and $b \equiv c \equiv b$.
- Not $G5$: $X = \{a, b, c\}$ and $a \sqsubseteq b \sqsubseteq a$ and $a \equiv c \equiv a$.
- Not $G6$: $X = \{a, b, c, d\}$ and $a \sqsubseteq b$ and $a \sqsubseteq cd \sqsubseteq b$ and $c \equiv d \equiv c$. \hfill \square

![Fig. 5 Diagrams]

Fig. 6. A visualisation of counterexamples in Proposition 5.

Proposition 6. Let $gmos = (X, \equiv, \sqsubseteq)$ be a GMO-structure. Then

\[
\begin{align*}
a \sqsubseteq^\wedge b & \Rightarrow a \sqsubseteq b \\
a \sqsubseteq b \sqsubseteq a & \Rightarrow a \not\equiv b.
\end{align*}
\]

Proof. If $a \sqsubseteq^\wedge b$ then there exists a sequence $c_1, \ldots, c_n$ such that $a = c_1$ and $b = c_n$ and, for $i = 1, \ldots, n - 1$, we have $c_i \sqsubseteq c_{i+1}$. Let $c_1, \ldots, c_n$ be the shortest
such sequence. Then $i \neq j$ implies $c_i \neq c_j$. Hence we can $n - 2$ times use axiom $G3$ and obtain $a \sqsubseteq b$.

Moreover, each GMO-structure satisfies axioms $G4$ and $G2$. Hence, as in the proof of Proposition 1, we obtain $a \sqsubseteq b \sqsubseteq a \implies a \neq b$. \hfill $\Box$

**Proposition 7.** Each GMO-structure is separable.

*Proof.* Let $gmos = (X, \sqsubseteq, \sqcap)$ be a GMO-structure. By Proposition 6 and axiom $G2$ we get $(\sqcap \circ \sqsubseteq) = \emptyset$. The reflexivity of $\sqsubseteq$ also follows from Proposition 6, while symmetry of $\sqsubseteq$ from axiom $G1$. \hfill $\Box$

**Proposition 8.** Every MO-structure is a GMO-structure.

*Proof.* Note that axioms $M1, M4$ and $M5$ are equivalent to axioms $G1, G3$ and $G4$, respectively. Moreover, by Proposition 2 and axiom $M2$ we get that every MO-structure satisfy also axioms $G2, G5$ and $G6$. \hfill $\Box$

The converse does not hold; for example, $\{(a, b), \{\{a, b\}, \{b, a\}\}, \emptyset\}$ is a GMO-structure but not an MO-structure, as $M3$ does not hold.

**Proposition 9.** If $gmos = (X, \sqsubseteq, \sqcap)$ is a GMO-structure, then $(X, \sqsubseteq \cap \sqcap, \sqcap)$ is an SO-structure.

*Proof.* Let $\prec'$ denote $\sqsubseteq \cap \sqcap$. Only $S4$ is not obvious. Assume $a \sqsubseteq b \prec' c$. From $G4$ we have $a \sqsupseteq c$. Thus $a \neq c$, and so from $G3$ we have $a \sqsubseteq c$. Hence $a \prec' c$. Similarly for $a \prec' b \sqsubseteq c$. \hfill $\Box$

Proposition 9 states that every GMO-structure is a GSO-structure. We observe that the converse is not true, with suitable counterexamples provided by the GSO-structures $S_{G5}$ and $S_{G6}$ in Figure 6.

### 4.1 Closure operator for generalised mutex order structures

We will now provide a method for deriving valid GMO-structures from other relational structures.

**Definition 4 (GMO-closure).** Let $S = (X, Q, R)$ be a relational structure. Then

\[
S^* = (X, Q^{[R]}, R^*)
\]

is its GMO-closure, where $Q^{[R]} = R^\circ (Q \cup (R^* \circ Q \circ R^*)^\text{sym}) \circ R^\circ$.

The GMO-closure operator introduced here can be seen as related to two different closure operators: (i) the transitive closure operator for acyclic reflexive binary relations; and (ii) the $\triangledown$-operator for $\triangledown$-acyclic order structures introduced in [11] in order to obtain SO-structures. It can also be seen as a generalisation of the GSO-closure introduced in [14] for GSO-acyclic structures in order to obtain GSO-structures.
The main property we want from the notion of $\text{gmo}$-closure is that whenever $S = (X, Q, R)$ is a separable relational structure, $S^\circ$ is a $\text{gmo}$-structure. Furthermore, if $S$ is already a $\text{gmo}$-structure, then we want $S^\circ = S$. The form of $Q[R]$ follows from the requirement that $S^\circ$ should be a $\text{gmo}$-structure and the axioms for $\text{gmo}$-structures. In particular the factor $(R^* \circ_Q R^*)^{\text{sym}}$ follows from axioms $G4$ and $G6$, while the factor $R^* \circ_Q R^*$ corresponds to $G5$.

![Diagram](image)

Fig. 7. A visualisation of the three cases of $(a, b) \in Q[R]$.

The next result corresponds to the property that the transitive closure of an acyclic relation is also acyclic.

**Proposition 10.** If $S$ is separable, then $S^\circ$ is also separable.

**Proof.** Let $S = (X, Q, R)$. We first note that $R^\circ$ is symmetric. Since a composition of symmetric relations is symmetric, we have that $Q[R]$ is symmetric. Moreover, $R^\circ$ is irreflexive by definition.

To prove that $Q[R] \cap (R^\circ)^\circ = \varnothing$, by $(R^\circ)^\circ = R^\circ$, it suffices to show that $Q[R] \cap R^\circ = \varnothing$. Suppose that $(a, b) \in Q[R] \cap R^\circ$. By $(a, b) \in Q[R]$, there are $c, d \in X$ such that $aR^\circ cZdR^\circ b$, where $Z = Q \cup (R^* \circ_Q R^*) \cup (R^* \circ_Q R^*)^{-1}$. We now consider three cases corresponding to the three parts of $Z$ (see Figure 8).

**Case 1:** $(c, d) \in Q$. Then $cR^* aR^\circ bR^\circ d$. Hence, by (8), $(c, d) \in R^\circ$. This, however, contradicts the separability of $S$.

**Case 2:** $(c, d) \in R^* \circ_Q R^*$. Then there is $(e, f) \in Q$ such that $eR^* fR^* d$. Hence $eR^* dR^\circ bR^\circ aR^\circ cR^* fR^* dR^\circ bR^\circ aR^\circ cR^* e$ which gives $(e, f) \in R^\circ$. This, however, contradicts the separability of $S$.

**Case 3:** $(c, d) \in (R^* \circ_Q R^*)^{-1}$. Similar to Case 2. $\square$

The next result shows that $\text{gmo}$-closure is a closure operation in the usual sense. First, however, we prove a technical lemma.

**Lemma 1.** If $S = (X, Q, R)$ is a relational structure, then

$$R^* \circ_Q Q[R] R^* \subseteq R^* \circ Q R^*.$$
Case 1: $\langle a, b \rangle \in R^* \circ_{Q[R]} R^*$. Then there is $\langle c, d \rangle \in Q^{[R]}$ such that $aR^*cdR^*b$. This is equivalent to saying that there are $c, d, e, f \in X$ such that:

$$a R^* cd R^* b \text{ and } c R^\circ e \text{ and } Z f R^\circ d,$$

where $Z = Q \cup (R^* \circ_Q R^*) \cup (R^* \circ_Q R^*)^{-1}$. Thus, by (6) and (8),

$$a R^* cdef R^* b.$$  \hfill (19)

We then consider three cases corresponding to three parts of $Z$ from which the relationship between $e$ and $f$ has been derived (see Figures 7 and 9).

Case 1: $\langle e, f \rangle \in Q$. Then, by (19), $aR^* ef R^* b$. Hence $\langle a, b \rangle \in R^* \circ_Q R^*$.

Case 2: $\langle e, f \rangle \in R^* \circ_Q R^*$. Then there is $\langle g, h \rangle \in Q$ such that $eR^* gh R^* f$. Thus, by (19) and (8), we have $aR^* gh R^* b$. Hence $\langle a, b \rangle \in R^* \circ_Q R^*$.

Case 3: $\langle e, f \rangle \in (R^* \circ_Q R^*)^{-1}$. Then there is $\langle g, h \rangle \in Q$ such that $fR^* gh R^* e$. Thus, by (19) and (8), we have $aR^* gh R^* b$. Hence $\langle a, b \rangle \in R^* \circ_Q R^*$.

Proposition 11. If $S$ is separable, then $S \subseteq S^\circ$ and $(S^\circ)^\circ = S^\circ$.

Proof. Let $S = (X, Q, R)$. The first part follows from the reflexivity of $R^\circ$ and irreflexivity of $R$. To show the second part, by (7), it suffices to prove that $(Q^{[R]})^{[R^*]} = Q^{[R]}$. We first observe that, by (7) and (8), $Q^{[R]} \subseteq (Q^{[R]})^{[R^*]}$ and:

$$\begin{align*}
(Q^{[R]})^{[R^*]} &= (Q^{[R]}) \cup (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \\
&= R^\circ \circ R^\circ \circ Q \circ R^\circ \circ R^\circ \cup \\
&\quad R^\circ \circ R^\circ \circ (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \circ R^\circ \cup \\
&\quad R^\circ \circ (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \\
&\subseteq R^\circ \circ Q \circ R^\circ \cup \\
&\quad R^\circ \circ (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \cup \\
&\quad R^\circ \circ (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \\
&= R^\circ \circ Q \circ R^\circ \cup R^\circ \circ (R^* \circ_{Q[R]} R^*)^{\text{sym}} \circ R^\circ \\
&= Q^{[R]}.
\end{align*}$$

Hence $(Q^{[R]})^{[R^*]} \subseteq Q^{[R]}$, and so $(Q^{[R]})^{[R^*]} = Q^{[R]}$. \hfill $\square$
Case 1:  

Case 2:  

Case 3:  

\( R^* \) (assumed): ---
\( R^* \) (induced): ----
\( Q \) (assumed): 
\( Q \) (induced): 

Fig. 9. A visualisation of the proof of Lemma 1.

The next result corresponds to saying that the transitive closure of an acyclic relation yields a poset.

**Proposition 12.** If \( S \) is separable, then \( S^\bullet \) is a GMO-structure.

*Proof.* Let \( S = (X, Q, R) \). By Propositions 10 and 11, it suffices to show that \( S^\bullet \) satisfies all the axioms in Definition 3 in the case that \( S^\bullet = S \), i.e., \( Q = Q^{[R]} \) and \( R = R^\circ \).

To show \( G1 \), we observe that, by Proposition 10, \( Q \) is symmetric.

To show \( G2 \), we first observe that, by definition, \( \langle a, a \rangle \notin R^\circ = R \). Suppose that \( \langle a, a \rangle \in Q \). Then, since \( \langle a, a \rangle \in R^\circ \), we have \( \langle a, a \rangle \in Q^{[R]} \cap (R^\circ)^\circ \), contradicting Proposition 10.

To show \( G3 \), suppose that \( aRbRc \) and \( a \neq c \). Then \( \langle a, c \rangle \in R^\circ = R \).

To show \( G4 \), suppose that \( aRbRc \) and \( aQc \). Since \( \langle c, c \rangle \in R^\circ \), we obtain that \( \langle b, c \rangle \in R^\circ \circ Q \circ R^\circ \subseteq Q^{[R]} = Q \).

Case 6: \( G6 \). Suppose that \( aRcdRb \) and \( cQd \). Then \( \langle a, b \rangle \in R^\circ \circ Q \circ R^\circ \subseteq Q^{[R]} = Q \). \( \square \)

The next result corresponds to saying that posets are transitively closed.

**Proposition 13.** If \( \text{gmos} \) is a GMO-structure, then \( \text{gmos}^\bullet = \text{gmos} \).

*Proof.* Let \( \text{gmos} = (X, \sqsubseteq, \sqsupseteq) \). By Propositions 6 and 7 we get that \( \text{gmos} \) is separable, and \( \sqsubseteq \subseteq \sqsupseteq^\circ \) and \( a \sqsubseteq^\circ b \) iff \( a \sqsubseteq b \sqsupseteq a \sqsupseteq b \).
To show that $\equiv$ is equal to $\equiv^{[\cdot]}$, we first observe that, by Definition 4 and the reflexivity of $\subseteq$, we have that $\equiv$ is contained in $\equiv^{[\cdot]}$. To show that $\equiv^{[\cdot]}$ is contained in $\equiv$, suppose that $a \equiv^{[\cdot]} b$, which means that there are $c, d \in X$ such that
\[
a \cap c \cap a \lor a = c \quad \text{and} \quad b \cap d \cap b \lor b = d
\] (20)
and one of the following is satisfied:
\[
\begin{align*}
a \cap^\circ c & \Rightarrow d \cap^\circ b \\
a \cap^\circ c \ (\cap^* \circ_{=0} \cap^*) & \ d \cap^\circ b \\
a \cap^\circ c \ (\cap^* \circ_{=0} \cap^*)^{-1} & \ d \cap^\circ b
\end{align*}
\] (21) (22) (23)
If (21) holds then, by (20) and $G5$, we get $a \equiv d$. Hence, by $G1$, $d \equiv a$. Therefore, by (20) and $G5$, $b \equiv a$. Hence, by $G1$, we obtain $a \equiv b$.

If (22) holds, then there are $e \equiv f$ such that $c \cap^* e \cap^* d$. Hence $a \cap^* e \cap^* b$. By Proposition 6, we need to consider sixteen different cases, as $x \cap^* y$ is equivalent to $x \cap y \lor x = y$. Most of them may be excluded, as the roles of $e$ and $f$ are symmetric and, by $G2$, we have $e \neq f$. Moreover, $a \neq b$, as otherwise $e \cap f \cap e$. Hence, together with $e \equiv f$, we get a contradiction with the separability of $\text{gmos}$. As a result we have to consider only four cases.

Case 1: $a = e$ and $b = f$. Then $a \equiv b$.

Case 2: $a = e$ and $b \neq f$. Then $a \cap f \cap b$ and $a \equiv f$. Hence, by $G4$, $a \equiv b$.

Case 3: $a \neq e$ and $b = f$. Then $a \cap e \cap b$ and $e \equiv b$. Hence, by $G4$, $a \equiv b$.

Case 4: $a, b, e$ and $f$ are all distinct. Then $a \cap e f \cap b$. Hence, by $G6$, $a \equiv b$.

Finally, if (23) holds, then $(b, a) \in \cap^\circ \circ(\cap^* \circ_{=0} \cap^*) \cap^\circ$, as $\cap^\circ$ is symmetric. Hence, by (22), we get $b \equiv a$. Thus, by $G1$, we obtain $a \equiv b$. \hfill \Box

As layered order structures and mutex order structures are special cases of generalised mutex order structures, we obtain an immediate

**Corollary 1.** Let $\text{los}$ be an $\text{LO-structure}$ and $\text{mos}$ be an $\text{MO-structure}$. Then $\text{los}^\bullet = \text{los}$ and $\text{mos}^\bullet = \text{mos}$.

The following technical lemma describes a single stage of the saturation process for a GMO-structure leading to an LO-structure. In such a process, we may add an arbitrary link between two elements that do not yet satisfy (14). We only need to remember that in the case of extending the relation $Q$, together with $(a, b)$ we have to add $(b, a)$. After such an addition, we get a separable order structure that may be closed. As a result, we obtain one of possible extensions of an initial $\text{gmos}$. The above process terminates after obtaining an LO-structure and it is central to the proof of the main Theorem 3.

In what follows, we denote
\[
R_{xy} = R \cup \{\langle x, y \rangle\} \quad \text{and} \quad Q_{xy} = Q \cup \{\langle x, y \rangle, \langle y, x \rangle\}.
\]
Lemma 2. Let $gmos = (X, Q, R)$ be a $gmo$-structure, $a, b \in X$ and $a \neq b$. Then

\[
\langle a, b \rangle \notin R \land \langle b, a \rangle \notin R \implies (X, Q, R_{ab}^\ast) \text{ is a } gmo\text{-structure}
\]
\[
\langle a, b \rangle \notin R \land \langle a, b \rangle \notin Q \implies (X, Q, R_{ab}^\ast) \text{ is a } gmo\text{-structure}
\]
\[
\langle a, b \rangle \notin R \land \langle a, b \rangle \notin Q \implies (X, Q_{aba}, R) \text{ is a } gmo\text{-structure}.
\]

Proof. By Proposition 13, $gmos^\ast = gmos$, hence $R = R^\ast$ and $Q = Q^{1[R]}$. To obtain the thesis, it suffices to prove the separability of enriched structures:

\[
\langle a, b \rangle \notin R \land \langle b, a \rangle \notin R \implies Q \cap R_{ab}^\ast = \emptyset \quad (i)
\]
\[
\langle a, b \rangle \notin R \land \langle a, b \rangle \notin Q \implies Q \cap R_{ab}^\ast = \emptyset \quad (ii)
\]
\[
\langle a, b \rangle \notin R \land \langle a, b \rangle \notin Q \implies Q_{aba} \cap R^\ast = \emptyset \quad (iii).
\]

Case (i): By (5) we have $R_{ab}^\ast = R^\ast$, and so $Q \cap R_{ab}^\ast = Q \cap R^\ast = \emptyset$.

Case (ii): If $\langle b, a \rangle \notin R$, then we have Case (i). Hence, we can assume that $\langle b, a \rangle \in R$. Suppose that $\langle c, d \rangle \in Q \cap (R_{ab}^\ast \setminus R^\ast)$. Then, without loss of generality, we may assume that $\langle c, d \rangle \notin R^\ast$. Hence, by (4), we get $\langle c, a \rangle \in R^\ast$ and $\langle b, d \rangle \in R^\ast$.

We now consider two cases:

Case 1: $\langle d, c \rangle \in R^\ast$. Then $bR^\ast dR^\ast a$, and so $\langle a, b \rangle \in Q^{1[R]} = Q$, a contradiction.

Case 2: $\langle d, c \rangle \notin R^\ast$. Then, by $\langle d, c \rangle \in R_{ab}^\ast$, we have $dR^\ast a$ and $bR^\ast c$. Hence $bR^\ast dR^\ast a$, and so $\langle a, b \rangle \in Q^{1[R]} = Q$, a contradiction.

As a result, $Q \cap R_{ab}^\ast = \emptyset$ since $Q \cap R^\ast = \emptyset$.

Case (iii): Suppose that $\langle c, d \rangle \in Q_{aba} \cap R^\ast \neq \emptyset$. Since $Q \cap R^\ast = \emptyset$, we have one of the following three cases:

\[
cR^\ast a \land bR^\ast d \; \text{ or } \; cR^\ast \circ R^\ast \circ abR^\ast \circ R^\ast d \; \text{ or } \; dR^\ast \circ R^\ast \circ abR^\ast \circ R^\ast c.
\]

Then, by $cR^\ast d$, we obtain in each case $aR^\ast b$, contradicting $\langle a, b \rangle \notin R$. \qed

Fig. 10. A visualisation of the proof of Lemma 2.

To complete the properties of the saturation process described in Lemma 2 and used in the proof of Theorem 3, we formulate the following...
Lemma 3. Let $gmos = (X, Q, R)$ be a GMO-structure such that $a, b \in X$, $a \neq b$, $\langle a, b \rangle \notin R$ and $\langle a, b \rangle \notin Q$ and $S' = (X, Q', R_{ab}') = (X, Q', R_{ab}')$. Then $\langle a, b \rangle \notin Q'$.

Proof. We first observe that, by Lemma 2, $S'$ is GMO-structure. Suppose that $\langle a, b \rangle \in Q'$. If $\langle b, a \rangle \in R_{ab}$, then $\langle a, b \rangle \in R_{ab}$, contradicting the separability of $S'$. Hence $\langle b, a \rangle \notin R_{ab}$, and so $\langle b, a \rangle \notin R$. The latter means that $R_{ab}^* = R^*$. We then consider three cases:

If (24) holds then, since $gmos$ is a GMO-structure, we have $\langle a, b \rangle \in Q$, a contradiction.

If (25) holds, then there exists $\langle c, d \rangle \in R^*$ such that $a R_{ab}^* c R_{ab}^* b$. Since $\langle a, b \rangle \notin Q$ and $gmos$ is a GMO-structure, $a R_{ab}^* c R_{ab}^* b$ does not hold. Hence, since the roles of $c$ and $d$ are symmetric, we consider two cases:

Case 1: $\langle a, c \rangle \notin R^*$. Then, by (4), $\langle b, c \rangle \in R^*$. Hence $\langle b, c \rangle \in R_{ab}^* = R^*$. Hence, by $G5$, we obtain $\langle b, d \rangle \in Q$.

If $\langle a, d \rangle \notin R^*$ then, similarly as in the case of $\langle a, c \rangle \notin R^*$, we have $\langle b, d \rangle \in R^*$, which contradicts the separability of $gmos$. Hence $\langle a, d \rangle \in R^*$.

Now, if $\langle d, b \rangle \in R^*$ then $a R_{ab} d R_{ab} b$. Hence, by $G4$, we have $a Q b$, which contradicts our initial assumption. Hence $\langle d, b \rangle \notin R^*$. Thus, by (4), $\langle d, a \rangle \in R^*$. Hence $\langle a, d \rangle \in R_{ab}^* = R^*$, and so by $G5$ we obtain $\langle a, b \rangle \in Q$, yielding a contradiction with our initial assumption.

Case 2: $\langle d, b \rangle \notin R^*$. Then similarly $\langle a, d \rangle \in R^*$ and $\langle b, c \rangle \in R^*$, and so $\langle a, b \rangle \in Q$. Summing up, (25) implies $\langle a, b \rangle \in Q$ and we obtain a contradiction.

If (26) holds, then $\langle b, a \rangle \in R_{ab}^*$, and we obtain a contradiction with $\langle b, a \rangle \notin R_{ab}^*$.

□

Fig. 11. A visualisation of the proof of Lemma 3.

In Lemmas 2 and 3 we have captured a method of saturating GMO-structures that are not lo-structures. It moreover allows us to formulate an immediate
Corollary 2. Every relational structure saturated among all separable relational structures is a layered order structure.

The relationships between the relational structures considered in this paper are depicted in Figure 12.

![Figure 12](image)

**Fig. 12.** Relationships between different relational structures and their properties: SEP are separable relational structures, CLO are relational structures closed in the sense of Definition 4, GMOS are GMO-structures, GSOS are GSO-structures, MOS are MO-structures, SOS are SO-structures, SO are stratified orders, and SAT are relational structures saturated among SEP.

### 4.2 General concurrent histories

We now return to our original goal which was to provide a structural characterisation of all histories comprising step sequence executions. Recall that $\text{sr2los}(\text{gmos})$ is the set of all LO-structures associated with a GMO-structure $\text{gmos}$. Then we obtain a result corresponding to Szpirajn’s Theorem:

**Theorem 3.** For every GMO-structure $\text{gmos}$,

$$\text{sr2los}(\text{gmos}) \neq \emptyset \quad \text{and} \quad \text{gmos} = \bigcap \text{sr2los}(\text{gmos}).$$

**Proof.** Let $\mathcal{F} = \text{sr2los}(\text{gmos})$. The first part is nothing but Corollary 2. Let $\text{gmos} = (X, =, \sqsubset)$. We will denote $S = (X, =_S, \sqsubset_S)$, for any layered extension $S$ of $\text{gmos}$.

Since $\mathcal{F}$ is the set of all layered extensions of $\text{gmos}$, we know that $\text{gmos} \subseteq S$, for all $S \in \mathcal{F}$. Hence $\text{gmos} \subseteq \bigcap_{S \in \mathcal{F}} S$. We need to shown the reverse inclusion.

We start by showing that $\bigcap_{S \in \mathcal{F}} \sqsubset_S$ is included in $\sqsubset$. Suppose that $a \not\sqsubset b$. We will now define two auxiliary GMO-structures, $\text{gmos}'$ and $\text{gmos}''$, in the following way. If $a = b$ then $\text{gmos}' = \text{gmos}$. Otherwise,

$$\text{gmos}' = (X, =_{aba}, \sqsubset) \quad \text{and} \quad (X, =_{aba}, \sqsubset)$$
is a GMO-structure, by Lemma 2. If \( b \sqsupset a \) then \( gmos'' = gmos' \). Otherwise,

\[
gmos'' = (X, \geq_{ab} \sqsubseteq \sqsubset_{ba})
\]
is a GMO-structure, by Lemma 2. Let \( gmos'' = (X, \geq'', \sqsubseteq'') \).

We have \( a \Rightarrow'' b \sqsupset'' a \). As a result, for every layered extension \( S \) of \( gmos'' \), we get \( a \Rightarrow_S b \sqsubseteq_S a \). Hence, by (14), we have that \( a \not\sqsubseteq_S b \). Moreover, by \( gmos \subseteq gmos' \subseteq gmos'' \), each layered extension of \( gmos'' \) is also a layered extension of \( gmos \). Consequently, \( (a, b) \) is not included in \( \bigcap_{S \in \mathcal{F}} \sqsubseteq_S \), and so the latter is a subset of \( \sqsubseteq \).

Next we show that \( \bigcap_{S \in \mathcal{F}} \Rightarrow_S \) is included in \( \Rightarrow \). Suppose that \( a \not\Rightarrow b \). We will again define two auxiliary GMO-structures, \( gmos' \) and \( gmos'' \), in the following way. If \( a \sqsubseteq b \) then \( gmos' = gmos \). Otherwise

\[
gmos' = (X, \Rightarrow, \sqsubseteq_{ab})
\]
is a GMO-structure, by Lemma 2. Let \( gmos' = (X, \Rightarrow', \sqsubseteq') \). We observe that, by Lemma 3, \( (a, b) \not\Rightarrow' \). If \( b \sqsubseteq' a \) then \( gmos'' = gmos' \). Otherwise

\[
gmos'' = (X, \Rightarrow', \sqsubseteq'_{ba})
\]
is a GMO-structure, by Lemma 2. Let \( gmos'' = (X, \Rightarrow'', \sqsubseteq'') \).

We have \( a \sqsubseteq'' b \sqsubseteq'' a \). As a result, for every layered extensions \( S \) of \( gmos'' \), we get \( a \sqsubseteq_S b \sqsubseteq_S a \). Hence, by (14), we have that \( a \not\sqsubseteq_S b \). Moreover, by \( gmos \subseteq gmos' \subseteq gmos'' \), each layered extension of \( gmos'' \) is also a layered extension of \( gmos \). Consequently, \( (a, b) \) is not included in \( \bigcap_{S \in \mathcal{F}} \Rightarrow_S \), and so the latter is a subset of \( \Rightarrow \).

Together with the fact that, for every nonempty set \( LOS \) of LO-structures with the same domain, \( \bigcap LOS \) is a GMO-structure, it leads to the conclusion that all concurrent histories in \( \pi_1 \) are represented by GMO-structures.

### 5 Concluding remarks

We can finally clarify the relationship between GSO-structures and GMO-structures. In general, in order to accept an order structure \( os = (X, \Rightarrow, \sqsubseteq) \) as an invariant representation of a concurrent history, we require that

\[
\text{sr2los}(os) \neq \emptyset \quad \text{and} \quad os = \bigcap \text{sr2los}(os).
\]

We demonstrated that this property holds whenever \( os \) is a GMO-structure, and that it may fail to hold for a GSO-structure. We have further shown that GMO-structures are GSO-structures, but that the converse does not hold. However, what is the case is that each GSO-structure \( gsos \) is separable, and so its GMO-closure \( gsos^\bullet \) is a GMO-structure satisfying \( \text{sr2los}(gsos^\bullet) = \text{sr2los}(gsos) \). In other
words, concurrent histories described by separable order structures and their GMO-closures are the same. The importance of GSO-structures comes from the fact that they paved the way for GMO-structures, by exposing the fundamental property that causal ordering is a combination of mutex and weak ordering.

A key motivation for the research presented in this paper comes from concurrent behaviours as exhibited by safe Petri nets with mutex arcs. The resulting semantical approach is based on GMO-structures which characterise all concurrent histories comprising step sequence executions. A natural direction for further work is to provide a compatible language-theoretic representation of concurrent histories, by generalising Mazurkiewicz traces [2] which correspond to causal posets, and comtraces [11] which correspond to SO-structures (or MO-structures). This development would also allow to link the dynamic notions of mutex and weak causality with the static properties of Petri nets with mutex arcs. Apart from the original motivation, an interesting extension of GMO-structures could be to allow executions where steps are multisets rather than sets. Such an extension would lead to the generalisation of causal pomsets, which are a useful tool in describing and verifying concurrent systems.

Acknowledgement

This research was supported by a fellowship funded by the “Enhancing Educational Potential of Nicolaus Copernicus University in the Disciplines of Mathematical and Natural Sciences” Project POKL.04.01.00-00-081/10, the EPSRC GAELS and UNCOVER projects, and by an NSERC of Canada grant.

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