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Embedddings between partially commutative groups: two counterexamples☆

Montserrat Casals-Ruiz[a,1,2], Andrew Duncan[b,3], Ilya Kazachkov[a,2,4]

[a] *Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford, OX1 3LB, UK*

[b] *School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne, NE1 7RU, United Kingdom*

Abstract

In this note we give two examples of partially commutative subgroups of partially commutative groups. Our examples are counterexamples to the Extension Graph Conjecture and to the Weakly Chordal Conjecture of Kim and Koberda, [KK]. On the other hand we extend the class of partially commutative groups for which it is known that the Extension Graph Conjecture holds, to include those with commutation graph containing no induced C_4 or P_3. In the process, some new embeddings of surface groups into partially commutative groups emerge.

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1. Introduction

Partially commutative groups are a class of groups widely studied on account of their simple definition, their intrinsically rich structure and their natural appearance in several branches of computer science and mathematics. Crucial examples, which shape the theory of presentations of groups, arise from study of their subgroups: notably Bestvina and Brady's example of a group which is homologically finite (of type FP) but not geometrically finite (in fact not of type F_2); and Mihailova's example of a group with unsolvable subgroup membership problem. More recently results of Wise and others have lead to Agol's proof of the virtual Haken conjecture: that every hyperbolic Haken 3-manifold is virtually fibred. An essential step in the argument uses the result that the fundamental groups of so-called "special" cube complexes embed into partially commutative groups. (It turns out that the manifolds in question have finite covers which are special.) In the light of such

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Email addresses: montsecasals@gmail.com (Montserrat Casals-Ruiz), andrew.duncan@ncl.ac.uk (Andrew Duncan), ilya.kazachkov@gmail.com (Ilya Kazachkov)

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results it is natural to ask which groups arise as subgroups of partially commutative groups, and, in particular, when one partially commutative group embeds in another.

Let Γ be a finite (undirected) simple graph, with vertex set X and edge set E, and let $F(X)$ be the free group on X. For elements g, h of a group denote the commutator $g^{-1}h^{-1}gh$ of g and h by $[g, h]$. Let

$$R = \{[x, y] \in F(X) \mid x, y \in X \text{ and } \{x, y\} \in E\}.$$

Then the *partially commutative (pc) group with commutation graph* Γ is the group $\mathbb{G}(\Gamma)$ with presentation $\langle X \mid R\rangle$. The complement of a graph Γ is the graph $\overline{\Gamma}$ with the same vertex set as Γ and an edge joining vertices u and v if and only if there is no such edge in Γ. The partially commutative group $\mathbb{G}(\Gamma)$ is said to have *non-commutation graph* $\overline{\Gamma}$.

Note that the graph Γ uniquely determines the partially commutative group up to isomorphism; that is two pc groups $\mathbb{G}(\Gamma)$ and $\mathbb{G}(\Lambda)$ are isomorphic if and only if Γ is isomorphic to Λ, see [D87.3]. Since isomorphism of pc groups can be characterised in terms of their defining graphs is natural to ask for an analogous characterisation of embedding between pc groups.

Question 1 (Problem 1.4 in [CSS])**.** Is there a natural graph theoretic condition which determines when one partially commutative group embeds in another?

If Λ and Γ are graphs such that Λ is isomorphic to a full subgraph of Γ, then Λ is said to be an *induced subgraph* of Γ and we write $\Lambda \leq \Gamma$. If Λ is not an induced subgraph of Γ then Γ is said to be Λ-*free*. It is easy to see that if Λ is an induced subgraph of Γ then $\mathbb{G}(\Lambda) \leq \mathbb{G}(\Gamma)$. However, unless Γ is a complete graph, $\mathbb{G}(\Gamma)$ will contain subgroups which do not correspond to induced subgraphs in this way. Moreover, in general pc groups contain subgroups which are not pc groups. In fact Droms proved [D87.2] that every subgroup of a partially commutative group $\mathbb{G}(\Gamma)$ is partially commutative if and only if Γ is both C_4 and P_3 free, where C_n is the cycle graph on n vertices and P_n is the path graph with n edges.

Significant progress towards answering Question 1 has been made by Kim and Koberda [KK], exploiting the notion of the extension of a graph. The *extension graph* Γ^e of a graph Γ is the graph with vertex set $V^e = \{g^{-1}xg \in \mathbb{G}(\Gamma) \mid x \in V(\Gamma), g \in \mathbb{G}(\Gamma)\}$ and an edge joining u to v if and only if $[u, v] = 1$ in $\mathbb{G}(\Gamma)$. In [KK] it is shown that if Λ is a subgraph of Γ^e then $\mathbb{G}(\Lambda)$ embeds in $\mathbb{G}(\Gamma)$. However, the converse is shown to hold only if Γ is triangle-free (C_3 free): in which case if $\mathbb{G}(\Lambda)$ embeds in $\mathbb{G}(\Gamma)$, then Λ is a subgraph of Γ^e ([KK, Theorem 10]). This suggests the following conjecture.

Conjecture 1. The Extension Graph Conjecture, [KK, Conjecture 4]] Let Γ_1 and Γ_2 be finite graphs. Then $\mathbb{G}(\Gamma_1) < \mathbb{G}(\Gamma_2)$ if and only if $\Gamma_1 < \Gamma_2^e$.

In Section 5 we extend the class of graphs for which Conjecture 1 is known to hold, to include in addition to the triangle-free graphs, a family lying at the opposite end of the spectrum; being "built out of triangles". More precisely, we generalise one direction of Droms' theorem to prove that if Γ is both C_4 and P_3 free, then the Extension Graph Conjecture holds for $\mathbb{G}(\Gamma)$. However, in Section 3 below we give a counterexample to the Extension Graph Conjecture. We note that Droms also showed that a pc group is coherent if and only if it is chordal [D87.1], i.e. C_n-free, for all $n \geq 4$. Our example in Section 3, shows that there are chordal graphs, and so coherent pc groups, for which the Extension Graph Conjecture fails.

As mentioned above, one reason that so much attention has been given to pc groups is that several classes of important groups embed into them. For example, motivated by 3-manifold theory, there has been a body of research investigating which partially commutative groups contain closed hyperbolic surfaces as subgroups (that is, fundamental groups of closed surfaces of Euler characteristic less than -1; see [CSS] and the references there). In [DSS] it was shown that many orientable surface groups embed into pc groups and subsequently Crisp and Wiest [CW] showed that the fundamental group of every compact surface embeds into some pc group, except in the cases of the compact, closed, non-orientable, surfaces of Euler characteristic $-1, 0$ and 1, which do not embed in any pc group. As observed in [R07], every orientable surface group in fact embeds in $\mathbb{G}(C_5)$. In Section 4 we establish similar results for the complement $\overline{P}_7$ of P_7, in place of C_5.

Although being an induced subgraph is not a necessary condition for embeddability, for some particular induced subgraphs it is a useful criterion. For instance, a pc group $\mathbb{G}(\Gamma)$ contains $\mathbb{G}(P_3)$ (resp. C_4), if and only if P_3 (resp. C_4) is an induced subgraph of Γ, see [Kambites, KK]. In [GLR], the authors ask if the presence of an induced n-cycle, $n \geq 5$, in the defining graph Γ is a necessary condition for the group $\mathbb{G}(\Gamma)$ to contain a hyperbolic surface group. However, counterexamples were given in [K08] and [CSS]. For example, the fundamental group of the closed, compact, orientable surface of genus 2 embeds in the pc group $\mathbb{G}(P_2(6))$, although $P_2(6)$ has no induced n-cycle, for $n \geq 5$. Note however that in the case when the graph Γ is triangle-free, $\mathbb{G}(C_n) \leq \mathbb{G}(\Gamma)$ for some $n \geq 5$ if and only if $C_m \leq \Gamma$ for some $5 \leq m \leq n$, see [KK, Corollary 48].

In spite of these examples, the question raised by Gordon, Long and Reid survives in the following form.

Question 2. Is there a partially commutative group $\mathbb{G}(\Gamma)$ that contains the fundamental group of a closed hyperbolic surface but contains no subgroup isomorphic to $\mathbb{G}(C_n)$, for $n \geq 5$?

In this weaker form, the question is still open and emphasises the importance of resolving Question 1 in the particular case when Λ is a n-cycle, $n \geq 5$.

One approach to Question 2 is to prove that certain classes of graphs give rise to pc groups containing no subgroups $\mathbb{G}(C_n)$, $n \geq 5$. A possible candidate is the class of weakly chordal graphs: a graph Γ is called *weakly chordal* if Γ does not contain an induced C_n or $\overline{C}_n$ for $n \geq 5$, where $\overline{C}_n$ denotes the complement of C_n. Since the extension graph of a weakly chordal graph is weakly chordal, see [KK, Lemma 30], one can weaken Conjecture 1 as follows.

Conjecture 2 (The Weakly Chordal Conjecture, [KK, Conjecture 13])**.** If Γ is a weakly chordal graph, then $\mathbb{G}(\Gamma)$ does not contain $\mathbb{G}(C_n)$ for any $n \geq 5$.

If this were true then, as $P_2(6)$ is weakly chordal, and contains hyperbolic surface groups, we would have a positive answer to Question 2. However, our second example, in Section 4, gives a counterexample to the Weakly Chordal Conjecture.

2. Preliminaries

2.1. Partially commutative groups

By *graph* we mean simple, undirected graph. Let Γ be a finite graph with vertices X and let $\mathbb{G}(\Gamma)$ be the partially commutative group with commutation graph Γ.

If w is a word in the free group on X then we say that w is *reduced in* $\mathbb{G}(\Gamma)$ if w has minimal length amongst all words representing the same element of $\mathbb{G}(\Gamma)$ as w. If w is reduced in $\mathbb{G}(\Gamma)$ then we define $\mathrm{alph}_\Gamma(w)$ to be the set of elements of X such that x or x^{-1} occurs in w. It is well-known that all words which are reduced in $\mathbb{G}(\Gamma)$ and represent a particular element g of $\mathbb{G}(\Gamma)$ have the same length, and that if $w = w'$ in $\mathbb{G}(\Gamma)$ then $\mathrm{alph}_\Gamma(w) = \mathrm{alph}_\Gamma(w')$.

For a word w in the free monoid on $X \cup X^{-1}$ we write $\mathrm{alph}(w)$ for the set of elements of X such that x or x^{-1} occurs in w. We write $u \doteq v$ to indicate that u and v are equal as words in the free monoid on $X \cup X^{-1}$.

2.2. Van Kampen diagrams

By van Kampen's Lemma (see [BSR07]) the word w represents the trivial element in a fixed group G given by the presentation $\langle A \mid R\rangle$ if and only if there exists a finite connected, oriented, based, labelled, planar graph $\mathcal{D}$, where each oriented edge is labelled by a letter in $A^{\pm 1}$, each bounded region (2-*cell*) of $\mathbb{R}^2 \setminus \mathcal{D}$ is labelled by a word in R (up to shifting cyclically or taking inverses) and w can be read on the boundary of the unbounded region of $\mathbb{R}^2 \setminus \mathcal{D}$ from the base vertex. Then we say that $\mathcal{D}$ is a *van Kampen diagram* for the boundary word w over the presentation $\langle A \mid R\rangle$. Note that any van Kampen diagram can also be viewed as a 2-complex, with a 2-cell attached for each bounded region. A van Kampen diagram for the word w is *minimal* if no other van Kampen diagram for w has fewer 2-cells.

Following monograph [O], if we complete the set of defining relations adding the trivial relations $1 \cdot a = a \cdot 1$ for all $a \in A$, then every van Kampen diagram can be transformed so that its boundary is a simple curve. In other words, as a 2-complex the van Kampen diagram is homeomorphic to a disc tiled by 2-cells which are also homeomorphic to a disc. We shall assume that all van Kampen diagrams are of this form.

From now on we shall restrict our considerations to the case when G is a partially commutative group. Let $\mathcal{D}$ be a minimal van Kampen diagram for the boundary word w. Given an occurrence of a letter $a \in A^{\pm 1}$ in w, there is a 2-cell C, in the 2-complex $\mathcal{D}$, attached to a. Since every 2-cell in a van Kampen diagram is either labelled by a relation of the form $a^{-1}b^{-1}ab$ or is a *padding* 2-cell, i.e. a 2-cell labelled by $1 \cdot a \cdot 1 \cdot a^{-1}$, there is just one occurrence of a and one occurrence of a^{-1} on the boundary of C.

Since $\mathcal{D}$ is homeomorphic to a disc, if the occurrence of a^{-1} on the boundary of C is not on the boundary of $\mathcal{D}$, there exists a unique 2-cell $C' \neq C$ attached to this occurrence of a^{-1} in $\mathcal{D}$. Repeating this process, we obtain a unique *band* in $\mathcal{D}$.

Because of the structure of the 2-cells and the fact that $\mathcal{D}$ is homeomorphic to a disc, a band never self-intersects in a 2-cell; indeed, since $\mathcal{D}$ is homeomorphic to a disc, a 2-cell corresponding to a self-intersection of a band would be labelled by the word $aaa^{-1}a^{-1}$, a contradiction.

Thus, since the number of 2-cells in $\mathcal{D}$ is finite, in a finite number of steps the band will again meet the boundary in an occurrence of a^{-1} in w.

We will use the notation L_a to indicate that a band begins (and thus ends) in an occurrence of a letter $a \in A^{\pm 1}$. Fix an orientation of the boundary of $\mathcal{D}$ so that the label of the boundary, read from the base vertex, with this orientation, is the word w. The band L_a determines a *boundary subword* $w(L_a)$ of w: namely the subword of w that is read on the boundary between the ends of L_a, in the direction of fixed orientation. In other words, every band L_a induces the following decomposition of the boundary word w:

$$w = w_1 a^{\epsilon} w(L_a) a^{-\epsilon} w_2,$$

where $\epsilon \in \{\pm 1\}$.

Furthermore the band L_a determines a *band word* $z(L_a)$: namely the word obtained by reading along the boundary of the band from the terminal end of a^{ϵ} to the initial end of $a^{-\epsilon}$.

Note that if two bands L_a and L_b cross then the intersection 2-cell realises the equality $a^{-1}b^{-1}ab = 1$ and so $a \neq b$ and $[a, b] = 1$. It follows that if a letter c occurs in $w(L_a)$ and $c = a^{\pm 1}$ or $[c, a] \neq 1$, then the ends of the band L_c both occur in $w(L_a)$, i.e. the band L_c is contained in the region bounded by $z(L_a)$, and $w(L_a)$. Notice that since $z(L_a)$ labels the boundary of the band L_a, it follows that every letter $b \in z(L_a)$ we have that $b \neq a^{\pm 1}$ and $[b, a] = 1$.

The following will be useful in the next section. If L_u is a band, where $u = a^{\pm 1}$, for some $a \in A$ such that $a \notin \mathrm{alph}(w(L_u))$, then we say that L_u is an *outside band.*

Lemma 2.1. *Let $\mathcal{D}$ be a van Kampen diagram for the boundary word w.*

1. *If $x \in \mathrm{alph}(w)$ then there exists an outside band L_u, $u = x^{\pm 1}$.*
2. *Let L_v be a band, where $v \in A^{\pm 1}$. If $y \in \mathrm{alph}(w(L_v))$, such that $[y, v] \neq 1$ or $y = v^{\pm 1}$, then there exists an outside band L_z, with $z = y^{\pm 1}$, which has both ends in $w(L_v)$.*

Proof.

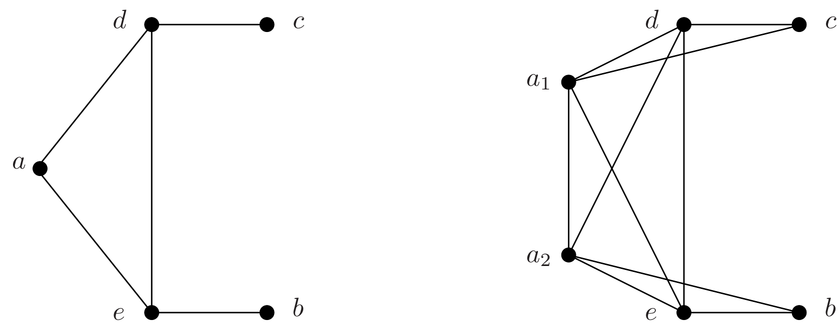
1. Amongst all bands beginning and ending in $x^{\pm 1}$, choose a band L_u, $u = x^{\pm 1}$, such that the length of $w(L_u)$ is minimal. As no two bands with ends $x^{\pm 1}$ can cross it follows that $x \notin \mathrm{alph}(w(L_u))$. Hence L_u is an outside band.
2. As $y \in \mathrm{alph}(w(L_v))$, there exists a band L with one end at an occurrence of $y^{\pm 1}$ in $w(L_v)$. As $[y, v] \neq 1$ or $y = v^{\pm 1}$, bands L_v and L cannot cross, so L has both ends in $w(L_v)$. That is, every band with one end at an occurrence of $y^{\pm 1}$ in $w(L_v)$ has both ends in $w(L_v)$. Now choose such a band L' with $|w(L')|$ minimal over lengths of boundary words of all bands ending at occurrences of $y^{\pm 1}$ in $w(L_v)$. As in the proof of the first part of the Lemma, L' is an outside band.

□

3. A counterexample to the Extension Graph Conjecture

In this Section we give a counterexample to Conjecture 1. Let Γ_1 and Γ_2 be the graphs of Figure 1a and Figure 1b, respectively. Let $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ be the corresponding partially commutative groups:

$$\mathbb{G}(\Gamma_1) = \langle a, b, c, d, e \mid [a, d], [a, e], [b, e], [c, d], [d, e] \rangle,$$
$$\mathbb{G}(\Gamma_2) = \langle a_1, a_2, b, c, d, e \mid [a_1, a_2], [a_1, c], [a_1, d], [a_1, e], [a_2, b], [a_2, d], [a_2, e], [b, e], [c, d], [d, e] \rangle.$$



(a) Γ_1 (b) Γ_2

Figure 1

Define a map

$$\varphi = \begin{cases} a \mapsto a_1a_2; \\ b \mapsto b; \\ c \mapsto c; \\ d \mapsto d; \\ e \mapsto e. \end{cases}$$

Lemma 3.1. *The map φ defined above extends to a monomorphism $\varphi : \mathbb{G}(\Gamma_1) \to \mathbb{G}(\Gamma_2)$.*

Proof. It is immediate that

$$\varphi([a,d]) = \varphi([a,e]) = \varphi([b,e]) = \varphi([c,d]) = \varphi([d,e]) = 1$$

in $\mathbb{G}(\Gamma_2)$, hence φ extends to a homomorphism.

We show that φ is injective. For a contradiction assume that $\ker(\varphi)$ contains a non-trivial element g and let $w = w(a,b,c,d,e) \in \mathbb{G}(\Gamma_1)$ be a minimal length word representing g. Consider a minimal van Kampen diagram $\mathcal{D}$ for $\varphi(w) = w(a_1a_2, b, c, d, e) = 1$.

Since the canonical parabolic subgroup generated by $\{b,c,d,e\}$ is isomorphic to its image by φ, it follows that the word w must contain at least one occurrence of the letter a (and its inverse a^{-1}). Since $\varphi(a) = a_1a_2$, $\varphi(w)$ contains at least one occurrence of a_1 and of a_2 and every occurrence of $a_1^{\pm1}$ (or $a_2^{\pm1}$) in $\varphi(w)$ occurs in a subword $(a_1a_2)^{\pm1}$. Moreover, if w has subword of the form $(a_1a_2)^{\epsilon}u(a_1a_2)^{-\epsilon}$, with $\epsilon \in \{\pm1\}$ and $a_1 \notin \mathrm{alph}(u)$, then it follows that $a_2 \notin \mathrm{alph}(u)$ (and if $a_2 \notin \mathrm{alph}(u)$ then $a_1 \notin \mathrm{alph}(u)$).

We claim that if L_x is an outside band, with $x = a_1^{\pm1}$, then there exists a band L with one end at an occurrence of $c^{\pm1}$ in $w(L_x)$ and its other end not in $w(L_x)$. Similarly, if L_y is an outside band, with $y = a_2^{\pm1}$, then there exists a band L with one end at an occurrence of $b^{\pm1}$ in $w(L_y)$ and its other end not in $w(L_y)$. As there is an isomorphism of Γ_2 interchanging a_1 and a_2 and b and c, it suffices to prove the first of these claims.

To see that the first claim holds, suppose that L_x is an outside band, with $x = a_1^{\pm1}$. Without loss of generality (replacing w with w^{-1} if necessary), we may assume that $x = a_1$. Then the band L_x induces the decomposition $\varphi(w) \doteq w_1a_1a_2w_2a_2^{-1}a_1^{-1}w_3$ and, since $a_1 \notin \mathrm{alph}(w_2)$, it follows that

$a_1, a_2 \notin \mathrm{alph}(w_2)$. Then $\mathrm{alph}(w_2)$ must contain either b or c. Indeed, otherwise, since w_2 belongs to the subgroup generated by $\{b, c, d, e\}$ and so $w_2 = \varphi(w_2)$, the subword aw_2a^{-1} of w would not be reduced in $\mathbb{G}(\Gamma_1)$, a contradiction. Also, as $w_2 = \varphi(w_2)$, the word w_2 is reduced in $\mathbb{G}(\Gamma_2)$.

Assume to begin with that $\mathrm{alph}(w_2)$ does not contain c; so does contain b. In this case, since $[a_1, b] \neq 1$, Lemma 2.1.2 implies the existence of an outside band L_z, where $z = b^{\pm 1}$, with both ends in w_2. As L_z is an outside band $b \notin \mathrm{alph}(w(L_z))$ and hence, since $w(L_z)$ is a subword of w_2, we have $\mathrm{alph}(w(L_z)) \subseteq \{d, e\}$. Moreover, w_2 is reduced in $\mathbb{G}(\Gamma_2)$ so $w(L_z)$ must contain an occurrence $d^{\pm 1}$. From Lemma 2.1.2 again, there is an outside band L_v, where $v = d^{\pm 1}$, with both ends in $w(L_z)$. However, this means that $\mathrm{alph}(w(L_v)) \subseteq \{e\}$, and so w_2 is not reduced in $\mathbb{G}(\Gamma_2)$, a contradiction.

Therefore we conclude that $c \in \mathrm{alph}(w_2)$ and there is a band L_u, where $u = c^{\pm 1}$, with one end in w_2. If both ends of L_u are in w_2 then there exists an outside band L_z, $z = c^{\pm 1}$, with both ends in w_2 (Lemma 2.1.2). As w_2 is reduced in $\mathbb{G}(\Gamma_2)$, $\mathrm{alph}(w(L_z))$ must contain either b or e. If $b \in \mathrm{alph}(w(L_z))$ then, since $c \notin \mathrm{alph}(w(L_z))$, we obtain a contradiction, using the argument of the previous paragraph. Therefore $b \notin \mathrm{alph}(w(L_z))$ and $e \in \mathrm{alph}(w(L_z))$. This implies the existence of an outside band L_v, $v = e^{\pm 1}$, with both ends in $w(L_z)$. However, in this case $\mathrm{alph}(w(L_v)) \subseteq \{d\}$, and again w_2 is not reduced in $\mathbb{G}(\Gamma_2)$, a contradiction. Thus one end of L_u lies in w_1 or w_3. This proves the claim, for L_x, with $L = L_u$.

Returning to the proof of the lemma, let L_{x_1} be an outside band, with $x_1 = a_1^{\pm 1}$, and let $w = w_1(a_1a_2)^\epsilon w_2 (a_1a_2)^{-\epsilon} w_3$, be the corresponding decomposition of w, where $w(L_{x_1}) = w_2$, if $\epsilon = 1$ and $w(L_{x_1}) = a_2^{-1} w_2 a_2$, otherwise. From the claim above, there exists a band L_{c_1}, where $c_1 = c^{\pm 1}$, with one end in w_2 and the other end in w_1 or w_3. Taking a cyclic permutation of w (and $\varphi(w)$), if necessary, we may assume that L_{c_1} has its other end in w_1.

Let L' be the band which has one end at the occurrence of $a_2^{\pm 1}$ in $(a_1a_2)^\epsilon w_2$. Since $[a_2, c] \neq 1$ the bands L_{c_1} and L' cannot cross and, as $a_2 \notin \mathrm{alph}(w_2)$, the other end of L' must lie in w_1 (see Figure 2). Now we find an outside band L_{y_1}, where $y_1 = a_2^{\pm 1}$, with both ends in $w(L')$. Hence w_1 decomposes as

$$w_1 \doteq w_{11} c^{-\delta} w_{12} a_2^{-\epsilon} w_{13} a_2^{\gamma} w_{14} a_2^{-\gamma} w_{15},$$

for some $\gamma, \delta \in \{\pm 1\}$, as shown in Figure 2. Thus, the claim above implies there exists a band L_{b_1}, where $b_1 = b^{\pm 1}$, with exactly one end in $w(L_y) = w_{14}$. As $[b, c] \neq 1$ and $[b, a_1] \neq 1$ the band L_{b_1} cannot cross L_{c_1} or L_x, so the other end of L_{b_1} lies in in w_{12}, w_{13} or w_{15}. In particular $w(L_{b_1})$ is a subword of $w(L_{c_1})$. Moreover, since $w(L_{b_1})$ contains an occurrence of $a_2^{\pm 1}$ we have $a_1, a_2 \in \mathrm{alph}(w(L_{b_1}))$.

Now suppose that we have occurrences of letters $c_1, \ldots, c_n$, $b_1, \ldots, b_n$, with $c_i = c^{\pm 1}$ and $b_i = b^{\pm 1}$, and corresponding bands L_{c_i}, L_{b_i}, such that $w(L_{b_i})$ is a subword of $w(L_{c_i})$, for $i = 1, \ldots, n$, $w(L_{c_{i+1}})$ is a subword of $w(L_{b_i})$, for $i = 1, \ldots, n-1$, and $a_1, a_2 \in w(L_{b_n})$.

As $[a_1, b] \neq 1$ there exists an outside band $L_{x_{n+1}}$, where $x_{n+1} = a_1^{\pm 1}$, with both ends in $w(L_{b_n})$. Thus $w(L_{b_n})$ decomposes as

$$w(L_{b_n}) \doteq u_1 (a_1a_2)^\zeta u_2 (a_1a_2)^{-\zeta} u_3,$$

where $L_{x_{n+1}}$ has both ends at occurrences of $a_1^{\pm 1}$ in $(a_1a_2)^\zeta u_2 (a_1a_2)^{-\zeta}$, for some $u_2 \in \langle b, c, d, e \rangle$ and $\zeta \in \{\pm 1\}$. From the claim above, there is a band $L_{c_{n+1}}$, where $c_{n+1} = c^{\pm 1}$, with one end in u_2

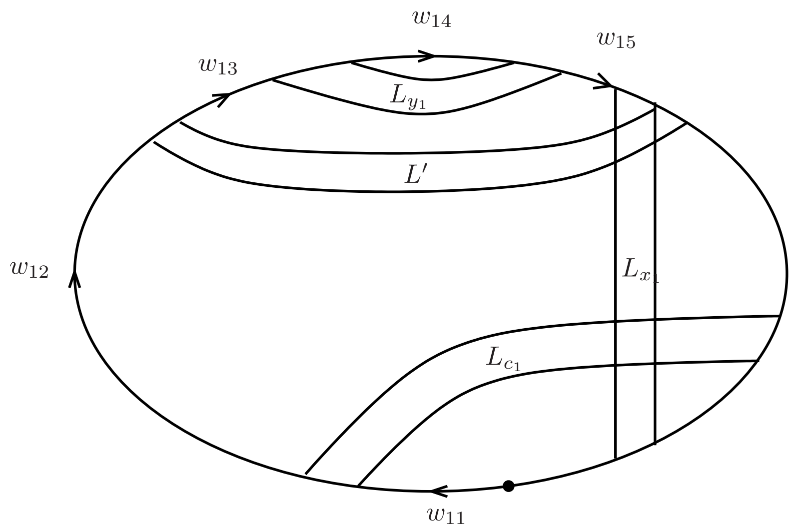


Figure 2

and the other end not in u_2. As $[b,c] \neq 1$, the other end of $L_{c_{n+1}}$ lies in u_1 or u_3. In particular, $w(L_{c_{n+1}})$ is a subword of $w(L_{b_n})$.

If $L_{c_{n+1}}$ has one end in u_1 then let L'' be the band which has one end at the occurrence of $a_2^{\pm 1}$ in $(a_1a_2)^\zeta u_2$, as shown in Figure 3. Then $w(L'')$ is a subword of $u_1(a_1a_2)^\zeta$ and there exists an outside band $L_{y_{n+1}}$, where $y_{n+1} = a_2^{\pm 1}$, with both ends in $w(L'')$.

From the claim again, we then have a band $L_{b_{n+1}}$, where $b_{n+1} = b^{\pm 1}$, with one end in $w(L_{y_{n+1}})$ and the other end outside $w(L_{y_{n+1}})$. As $[b,c] \neq 1$, it follows that the other end of $L_{b_{n+1}}$ lies in $w(L_{c_{n+1}})$, so $w(L_{b_{n+1}})$ is a subword of $w(L_{c_{n+1}})$.

If $L_{c_{n+1}}$ has one end in u_3 then let L'' be the band which has one end at the occurrence of $a_2^{\pm 1}$ in $u_2(a_1a_2)^{-\zeta}$; see Figure 4. A similar argument shows that we can find a band $L_{y_{n+1}}$, where $y_{n+1} = a_2^{\pm 1}$, with both ends in $w(L'')$, and then a band $L_{b_{n+1}}$, where $b_{n+1} = b^{\pm 1}$, with exactly one end in $w(L_{y_{n+1}})$ and the other end in $w(L_{c_{n+1}})$, so in this case as well, $w(L_{b_{n+1}})$ is a subword of $w(L_{c_{n+1}})$. As only one end of $L_{b_{n+1}}$ lies in $w(L_{y_{n+1}})$ it follows, as above, that $a_1, a_2 \in \text{alph}(w(L_{b_{n+1}}))$. Hence we can extend the list above, of occurrences c_i and b_i, to c_{n+1} and b_{n+1}.

Therefore, if such a diagram exists, there is an infinite sequence of subwords $w(L_{c_i})$ of $\varphi(w)$, no two of which are equal, such that $w(L_{c_{i+1}})$ is a subword of $w(L_{c_i})$. Since $\varphi(w)$ is of finite length, this is a contradiction. □

Lemma 3.2. *The graph Γ_1 is not an induced subgraph of the extension graph Γ_2^e.*

Proof. Suppose, for a contradiction, that Γ_1 is an induced subgraph of Γ_2^e. Then there are vertices $v_a, v_b, v_c, v_d, v_e, v_f$ of Γ_2^e, such that the map sending a to v_a, b to v_b and so on, in-

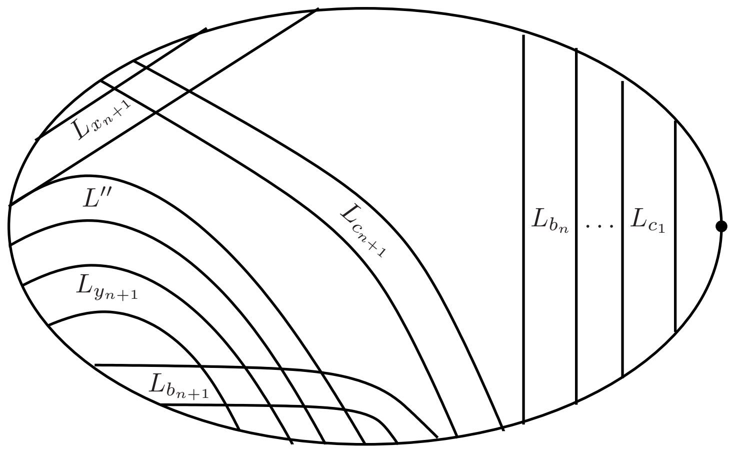


Figure 3

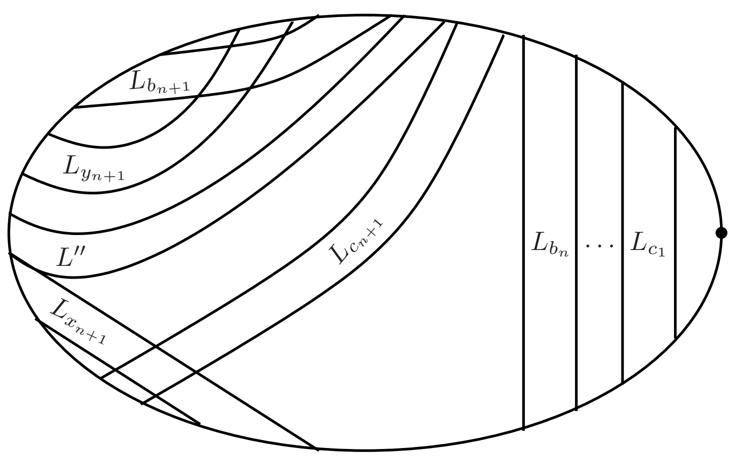


Figure 4

duces an embedding of graphs. Therefore, by definition of Γ_2^e there exist canonical generators $x_a, x_b, x_c, x_d, x_e, \in \{a_1, a_2, b, c, d, e\}$ and elements $w_a, w_b, w_c, w_d, w_e \in \mathbb{G}(\Gamma_2)$, such that

$$v_a = x_a^{w_a}, v_b = x_b^{w_b}, v_c = x_c^{w_c}, v_d = x_d^{w_d}, v_e = x_e^{w_e},$$

where the words $w_y^{-1} x_y w_y$, are reduced in $\mathbb{G}(\Gamma_2)$, for $y \in \{a, b, c, d, e, f\}$ and the graph spanned by the set $\{v_a, v_b, v_c, v_d, v_e\}$ in Γ_2^e is isomorphic to the graph Γ_1, i.e. the following commutation relations, and only these, hold between the v_y:

$$[v_a, v_d], [v_a, v_e], [v_b, v_e], [v_c, v_d], [v_d, v_e].$$

We perform a case-by-case analysis. We repeatedly use the fact that if Y is a subset of the canonical generating set of a pc group $\mathbb{G}(\Gamma)$ and $g \in \langle Y \rangle$ then $\mathrm{alph}_\Gamma(g) \subseteq Y$. That is, if w is a reduced word in $\mathbb{G}(\Gamma)$ and represents g then every letter occurring in w belongs to Y. In particular, if x is a generator and v is a reduced word in $\mathbb{G}(\Gamma)$ then $x \in \mathrm{alph}_\Gamma(x^v)$.

Note that the sets of vertices $\{v_a, \ldots, v_e\}$ and $\{v_a^w, \ldots, v_e^w\}$ in Γ_2^e, where $w \in \mathbb{G}(\Gamma_2)$, span isomorphic graphs, in this case both isomorphic to Γ_1. Hence, conjugating the vertices of the set $\{v_a, \ldots, v_e\}$ by w_d^{-1} we may assume that $w_d = 1$.

Case I: $x_d = c$. In this case, using the Centraliser Theorem, [S89, DK93], since $[a, d] = 1$, it follows that $x_a^{w_a} \in C_{\mathbb{G}(\Gamma_2)}(x_d) = \langle a_1, c, d \rangle$, a free Abelian group, so $w_a = 1$ and $x_a \in \{a_1, d\}$.

Similarly, since $[e, d] = [c, d] = 1$, it follows that $x_c, x_e \in \{a_1, d\}$ and $w_c, w_e = 1$. It is clear that x_a, x_e and x_c need to be pairwise distinct, so this is a contradiction.

Case II: $x_d = b$. This case follows from Case I, using the symmetry of the graph Γ_2 (interchanging b and c).

Case III: $x_d = d$. From the Centraliser Theorem again, $C_{\mathbb{G}(\Gamma_2)}(d) = \langle a_1, a_2, c, d, e \rangle$, which has centre $\langle a_1, d \rangle$. Since $[d, e] = 1$, we have $x_e^{w_e} \in \langle a_1, a_2, c, d, e \rangle$, and so, as $w_e^{-1} x_e w_e$ is reduced in $\mathbb{G}(\Gamma_2)$, the letters a_1 and d do not belong to $\mathrm{alph}(w_e)$ and $w_e \in \langle a_2, c, e \rangle$. This means that $w_e \in C_{\mathbb{G}(\Gamma_2)}(x_d)$, so we may conjugate again, this time by w_e^{-1}, and assume that $w_e = 1$. As x_d and x_e must be distinct, we must have $x_e \in \{a_1, a_2, c, e\}$. If $x_e = c$, then the statement follows from Case I using the symmetry of Γ_2 (interchanging d and e).

Hence we may assume that $x_e \in \{a_1, a_2, e\}$.

Case III.1: $x_e = a_1$. In this case $x_e \in Z(C_{\mathbb{G}(\Gamma_2)}(d))$. As above, since $[c, d] = 1$, it follows that $x_c^{w_c} \in C_{\mathbb{G}(\Gamma_2)}(d)$ and so $[v_e, v_c] = [x_e, x_c^{w_c}] = 1$, a contradiction.

Case III.2: $x_e \in \{a_2, e\}$. As above, since $[c, d] = [a, d] = 1$, it follows that $x_c, x_a \in \{a_1, a_2, c, e\}$. If $x_c = a_1$, then $[v_c, v_e] = [x_c^{w_c}, x_e^{w_e}] = 1$, a contradiction. It follows that $x_c \in \{a_2, c, e\}$. Since $[c, a] \neq 1$, it follows that $x_a \in \{a_2, c, e\}$. On the other hand, since $[a, e] = 1$, it follows that $[v_a, v_e] = [x_a^{w_a}, x_e] = 1$, so $x_a^{w_a} \in C_{\mathbb{G}(\Gamma_2)}(\{a_2, e\}) = C_{\mathbb{G}(\Gamma_2)}(e) = \langle a_1, a_2, e, d, b \rangle$. Thus $x_a \in \{a_2, e\}$ and $w_a = 1$. However, this means that $C_{\mathbb{G}(\Gamma_2)}(v_a) = C_{\mathbb{G}(\Gamma_2)}(\{a_2, e\}) = C_{\mathbb{G}(\Gamma_2)}(v_e)$, contrary to the hypotheses on the v_y's.

Case IV: $x_d = e$. Follows from Case III using symmetry of the graph Γ_2 (interchanging e and d).

Case V: $x_d = a_1$. This follows from Case III using the symmetry of Γ_2 (interchanging a_1 and d).

Case VI: $x_d = a_2$. Follows from Case IV using symmetry of the graph Γ_2.

□

4. A counterexample to the Weakly Chordal Conjecture

In this section we give a counterexample to Conjecture 2.

Remark 4.1. We note that the graph Γ is weakly chordal if and only if the complement graph $\overline{\Gamma}$ is weakly chordal.

Let Γ_1 be the graph C_5 (the 5-cycle)and Γ_2 the graph P_7 (the path of length 7), as shown in Figure 5. Let $\mathbb{G}(\overline{C}_5)$ and $\mathbb{G}(\overline{P}_7)$ be partially commutative groups with the underlying non-commutation graphs Γ_1 and Γ_2, respectively. That is

$$\mathbb{G}(\overline{C}_5) = \langle a, b, c, d, e \mid [a, c], [a, d], [b, d], [b, e], [c, e]\rangle \tag{1}$$

and

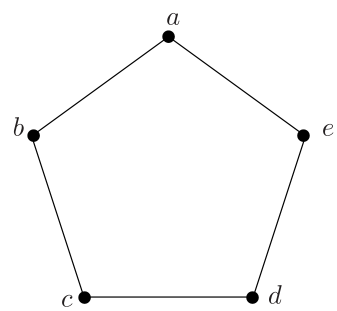
$$\begin{aligned}\mathbb{G}(\overline{P}_7) = \langle a, b, c_1, c_2, d_1, d_2, e_1, e_2 \mid & [a, c_1], [a, c_2], [a, d_1], [a, d_2], [a, e_2],\\ & [b, c_1], [b, d_1], [b, d_2], [b, e_1], [b, e_2],\\ & [c_1, c_2], [c_1, d_2], [c_1, e_1], [c_1, e_2],\\ & [c_2, d_1], [c_2, e_1], [c_2, e_2],\\ & [d_1, d_2], [d_1, e_2], [d_2, e_1],\\ & [e_1, e_2]\rangle.\end{aligned}$$

Remark 4.2. Observe that any for $n \geq 0$ the path graph P_n, and thus also its complement $\overline{P}_n$, are weakly chordal. Indeed, P_n is C_m free, for all $m, n \geq 1$, and for $n \geq 3$, the graph $\overline{P}_n$ has 2 vertices of degree $n-1$ and $n-1$ vertices of degree $n-2$. Therefore, for $n \geq 5$, $\overline{P}_n$ contains no induced C_m; and so its complement P_n contains no $\overline{C}_m$, $m \geq 1$. Hence P_n is weakly chordal, for all integers $n \geq 0$. Furthermore, the complement of C_5 is again C_5. Moreover, $\mathbb{G}(\overline{P}_n) \leq \mathbb{G}(\overline{P}_m)$, as $P_n \leq P_m$, for all $n \leq m$, and $\mathbb{G}(C_n) \leq \mathbb{G}(C_5)$, for all $n \geq 5$, see [KK, Theorem 11].

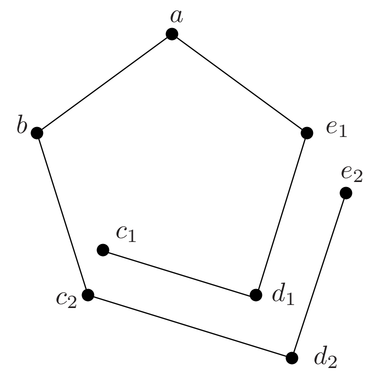
Define a map

$$\varphi = \begin{cases} a \mapsto a; \\ b \mapsto b; \\ c \mapsto c_1c_2; \\ d \mapsto d_1d_2; \\ e \mapsto e_1e_2. \end{cases}$$

Proposition 4.3. *The map φ defined above induces an embedding of $\mathbb{G}(\overline{C}_5)$ into $\mathbb{G}(\overline{P}_7)$.*



(a) $\Gamma_1 = C_5$



(b) $\Gamma_2 = P_7$

Figure 5

The following corollary follows immediately from Remark 4.2 and Proposition 4.3.

Corollary 4.4. *For all $n \geq 5$ and $m \geq 7$, the group $\mathbb{G}(C_n)$ embeds into $\mathbb{G}(C_5) = \mathbb{G}(\overline{C}_5)$, the group $\mathbb{G}(\overline{P}_7)$ embeds into $\mathbb{G}(\overline{P}_m)$ and the group $\mathbb{G}(C_n)$ embeds into $\mathbb{G}(\overline{P}_m)$.*

Proposition 4.3 may also be used to show that many surface groups embed into $\mathbb{G}(\overline{P}_m)$, for $m \geq 7$.

Corollary 4.5. *The fundamental group of a compact surface of even Euler characteristic at most -2 embeds into $\mathbb{G}(C_5) \leq \mathbb{G}(\overline{P}_m)$, for all $m \geq 7$.*

Proof. The compact orientable and non-orientable surfaces of Euler characteristic -2 are shown by Crisp and Wiest [CW] to embed in $\mathbb{G}(C_5)$. As observed in [R07] the orientable surface of Euler characteristic -2 is finitely covered by all orientable surfaces of smaller Euler characteristic; so the latter all have fundamental groups which embed in $\mathbb{G}(C_5)$. The construction of these finite covers appears in [M, Example 2.6] and a similar construction gives a finite cover of the compact non-orientable surface of Euler characteristic -2 by any compact non-orientable surface of even Euler characteristic, less than -2. □

(Using Stallings' definition of the genus of a closed surface S, namely genus$(S) = (2 - \chi(S))/2$, the Corollary says that all surface groups of integer genus g, such that $g \geq 2$, embed in $\mathbb{G}(\overline{P}_m)$, $m \geq 7$.)

Proof of Proposition 4.3. As φ maps each of the relators of the presentation (1) of $\mathbb{G}(\overline{C}_5)$ to the identity of $\mathbb{G}(\overline{P}_7)$, it is immediate that φ is a homomorphism. We show that φ is injective.

Let w be a reduced, non-trivial, word in $\mathbb{G}(\overline{C}_5)$, suppose that $\varphi(w) = 1$, and assume that w is of minimal length, among all such words. Let $\mathcal{D}$ be a minimal van Kampen diagram for $\varphi(w)$. As in the proof of Lemma 3.1, $\varphi(w)$ must contain a letter from $\{c_i^{\pm 1}, d_i^{\pm 1}, e_i^{\pm 1}\}$. Suppose first that $\varphi(w)$ contains an occurrence of $e_i^{\pm 1}$. Then $\varphi(w)$ contains $e_1^{\pm 1}$ and, as in the proof of Lemma 3.1, without loss of generality we may assume there exists a band L_v such that $v = e_1$ and $w(L_v) = e_2 w_2 e_2^{-1}$, where w_2 contains no occurrences of $e_i^{\pm 1}$, $i = 1, 2$. As w is reduced in $\mathbb{G}(\Gamma_1)$ it follows that $w(L_v)$ must contain a letter from $\{a^{\pm 1}, d_i^{\pm 1}\}$.

In the case where $w(L_v)$ contains no letter $d_i^{\pm1}$, the word w_2 contains an occurrence of $a^{\pm1}$ and, as $[a, e_1] \neq 1$, no band L_a can cross L_v, so a band with one end at an occurrence of $a^{\pm1}$ in w_2 must have both ends in w_2. Hence there must be a band L_x such that $x = a^{\pm1}$, $w(L_x)$ is a subword of w_2 and no letter $a^{\pm1}$ occurs in $w(L_x)$. As w is reduced, $\mathrm{alph}(w(L_x))$ must contain a letter from $\{b, e_i\}$; but $w(L_x)$ is a subword of $w(L_v)$, so this implies $b^{\pm1}$ must occur in $w(L_x)$. Now we find a band L_y, where $y = b^{\pm1}$, $w(L_y)$ is a subword of $w(L_x)$ and contains no occurrence of $b^{\pm1}$. We are forced to conclude that $w(L_y) \in \langle c_1, c_2\rangle$. As w is reduced in $\mathbb{G}(\Gamma_1)$ the word $w(L_y)$ contains an occurrence of $c_i^{\pm1}$ and, as all such letters occur in subwords $(c_1c_2)^{\pm1}$, it follows that $w(L_y)$ contains an occurrence of $c_2^{\pm1}$. As $[b, c_2] \neq 1$, there is a band L_z, where $z = c_2^{\pm1}$, with both ends in $w(L_y)$. In this case $w(L_y)$ contains the ends c_2^{ϵ} and $c_2^{-\epsilon}$ of L_z, and, as $w(L_y) \in \langle c_1, c_2\rangle$, w cannot be reduced, a contradiction

Hence we assume that $w(L_v)$ contains an occurrence $d_i^{\pm1}$. As $[d_1, e_1] \neq 1$, we may choose a band L_x, such that such that $x = d_1^{\pm1}$, and $w(L_x)$ is a subword of $w(L_v)$ which either contains no letter $d_i^{\pm1}$, if $x = d_1^{-1}$, or factors as $w(L_x) = d_2w_3d_2^{-1}$, where no letter $d_i^{\pm1}$ occurs in w_3, if $x = d_1$. Next, we may choose a band L_y, such that $y = c_1^{\pm1}$, $w(L_y)$ is a subword of $w(L_x)$ and contains no letter $c_i^{\pm1}$, except possibly the first and last (which may be c_2 and c_2^{-1}). Finally there must be a band L_z, with $z = b^{\pm1}$, such that $w(L_z)$ is a subword of $w(L_y)$ and there is no occurrence of $b^{\pm1}$ in $w(L_z)$. This forces $w(L_z)$ to be an element of $\langle a\rangle$, which cannot be trivial as the ends of L_z must be separated by at least one letter $a^{\pm1}$, if w is reduced. However, there is then a band with ends a^{ϵ} and $a^{-\epsilon}$, both lying in $w(L_z)$; and so w is not reduced, a contradiction.

If $e_i \notin \mathrm{alph}(w)$, $i = 1, 2$, then begin above with d_i instead of e_i and the same argument gives the result. The case where both $e_i, d_i \notin \mathrm{alph}(w)$ follows similarly. □

5. Subgroups of C_4 and P_3 free graphs

Let us call graphs which are C_4 and P_3 free *thin-chordal graphs.* The following theorem is a generalisation of part of the result of Droms [D87.2, Theorem].

Theorem 5.1. *The extension graph conjecture holds for finite, thin-chordal graphs, in the following strong form. Let Γ be a finite thin-chordal graph. Then H is a subgroup of $\mathbb{G}(\Gamma)$ if and only if $H \cong \mathbb{G}(\Gamma')$, for some induced subgraph Γ' of Γ^e.*

Proof. Let Γ be a thin-chordal graph. If $H \cong \mathbb{G}(\Gamma')$, for an induced subgraph Γ' of Γ^e, then it follows from [KK, Theorem 2] that H is a subgroup of $\mathbb{G}(\Gamma)$. To prove the converse, we use induction on the number of vertices of Γ. If Γ is not connected then $\mathbb{G}(\Gamma)$ is a free product of partially commutative groups of thin-chordal graphs, each of which has fewer vertices than Γ. Hence the result holds for each factor, by induction. The result for $\mathbb{G}(\Gamma)$ follows from the Kurosh subgroup theorem.

To see this, assume that the connected components of Γ are $\Gamma_1, \ldots, \Gamma_n$ and let $\mathbb{G}_i = \mathbb{G}(\Gamma_i)$. From [KK, Lemma 26 (2)], it follows that Γ^e is the disjoint union of countably many copies of Γ_i^e, $i = 1, \ldots, n$. That is

$$\Gamma^e = \coprod_{i=1}^{n} \left(\coprod_{j\in\mathbb{N}} \Gamma_{i,j}^e \right),$$

where $\Gamma^e_{i,j} = \Gamma^e_i$, for all $j \in \mathbb{N}$ and $i = 1, \ldots, n$. Let D_i be a set of representatives of double cosets $Hx\mathbb{G}_i$ of H and $\mathbb{G}_i$ in $\mathbb{G}(\Gamma)$. Then, from the Kurosh subgroup theorem,

$$H \cong \mathbb{F} * *_{i=1}^{n} \left(*_{d\in D_i} H \cap d\mathbb{G}_i d^{-1}\right),$$

where $\mathbb{F}$ is a free group of finite or countably infinite rank.

By induction, we have

$$H \cap d\mathbb{G}_i d^{-1} \cong d^{-1}Hd \cap \mathbb{G}_i \cong \mathbb{G}(\Lambda_{i,d}),$$

where $\Lambda_{i,d} \leq \Gamma^e_i$, for all $d \in D_i$, $i = 1, \ldots, n$. Let X_0 be a free generating set for $\mathbb{F}$. Then, for each i, there exists a bijection $\alpha_i : X_0 \coprod D_i \to \mathbb{N}$. Hence we have an embedding of graphs,

$$X_0 \coprod \coprod_{d\in D_i} \Lambda_{i,d} \leq \coprod_{j\in\mathbb{N}} \Gamma^e_{i,j},$$

where $x \in X_0$ maps to a vertex of $\Gamma_{i,\alpha_i(x)}$ and $\Lambda_{i,d}$ is embedded in $\Gamma^e_{i,\alpha(d)} = \Gamma^e_i$, for all $d \in D_i$. Hence

$$\mathbb{F} * \left(*_{d\in D_i} H \cap d\mathbb{G}_i d^{-1}\right) \cong \mathbb{G}(X_0) * (*_{d\in D_i}\mathbb{G}(\Lambda_{i,d})) \cong \mathbb{G}\left(X_0 \coprod \coprod_{d\in D_i} \Lambda_{i,d}\right),$$

where

$$X_0 \coprod \coprod_{d\in D_i} \Lambda_{i,d} \leq \coprod_{j\in\mathbb{N}} \Gamma^e_{i,j}.$$

Therefore

$$H \cong \mathbb{G}(X_0) * (*_{i=1}^{n} (*_{d\in D_i}\mathbb{G}(\Lambda_{i,d}))) \cong \mathbb{G}\left(X_0 \coprod \left(\coprod_{i=1}^{n} \coprod_{d\in D_i} \Lambda_{i,d}\right)\right),$$

where

$$X_0 \coprod \left(\coprod_{i=1}^{n} \coprod_{d\in D_i} \Lambda_{i,d}\right) \leq \coprod_{i=1}^{n} \left(X_0 \coprod \coprod_{d\in D_i} \Lambda_{i,d}\right) \leq \coprod_{i=1}^{n} \coprod_{j\in\mathbb{N}} \Gamma^e_{i,j} = \Gamma^e.$$

Suppose then that Γ is connected and that the result holds for all graphs of fewer vertices, connected or not. Let H be a subgroup of $\mathbb{G}(\Gamma)$. Then, as shown in [D87.2, Lemma], Γ has a vertex z which is connected to every other vertex of Γ. Let $V(\Gamma) = Y \cup \{z\}$ and let Λ be the induced subgraph of Γ with vertices Y. Then Λ is a thin-chordal graph and, from [KK, Lemma 26 (1)], $\Gamma^e = \{z\} * \Lambda^e$, that is, the graph formed from Λ^e by adjoining a new vertex z connected to every other vertex. It is shown in [D87.2] that either $H \leq \mathbb{G}(\Lambda)$ or $H \cong \langle z^d\rangle \times H'$, where $H' \leq \mathbb{G}(\Lambda)$, and d is a positive integer. By induction on the number of vertices, H in the first case, or H' in the second case, is a partially commutative group isomorphic to $\mathbb{G}(\Lambda')$, for some induced subgraph Λ' of Λ^e. As Λ^e is a full subgraph of Γ^e this completes the proof in the case $H \leq \mathbb{G}(\Lambda)$. In the second case $H \cong \langle z^d\rangle \times \Gamma(\Lambda') \cong \mathbb{G}(\{z\} * \Lambda')$ and $\{z\} * \Lambda' \leq \{z\} * \Lambda^e = \Gamma^e$, so the result holds in this case as well. □

[BSR07] N. Brady, H. Short, T. Riley *The Geometry of the Word Problem for Finitely Generated Groups (Advanced Courses in Mathematics - CRM Barcelona)* Birkhauser; 2007.

[CW] J. Crisp and B. Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups* , Algebraic Geometric Topology, **4** (2004) 439–472

[CSS] J. Crisp, M. Sageev and M. Sapir, *Surface Subgroups of Right-Angled Artin Groups* , Internat. J. Algebra Comput., **18** (2008), no. 3, 443–491.

[D87.1] C. Droms, *Graph Groups, Coherence, and Three-Manifolds*, J. Algebra **106** (1987), no. 2, 484–489.

[D87.2] C. Droms, *Subgroups of graph groups*, J. Algebra **110** (1987), no. 2, 519–522.

[D87.3] C. Droms, *Isomorphisms of graph groups*, Proc. Amer. Math. Soc.,**100** (1987), no. 3, 407–408.

[DSS] H. Servatius, C. Droms and B. Servatius, *Surface Subgroups of Graph Groups*, Proc. Amer. Math. Soc., **106** (1989) no.3, 573–578

[DK93] G. Duchamp, D. Krob, *Partially Commutative Magnus Transformations*, Internat. J. Algebra Comput., **3** (1993), 15–41.

[GLR] C. McA. Gordon, D. D. Long, and A. W. Reid, *Surface subgroups of Coxeter and Artin groups*, J. Pure Appl. Algebra **189** (2004), no. 13, 135–148.

[Kambites] M. Kambites *On commuting elements and embeddings of graph groups and monoids*, Proc. Edinb. Math. Soc., **52** (2009), 155–170.

[K08] S.-H. Kim, *Co-contractions of graphs and right-angled Artin groups*, Algebraic & Geometric Topology **8** (2008) 849–868

[KK] S.-H. Kim, T. Koberda, *Embedability between right-angled Artin groups*, Geometry and Topology **17** (2013), 493–530.

[M] W.S. Massey, *A basic course in Algebraic Topology*, Graduate Texts in Mathematics **127**, 1991, Springer-Verlag, New York.

[R07] C. Röver, *On Subgroups of the Pentagon Group*, Math. Proc. R. Ir. Acad., **107** (2007), no. 1, 11-13

[S89] H. Servatius, *Automorphisms of Graph Groups*, J. Algebra, **126** (1989), no. 1, 34–60.

[O] A.Yu. Ol'shankii, *Geometry of Defining Relations in Groups*, Mathematics and Its Applications Series, 1991, Kluwer Academic Publishers.