Three-dimensional inverse energy transfer induced by vortex reconnections

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In low-temperature superfluid helium, viscosity is zero, and vorticity takes the form of discrete, vortex filaments of fixed circulation and atomic thickness. We present numerical evidence of three-dimensional inverse energy transfer from small length scales to large length scales in superfluid turbulence generated by a flow of vortex rings. We argue that the effect arises from the anisotropy of the flow, which favours vortex reconnections of vortex loops of the same polarity, and that it has been indirectly observed in the laboratory. The effect open questions about analogies with related processes in ordinary turbulence.

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The phenomenology of three-dimensional turbulence is based on Richardson’s idea of the (forward) turbulent cascade. Kinetic energy, injected externally at large length scales, feeds large unstable eddies, which interact, become stretched and break up into smaller eddies. The process repeats, until, at sufficiently small length scales, viscous forces dissipate energy into heat. A reversed flux of energy, from the small scales to the large scales, is observed in two-dimensional turbulence. Such inverse cascade is more rare in three-dimensional turbulence, but can be observed in the presence of strong anisotropy. Such inverse cascade is more rare in three-dimensional turbulence, but can be observed in the presence of strong anisotropy. Secondly, at temperatures below 1 K, thermal excitations can be neglected and liquid helium (4He) is a pure superfluid. Unlike ordinary fluids (in which vorticity is a continuous field), the superfluid’s rotational motion is constrained by quantum mechanics to discrete vortex lines of fixed circulation and atomic thickness (the radius of the vortex core is only a0 ≈ 10−8 cm). Turbulence, easily excited by stirring the liquid helium, is a tangle of such vortex lines. An important property of vortex lines is that they reconnect when they come sufficiently close to each other, as predicted by theory and observed in experiments. Superfluid reconnects are similar to reconnections in ordinary fluids.

In this report, we exploit the singular nature of superfluid vorticity to examine the three-dimensional inverse energy transfer. Using numerical simulations, firstly we demonstrate that an inverse energy transfer is possible in superfluid helium (and, we argue, it has already been observed in the laboratory, although indirectly). Secondly, we show that vortex reconnections play a key role in this process.

We numerically model vortex lines as oriented space curves s(ξ, t) of infinitesimal thickness, where ξ is arc length and t is time. This approach is justified by the large separation of scales between a0 and the typical distance between vortices, ℓ ≈ L−1/2 (where L = Λ/V is the vortex line density, Λ the vortex length and V volume). Two physical ingredients determine the evolution of vortex lines. The first is Helmholtz’s theorem: a vortex at location s is swept by velocity field v generated by the entire vortex configuration L at s via the Biot-Savart law:

\[
\frac{ds}{dt} = v(s, t), \quad v(s, t) = -\frac{\kappa}{4\pi} \oint_L \frac{(s - r) \times dr}{|s - r|^3}, \quad (1)
\]

where \(\kappa = 9.97 \times 10^{-4} \text{ cm}^2/\text{s}\). The second ingredient, mentioned before, is vortex reconnections, instantaneous events which occur when vortex lines collide.

Our numerical simulations are performed in a periodic cube of size D. The techniques to discretize vortex lines into a variable number of points held at minimum separation δ/2, time-step Eq. 1, de-singularize the Biot-Savart integrals Eq. (1) (right), and evaluate them via a tree-method are described in the literature. The reconnection algorithm is described in and compared to other published algorithms.

In a pure superfluid, although viscosity is zero, the kinetic energy \(K(t)\) is not conserved, but is turned into sound (phonons) by rapidly rotating Kelvin waves (helicoidal perturbations of vortex lines) at length scales of the order of \(10^2 \text{ a}_0\). In our simulations it is impossible to discretize vortex lines down to almost the atomic scale. However, the finite numerical resolution qualitatively models phonon losses, because it damps out Kelvin waves at scales of the order of δ, and slightly reduces the vortex length (again at scale δ) at each reconnection event. Kelvin waves are often studied in the...
context of the decay of superfluid turbulence at very low temperatures. Interacting Kelvin waves (see [28] and references therein) form a one-dimensional weakly nonlinear system in which a dual cascade in k-space takes place: a direct cascade of energy to large k and an inverse cascade of wave action to small k. Generation of long waves as well as short waves on individual vortices has been observed in numerical simulations [30], but this Kelvin cascade process is not directly relevant to the inverse energy transfer which we present here, which is three-dimensional in nature.

Our first numerical simulation [31] models experiments [32]. We start with an empty computational box and inject vortex rings at frequency f drawing their radius from a normal distribution. The rings are in the yz-plane and travel in the positive x direction; their evolution is computed using the full Biot–Savart law and the reconnection algorithm. After an initial transient, the vortex system settles to a statistical steady state. A snapshot of the vortex tangle is shown in Fig. 1. In this regime forcing is balanced by dissipation, and the vortex line density fluctuates about a saturated value, as shown in the inset of Fig. 2.

To determine the distribution of the kinetic energy over the length scales, we Fourier–transform the velocity field \( \mathbf{v} \) and define the energy spectrum \( E(k) \) by

\[
K(t) = \frac{1}{V} \int \frac{1}{2} \mathbf{v}^2 dV = \int_0^\infty E(k)dk,
\]

where \( k = |k| \) is the magnitude of the three-dimensional wavenumber and \( V = D^3 \) is volume. Large (small) length scales correspond to small (large) wavenumbers \( k \) respectively. We find that, during the evolution, \( E(k) \) progressively increases at small \( k \). To quantify this energy transfer to large length scales, we compute the energy flux \( \langle \epsilon \rangle \) of energy is transferred from large to small length scales, \( \langle \epsilon \rangle = \frac{1}{V} \int dV \left( \frac{\partial E}{\partial t} \right) \) with \( k = 2\pi/D \). Fig. 2 shows that, in the statistically steady regime, the time-averaged energy flux \( \langle \epsilon \rangle \) is negative for \( k < k_f \) and positive for \( k > k_f \), where \( k_f = 2\pi/(2\hat{R}) \approx 1300 \text{ cm}^{-1} \) is the forcing wavenumber based on the injected rings’ mean diameter \( 2\hat{R} \).

In other numerical simulations we examine the inverse energy transfer under different conditions. The second simulation [33] is inspired by numerical studies of homogeneous isotropic turbulence in which a forcing term is added to the governing Navier–Stokes equation to balance viscous dissipation and achieve a statistically-steady state, independent of the initial condition. We add a random, incompressible, isotropic velocity field \( \mathbf{v}_{\text{ext}} \) to the right-hand-side of Eq. (I) left), consisting of 100 random Fourier modes, narrow banded (\( \Delta k \approx 2 \text{ cm}^{-1} \)) around wavenumber \( k_f \approx 70 \text{ cm}^{-1} \), with \( \langle v_{\text{ext}}^2 \rangle^{1/2} = 3.1 \text{ cm/s} \). The seeding initial condition consists of a small number of randomly oriented vortex rings. During the evolution, energy is fed into the system by \( \mathbf{v}_{\text{ext}} \) and removed by the numerical dissipation. Fig. 3 shows the growth of \( E(k) \) during the initial evolution and the overall build up of \( K(t) \) (the area under \( E(k) \)). The dashed line is the spectrum of the forcing term \( \mathbf{v}_{\text{ext}} \). The transfer of energy from large \( k \) near the forcing \( k_f \) to small \( k \) is apparent: at small \( k \), \( E(k) \) grows by a factor of 500.

In the third numerical simulation [34] we examine the role of reconnections. We proceed as in the first simulation, injecting rings of random radius aligned in the

FIG. 1. First numerical simulation. Snapshots of the vortex tangle in the steady–state regime at \( t = 4 \text{ s} \) (L ≈ 6000 cm\(^{-2}\)).

FIG. 2. (Color online). First numerical simulation. Energy flux \( \langle \epsilon \rangle \) (averaged over the statistical steady regime \( 2 < t < 4 \text{ s} \)) vs wavenumber \( k \) (cm\(^{-1}\)). Notice that \( \langle \epsilon \rangle < 0 \) for \( k < k_f \) (energy is transferred from small to large length scales), and that \( \langle \epsilon \rangle > 0 \) for \( k > k_f \) (energy is transferred from large to small scales), where \( k_f \approx 1300 \text{ cm}^{-1} \) is the wavenumber corresponding to the average diameter of the injected rings. The inset shows the vortex line density \( L \) (cm\(^{-2}\)) vs time \( t \) (s).
yz plane and travelling in the \( x \) direction; we retain the reconnection algorithm, but replace the Biot-Savart law (Eq. \( \text{I} \) (right)) with its Local Induction Approximation (LIA) \( \text{[36, 37]} \):

\[
v(\mathbf{s}, t) \approx \frac{\kappa}{4\pi} \ln \left( \frac{R}{a_0} \right) \mathbf{s}' \times \mathbf{s}'',
\]

where a prime denotes derivative with respect to arc length and \( R = 1/|\mathbf{s}'| \) is the local radius of curvature. Under LIA, vortex lines move along the binormal direction with speed inversely proportional to \( R \), ignoring each other; in other words, vortices interact only when they collide. Fig. \( \text{I} \) shows that, in the absence of forcing, \( K(t) \) decreases, but energy is shifted from large \( k \) to small \( k \), as in the previous simulations. This result means that vortex-vortex interaction (represented by the Biot-Savart law) is not necessary to produce a reverse energy transfer: vortex reconnections are enough to drive the process.

A simple geometrical interpretation of this result is the following. Energy and speed of a vortex loop of size \( R \) are roughly proportional to \( R \) and \( 1/R \) respectively. Vortex loops travelling parallel or antiparallel to a given direction undergo two kinds of collisions: head-on and from behind. Head-one collisions leave the size of loops approximately unchanged after the reconnection, as in Fig. \( \text{I} \) collisions from behind create a larger loop and a smaller loop, as in Fig. \( \text{I} \). The large loop, which contains most of the energy, is more likely to become entangled with other vortices, while the small loop, which quickly moves away, is more likely to be absorbed by walls (in the presence of periodic boundary conditions, the small loops which collide with larger loops can only become smaller, without significantly increasing the size of the larger loops). In an isotropic tangle, collisions of either kind are equally likely. In an anisotropic system (a jet of rings, or a system in which loops are injected along a preferred direction), we expect more collisions from behind, particularly in the early stage, hence a shift of energy to larger length scales induced by reconnections alone.

In the fourth simulation \( \text{[38]} \) we proceed as in the first: we inject rings moving in the \( x \) direction continuously (drawing their radius from a normal distribution and compute the evolution using the Biot-Savart law and the reconnection algorithm); the difference is that now the forcing is relatively larger than in the first simulation \( \text{[39]} \).
When the vortex line density saturates, the tangle settles down to a steady state (a snapshot of the saturated vortex tangle is shown in Fig. 7). We notice (see Fig. 8) that so much energy has been shifted to wavenumbers smaller than the injection’s inverse lengthscale $k_f \approx 1/R$ that the spectrum has acquired a form which is consistent with the classical Kolmogorov scaling $E(k) \sim k^{-5/3}$ typical of ordinary turbulence, a result which is in agreement with existing numerical simulations of superfluid turbulence [40–44].

Now we put these numerical results in the context of experiments. Walmsley & Golov [32] created turbulence in $^4$He at very low temperatures (so that the normal fluid can be neglected) by injecting vortex rings with a high voltage tip. After the injection stage, they monitored the decay of the vortex line density $L$ and observed two regimes, $L \sim t^{-1}$ (called “ultraquantum”) and $L \sim t^{-3/2}$ (called “quasiclassical”), associated with short and long initial injection times respectively. Both regimes were also observed in $^3$He-B [45]. In our previous paper [46] we modelled the experiment of Walmsley & Golov as realistically as possible, numerically injecting vortex rings in the form of a narrow beam originating from a point source. Firstly, we reproduced ultraquantum and quasiclassical regimes at short and long injection times (in a related calculation [47], vortex injection was not strong enough to generate large length scales; this is consistent with the facts that turbulence decayed as $L \sim t^{-1}$ and the relative forcing [39] was ten times less than in our first simulation). Secondly, by computing the spectrum, we discovered [46] that the quasiclassical regime is the decay of a Kolmogorov spectrum, which forms as energy is transferred from the small injection length scale to larger length scales. However, the nonuniformity of the beam was (at least in principle) a possible origin of the observed inverse energy transfer.

In summary, the simulations which we present here, together with our previous result [46], show clearly the phenomenon of energy transfer from small to large scales and its relation with vortex reconnections. The negative energy flux is observed not only in transients but also in statistically steady state regimes, and occurs over a wide range of wavenumbers. The effect which we describe has implications for other superfluid turbulence experiments, in particular for the formation of developed turbulence past a grid, as in the towed-grid experiments by Donnelly and collaborators [48] which have been much discussed in the literature, the oscillating grid experiments performed at the University of Lancaster [49, 45], and the most recent experiments of Walmsley et al. [50] which seem to confirm the inverse transfer of energy which we have identified.

The analogies with related processes in classical fluid dynamics are also intriguing, but need further detailed
It is interesting to recall recent work by Biferale et al. [51] who numerically induced the classical three-dimensional inverse energy cascade by artificially restricting the nonlinearity of the governing Navier-Stokes equation to the interaction of Fourier modes of the same helical sign. Their result shows that, in principle, all three-dimensional turbulent flows contain nonlinearities which may lead to an inverse cascade: to make the effect apparent one has to break the mirror symmetry of the interactions. Our findings are apparently consistent with Biferale’s. It must be stressed that it is not the anisotropy of the configuration which matters, but rather the anisotropy of the interaction. In Biferale’s problem, the anisotropy is enforced at every time step by the numerical algorithm; in our problem, the anisotropy is introduced by the initial condition which favours one kind of vortex reconnections over the other, as we have described. In more isotropic conditions, the direct cascade generally may hide this effect (for example, a small inverse energy transfer is apparent in the energy spectrum of a decaying Taylor-Green flow [52], although the authors do not comment on it).

In conclusion, the natural question is whether the inverse energy transfer which we have described amounts to a cascade, or creates an equilibrium distribution at large scales as described for example in the simpler case of wave turbulence [53]. In a nonlinear system we expect that excitation in a spectral interval means transfer of energy to both larger and smaller scales. In our case, the interaction between vortex loops is not symmetric and far from trivial. Numerical simulations over a much wider range of wavenumbers and temporal scales than we can perform now will help answering this question.

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[22] The opening angle of the tree-algorithm is 0.3.
[31] Parameters: $D = 0.1$ cm, $\delta = 10^{-3}$ cm, rings injected with frequency $f = 10^3$ Hz, mean radius $\bar{R} = 0.0024$ cm and standard deviation $\sigma = 5 \times 10^{-4}$ cm.
[33] Spectra are computed from a 512$^2$ Cartesian mesh in the $x$-$z$-plane.
[34] Parameters: $D = 1$ cm, $\delta = 0.01$ cm;
[35] Parameters as in **34**.
[38] Parameters: $D = 1$ cm, $\delta = 0.01$ cm; rings injected with frequency $f = 125$ Hz, mean radius $\bar{R} = 0.021$ cm, standard deviation $\sigma = 0.017$ cm.
[39] Let $D$ be the unit of length and $\kappa/\ell$ the unit of speed. The unit of time is then $\tau = Df/\kappa$. In a characteristic time scale $\tau$, the injected vortex length relative to the total vortex length (using length as a proxy for energy for simplicity), $\delta\Lambda/\Lambda = 2\pi Rf\tau/\Lambda$, is almost three times larger in the fourth simulation than in the first.