Discrete-Time Adaptive Control for Systems With Input Time-Delay and Non-Sector Bounded Nonlinear Functions

KHALID ABIDI (Member, IEEE), AND IAN POSTLETHWAITE (Life Fellow, IEEE)

1 Electrical Power Engineering Program, Newcastle University in Singapore, Singapore 567739
2 School of Engineering, Newcastle University, Newcastle upon Tyne NE1 7RU, U.K

Corresponding author: Khalid Abidi (khalid.abidi@ncl.ac.uk)

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ABSTRACT This paper presents a discrete-time adaptive control approach for nonlinear systems with input delay. The nonlinearity is assumed to be non-sector bounded, resulting in the key technical lemma being inapplicable. The main aim of this paper is to present a general implementation inspired from Kanellakopoulos and Fu, et al. for uncertain scalar and multivariable input delay systems with uncertain parameters as well as uncertain input gain. While it has been shown by Kanellakopoulos and Fu, et al. that it is possible to design adaptive control laws that compensate for the growth of the nonlinearity for single parameter scalar systems, a rigorous analysis of multiple parameter systems is not shown. In this paper, it is shown that an adaptive controller design that compensates for the growth of the nonlinearity is possible for both multiple parameter scalar and multivariable systems with input delay. Rigorous stability proofs and simulations are presented to verify the validity of the approach.

INDEX TERMS Adaptive control, discrete-time systems, nonlinear control, time-delay systems.

I. INTRODUCTION

Stabilization of systems with actuator delays has always been a challenge in controller design. The celebrated Smith Predictor [3], proved to be the first practical solution to dealing with actuator delays although it was limited by the requirement of exact model parameters as well as the time-delay. Later on, adaptive control designs for uncertain linear time invariant systems with known time-delays were presented by Ortega and Lozano [4]. This was expanded further in [5]–[12], for various cases including input delays, state delays, distributed delays, time-varying delays, etc. In addition, various practical implementations have been presented in [13]–[15]. The survey paper [19] provides a comprehensive list of papers published prior to 2003 that discuss the stabilization of time delay systems. Also, the book [20] presents predictive feedback in delay systems with extensions to nonlinear systems, delay-adaptive control and actuator dynamics modeled by PDEs. More recently, compensation approaches for input delays using truncated predictor feedback are shown in [16]–[25].

Successful studies on the adaptive control of linear, discrete-time uncertain systems with time-delay can be found in [21]–[25]. For nonlinear discrete-time adaptive control, implementations have always been limited by the requirement that the system nonlinearities are sector bounded. This is a strict requirement of the Key Technical Lemma [26] (page 181) that guarantees asymptotic stability of the system. In order to eliminate this limitation a new approach was proposed in [1]. This approach allowed for the relaxation of the bound conditions on the nonlinearity while still guaranteeing asymptotic stability. The approach was developed for a scalar system (with a single uncertain parameter) without an uncertain input gain or input time-delay and it was highlighted that extension to more general cases is difficult. In [2], the same problem is addressed without assuming a growth condition on the nonlinearity, in the presence of bounded disturbances. The results are proven for a system similar to that in [1] and the algorithm for multivariable systems is given without any rigorous analysis or stability proofs.

In this paper, a more general implementation inspired by [1] and [2] is presented for uncertain scalar input delay systems with multiple uncertain parameters as well as uncertain input gain. The approach is further extended to multivariable input delay systems. For the scalar case, the approach is based on the prediction of future signals through successive substitution of the system model as is shown in [27]. Following the approach in [1], a coefficient is introduced into the adaptive law that guarantees asymptotic convergence in the presence of non sector bounded nonlinearities. The approach is further
extended to multivariable systems and it is shown that this extension is not trivial and needs to be investigated rigorously. Stability proofs are given with simulation results for a scalar and a multivariable system to verify the proposed approach.

The organization of this paper is as follows: In Section II, the main result and a discussion of scalar systems are presented. In Section III, an extension to multivariable systems is provided. In Section IV, simulation examples are presented and concluding remarks are given in Section V.

Throughout this paper, I - I denotes the Euclidean norm and $O(\cdot)$ denotes order of ‘$\cdot$’. For notational convenience, the mathematical expression “$f_k$” represents the value of the signal $f$ at the $k$’th sampling instant.

II. MAIN RESULT

In this section, the controller design is presented starting with a simple scalar first-order system.

A. CONTROL OF A SCALAR INPUT-DELAY SYSTEM IN DISCRETE-TIME

Consider the following discrete-time system with input delay

$$x_{k+1} = \varphi^T \xi(x_k) + b u_{k-p} + \delta_k$$

where $x_k \in \mathbb{R}$ is the system output, the parameters $\varphi \in \mathbb{R}^n$, the function $\xi(x_k) \in \mathbb{R}^m$, $q \in \mathbb{Z}^+$ is a known polynomial function of $x_k$, $q' \in \mathbb{Z}^+$ is the number of parameters, $b \in \mathbb{R}$ is assumed to be known, $p$ is the delay in number of steps and $|\delta_k| \in O(1)$ is an uncertain smooth time-varying disturbance. For the system (1), the following assumptions are made:

Assumption 1: The delay $p$ is known a priori.

Assumption 2: The function $\xi(x_k)$ is bounded for a bounded $x_k$. Furthermore, $|\xi(x_k)| \leq c_0 + c_1 |x_k|^g$ for some positive constant $c_0, c_1$ and $g \in \mathbb{Z}^+$ is the order of the polynomial function $\xi(x_k)$.

Assumption 3: From the structure of the system (1), there exist constants $\kappa_0$ and $\kappa_1$ such that the control input is bounded as $|u_{k-p}| \leq \kappa_0 + \kappa_1 \max_{i \in [0, k+1]} |x_i|$.

The goal is to force the system (1) to track the reference model

$$x_{m,k+1} = a_m x_{m,k-p} + b_m r_{k-p}$$

where $a_m \in \mathbb{R}$ is in the unit-disk. Extending the work in [3] and [28] a controller is chosen as

$$u_k = b^{-1} \left( x - \varphi^T \xi x + b r - \delta \right)$$

where $\delta_k$ is an estimate of the disturbance. Substitution of the controller (3) into (1) leads to error dynamics of the form

$$e_{k+1} = a_m e_{k-p} + \tilde{\delta}_k$$

where $e_k = x_{n,k} - x_k$ and $\tilde{\delta}_k$ is the disturbance estimation error. Since $|a_m| < 1$ and if the term $|\tilde{\delta}_k|$ is bounded such $|\tilde{\delta}_k| \leq \gamma$ for some constant $\gamma$, then (4) is stable. Note that the controller (3) is a function of $x_{k+p}$ and $\delta_{k+p}$. There-

control input $w_k$. Rather than deriving separate estimations for $x_{k+p}$ and $\delta_{k+p}$, the system (1) will be rewritten in a form that attenuates the influence of the disturbance $\delta_k$ and that form will be used to derive delay free system dynamics. From the delay free system dynamics, a causal control law is derived.

Consider the system (1), according to the assumptions on the disturbance $\delta_k$ and the results in [29], it follows that $\delta_k = 2\delta_{k-1} + \delta_{k-2} \in O(T^2)$ where $T < 1$ is the sampling interval. Using this result, the system can be written in a disturbance compensated form as

$$x_{k+1} - 2 x_k + x_{k-1} = \varphi^T \xi_k - 2 \xi_{k-1} + \xi_{k-2} + b u_{k-p} - 2 u_{k-p-1} + u_{k-p-2} + u_k$$

where $\xi_k \equiv \{ \xi(x_k) \}$ and $u_k = \delta_k - 2 \delta_{k-1} + \delta_{k-2} \in O(T^2)$.

Using successive substitutions a delay-free system is obtained as

$$x_{k+p+1} = \theta^T \xi(x_k, u_{k-1}, \ldots, u_{k-p+1}) + bu_k + p_{k-1}^T \phi_{k-1} + \tilde{u}_{k+p}$$

(7)

where $\theta$ is the augmented parameter vector, $\xi(\cdot)$ is the augmented nonlinear function that is a function of the state $x_k$ and control history $u_{k-1}, \ldots, u_{k-p+1}$, and $p_{k-1}^T \phi_{k-1}, \tilde{u}_{k+p}$ are the augmented disturbance terms due to the successive substitutions. Note that as a result of the successive substitutions $p_{k-1}$ will be a function of $\varphi$, $b$ and $u_k$ and that $\lambda_p \in O(T^2)$ for some constant $\lambda$. Consider the term $p_{k-1}^T \phi_{k-1}$, based on the structure of $\xi(x_k)$, the augmented nonlinear function $\xi(\cdot)$ and $\phi_k$ will contain cross terms of the states $x_{1,k}, x_{2,k}, \ldots$, the control inputs $u_{1,k}, u_{2,k}, \ldots$ and $u_k$. However, $p_{k-1}^T \phi_{k-1}$ can still be written in parametric form. Using Assumption 3, it can be shown that

$$\left| k_{-1}^T \phi_{k-1} \right| \leq k_2 + k_3 \max_{i \in [0, k]} |x_i|$$

where $k_2, k_3 \in O(T^2)$ are some positive constants. Furthermore, it can be shown that since $\tilde{u}_k$ is a function of $u_k$ and the uncertain parameter vector $\varphi$, a bound can be found such that $|\tilde{u}_k| \leq \phi \in O(T^2)$.

Proceeding with the control law design, subtracting (7) from a $k+p$ steps ahead form of (2) results in an error dynamics of the form

$$e_{k+p+1} = a_m e_k + a_m x_k - \theta^T \xi(x_k, u_{k-1}, \ldots, u_{k-p+1}) - bu_k + b_m r_k - p_{k-1}^T \phi_{k-1} = \tilde{u}_{k+p}$$

(8)

From (8) a control law is selected as

$$u_k = b^{-1} \left( a_m x_k - \theta^T \xi(x_k, u_{k-1}, \ldots, u_{k-p+1}) + b_m r_k \right)$$

(9)

such that an error dynamics is achieved as
fore, $x_{k+p}$ and $\hat{\delta}_{k+p}$ are needed for the computation of the

$$
e_{k+1} = a_m e_{k-p} + \rho_{k-p-1}^T \phi_{k-p-1} + \tilde{\nu}_k.
$$

(10)
Assume that $\kappa_2 \approx \kappa_2 + \tilde{\phi}$, then (10) is further written in the form

$$\begin{align*}
\left| e_{k+1} \right| & \leq \left| a_m \right| + \kappa_3 \left| x_{k-p} \right| (e^{\left(g_{k+1}-g_{k}\right)} - 1) + \left| e_{k-p} \right| + \kappa_2 + \left| \tilde{u}_k \right| \\
& \leq \left| a_m \right| + \kappa_3 \left| x_{k-p} \right| (e^{\left(g_{k+1}-g_{k}\right)} - 1) + \left| e_{k-p} \right| + \kappa_2 \\
& \quad + \kappa_3 \left| x_{k-p} \right| \\
\end{align*}$$

which is asymptotically stable if and only if the state $\dot{k}_e$ lies in a neighborhood that satisfies the condition

$$
\left| a_m \right| + \kappa_3 \left| x_{k-p} \right| \left( e^{\left(g_{k+1}-g_{k}\right)} - 1 \right) < 1.
$$

**Remark 1:** Upon careful inspection of the result (11), it can be seen that the term $\left| a_m \right| + \kappa_3 \left| x_{k-p} \right| \left( e^{\left(g_{k+1}-g_{k}\right)} - 1 \right)$ is a function of the delay $p$. Therefore, the stability of (11) is guaranteed if and only if

$$
\left| x_k \right| < \frac{\left( 1 + \left| a_m \right| \right)}{\kappa_3} \left( e^{\left(g_{k+1}-g_{k}\right)} - 1 \right).
$$

The condition (12) gives the neighborhood for the stability of (11). It is possible to select the sampling-interval $T$ such that $\kappa_3 \in O(\lambda T^2)$ is small enough resulting in a large enough neighborhood for stability.

**B. ADAPTIVE CONTROL OF AN INPUT-DELAY SYSTEM**

Consider now that the parameters $\varphi$ and $b$ in system (1) are uncertain constants. This will result in the parameter vector $\theta$ being uncertain and the control law (9) is revised as

$$u_k = \hat{\theta}_k + a_m x_k - \hat{\theta}_k^T \zeta(x_k, u_k, \ldots, u_{k-p+1}) + b \cdot r_m k
$$

where $\hat{\theta}_k$ and $\bar{b}_k$ are the estimates of $\theta$ and $b$, respectively.

The parameter estimates must be computed such that the system (1) tracks the reference model (2). Now that the goal of the adaptive law is defined, it is possible to proceed with the derivation. In order to derive the adaptive law, substituting the control law (13) in (7) it is obtained that

$$x_{k+1} = a x_k + \hat{\theta}_k^T \zeta_k + b r_k + \tilde{b} u + \rho_k \phi_k + \tilde{u}_k
$$

where $\zeta_k \equiv \zeta x_k, u_k, \ldots, u_{k-p+1}$ and $\hat{\theta}_k, \bar{b}_k$ are the parameter estimation errors respectively. Subtracting (2) from a $p$ steps delayed (14), it is obtained that

$$e_{k+1} = a e_k + \hat{\theta}_k^T \zeta_k + b \tilde{u} + \rho_k \phi_k + \tilde{u}_k + \tilde{u}_{k-p+1}
$$

and let $s = [0 \cdots 0 1]^T$ such that $\hat{b}_k = s^T \tilde{b}_k$. Then premultiplying both sides of (19) with $s^T$ it is obtained that

$$\dot{\hat{b}}_k = s^T \tilde{b}_k + a \hat{\theta}_k \psi_{k+1} + \tilde{b} \tilde{u} + \rho_k \phi_k
$$

and let the initial condition of $\hat{b}_k$ be nonsingular and $\beta_{k+1} = \frac{1}{\tilde{b}_k} \psi_{k+1} + \tilde{b} \tilde{u} + \rho_k \phi_k$ then $\hat{b}_k$ will be nonsingular long as it satisfies $\beta_{k+1} = \frac{1}{\tilde{b}_k} \psi_{k+1} + \tilde{b} \tilde{u} + \rho_k \phi_k$.

Before proceeding with the stability analysis it is necessary to define the following Lemmas:

**Lemma 1:** For the system (15) and the adaptive laws (16) and (17), the following conditions are true:

(a) \[ \lim_{{k \to \infty}} a_k + b_k \psi_{k+1} - \zeta_{k+1} = 0 \]

(b) \[ I_{\tilde{K}} \psi_{k+1} \leq c_0 I_{\tilde{K}} \psi_{k+1} \] for some constants $c_0$.

**Proof:** Consider the positive function

$$V_k = \psi_{k+1}^T P^{-1} \psi_{k+1}. \quad (21)$$
it is possible to formulate the adaptive law as follows
\[
\dot{\psi}_{k+1} = \dot{\psi}_k - \alpha k^p \beta P \zeta_k \bar{P} \forall k \in [k, \infty)
\]
\[
\forall k \in [0, k_0)
\]
(16)

The forward difference \( V_{k+1} - V_k \) can be found as, [22],
\[
V_k = V_{k+1} - V_k = \tilde{\psi}_{k+1} P^{1/2} \psi_k - \tilde{\psi}_{k} P^{1/2} \psi_k + \tilde{\psi}_k P^{-1} \psi_k - \tilde{\psi}_{k-p} P^{-1} \psi_k \]
(22)
Substitution of (16) in (22) and following the approach in [27], it is obtained that
\[ V_k = \psi^T \left( P - I \right) \psi_k + 2\alpha \beta \psi^T \zeta_k \psi_{k+1} \]
\[ + \zeta^T_{k+1} P_{k+1} \zeta_{k+1} \]
\[ + \zeta^T_{k} P_{k} \zeta_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ + \alpha \beta \gamma \psi^T \psi_{k} + \beta \gamma \psi^T \psi_{k+1} \]
\[ + \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
(23)

To proceed further, consider (47). According to [22], the covariance matrix \( P_k \) satisfies
\[ k+1 \quad k \]
\[ \alpha_k + \beta \kappa \leq \psi^T \psi \]
Using this condition and the fact that \( \alpha_k, \beta_k, \psi_k \) are positive constants, then (23) can be simplified further to obtain
\[ V_k \leq \frac{\alpha \beta \gamma^2}{1 + \alpha_k + \beta \kappa} \]
\[ + \frac{\alpha_k + \beta \kappa}{1 + \alpha_k + \beta \kappa} \]
\[ \zeta^T_{k+1} P_{k+1} \zeta_{k+1} \]
\[ \zeta^T_{k} P_{k} \zeta_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ \alpha \beta \gamma \psi^T \psi_{k} + \beta \gamma \psi^T \psi_{k+1} \]
\[ \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
(24)

which implies that \( \psi^T \psi \) is bounded and, therefore, \( \psi^T \) is also bounded, [22]. Note that for any \( k \in [k_0, \infty) \) the following is true
\[ V_{k+1} = V_{k} + \psi \psi_{k+1} \]
(25)

Substituting (24) in (25), it is obtained that
\[ \lim_{k \to \infty} V_{k+1} < V_{k} \]
\[ \lim_{k \to \infty} \frac{\alpha \beta \gamma^2}{1 + \alpha_k + \beta \kappa} \zeta^T_{k+1} P_{k+1} \zeta_{k+1} \]
\[ \frac{\alpha_k + \beta \kappa}{1 + \alpha_k + \beta \kappa} \zeta^T_{k} P_{k} \zeta_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
(26)

Since by definition, \( V_{k+1} \) is non-negative and \( V_{k_0} \) is finite, then according to the convergence theorem of the sum of series condition (a) of Lemma 1 is established as
\[ \lim_{k \to \infty} \frac{\alpha \beta \gamma^2}{1 + \alpha_k + \beta \kappa} \zeta^T_{k+1} P_{k+1} \zeta_{k+1} \]
\[ \frac{\alpha_k + \beta \kappa}{1 + \alpha_k + \beta \kappa} \zeta^T_{k} P_{k} \zeta_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
\[ \beta \gamma \psi^T \psi_{k} + \alpha \beta \gamma \psi^T \psi_{k+1} \]
(27)

Finally, to verify part (b), consider the definition of \( \zeta_k \) and the control law (13). It is obtained that
\[ \zeta^T_{k} = \frac{1}{\alpha \beta \gamma} \zeta^T_{k} \zeta_{k+1} \]
\[ = \frac{1}{\alpha \beta \gamma} \zeta^T_{k} \zeta_{k+1} \]
\[ = \zeta^T_{k} \zeta_{k+1} \]
\[ = \zeta^T_{k} \zeta_{k+1} \]
(28)

Consider (28), then from condition (a) it follows that the adaptive parameters \( \theta \) are bounded. Furthermore, the reference signal \( r_k \) is bounded and \( \zeta_k \) is not sector bounded w.r.t. \( x_k \). Then it is obtained that
\[ \zeta^T_{k} \zeta_{k+1} \]
(29)

if \( \alpha_k \) is selected such that
\[ \alpha_k \geq \frac{f_k - d_1 g_k}{h_k} \]
\[ \alpha_k \leq \frac{f_k - d_1 g_k}{h_k} \]
(31)

where \( f_k, g_k, h_k \) and \( l_k \) are functions of the elements of \( \zeta_k \) and \( \alpha_k \) history while \( d_0, d_1 \) are some positive constants.

Proof: The inverse of the covariance matrix satisfies the condition
\[ P_{k+1} = P_{k} + \alpha_k + \beta_k \zeta_k \zeta^T_k \]
(32)

where \( k_0 \) is an initial time step and \( L \) is the floor function. Rewriting (32) as
\[ P_{k+1} = P_{k} + \alpha_k \beta_k \zeta_k \zeta^T_k \]
\[ \alpha_k \zeta_k \zeta^T_k \]
\[ \alpha_k \zeta_k \zeta^T_k \]
\[ \alpha_k \zeta_k \zeta^T_k \]
\[ \alpha_k \zeta_k \zeta^T_k \]
(33)

where \( k_0 = 0 \) for the sake of simplicity and considering that the initial value of \( P_0 \) is selected such that \( P_{0} = diag(p_1, p_2, \ldots, p_q + 1) \), then the matrix \( P_k \) can be evaluated by computing the inverse of \( P_k \). Therefore, the expression of \( P_k \) is obtained as
\[ P_k = \frac{1}{\alpha_k h_k + g_k} \]
\[ \alpha_k M_{1,k} + M_{2,k} \]
(34)

where det \( P_{k} = \alpha_k h_k + g_k \) and adj \( P_{k} = \alpha_k M_{1,k} + M_{2,k} \). Premultiplying (34) with \( \zeta_k^T \) and postmultiplying with \( \zeta_k \), it is obtained that
\[ \zeta_k^T P_k \zeta_k = \frac{1}{\alpha_k h_k + g_k} \]
\[ \alpha_k M_{1,k} \zeta_k + M_{2,k} \zeta_k \]
\[ \zeta_k \]
\[ \zeta_k \]
\[ \zeta_k \]
(35)

Furthermore, using matrix and vector norms on the right-hand-side of (35), the upperbound on \( \zeta_k^T P_k \zeta_k \) is obtained as
\[ \zeta_k^T P_k \zeta_k \leq \alpha_k h_k + g_k \]
\[ \alpha_k M_{1,k} + M_{2,k} \zeta_k \]
\[ \zeta_k \]
\[ \zeta_k \]
\[ \zeta_k \]
\[ \zeta_k \]
(36)

Now consider (36). If \( \zeta_k \) is bounded such that
\[ \zeta_k \leq d_0 \]
\[ \zeta_k \leq d_0 \]
\[ \zeta_k \leq d_0 \]
(37)
\[ 1 \tilde{\zeta}_k I \leq c_0 I \zeta_k I \]  
(29)

\[
\text{and solving for } \alpha_k \text{ results in a condition on } \alpha_k \text{ that is given as}
\]

\[
\alpha_k \geq \frac{f_k d_1 g_k}{M \zeta^2 l_1 M \zeta^2} \text{ and } d
\]
(38)

\[
\text{where } f_{k} = \frac{2_k}{k} \quad l_{k} = \frac{1_k}{k} \quad d_1 = \frac{d_1}{c_0}
\]
Remark 2: Note that the constant $d_1$ can be adjusted to avoid division by zero. Furthermore, a lower bound on $\alpha_k$ can be imposed to ensure that $\alpha_k$ is always positive.

Remark 3: It is not possible to generalize the expression for $\alpha_k$ for multiple uncertain parameters, however, the procedure will be presented for a system with two uncertain parameters in order to illustrate the implementation of the adaptive law. The procedure is similar for any number of uncertain parameters.

Example 1: Consider the system given by (15) and assume that $\psi_k^T = \{\psi_1^T, \psi_2^T\} \in \mathbb{R}^{2k}$ and $\chi_k^T = \{\chi_1^T, \chi_2^T\} \in \mathbb{R}^{2k}$. Also let $P_p^{-1} = \text{diag}(p_1, p_2)$. Then the matrices $M_{1,k}$ and $M_{2,k}$ are obtained as

$$
M_{1,k} = \tilde{B}_k - \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} + \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} \tag{39}
$$

and

$$
M_{2,k} = \left[ \begin{array}{cc}
\frac{\psi_{2,k-1}^T \bar{B}_{k-1}}{\alpha_{k-1}} & \tilde{\zeta}_{2,k-1} \\
\tilde{\zeta}_{1,k-1} & \tilde{\zeta}_{2,k-1}
\end{array} \right] - \tilde{\zeta}_{1,k-1} \tilde{\zeta}_{2,k-1} \tag{40}
$$

where $\tilde{\zeta}_{2,k-1} = \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} + \bar{B}_{k-1} \tilde{\zeta}_{1,k-1} \tilde{\zeta}_{2,k-1}$.

Furthermore, the functions $h_k$ and $g_k$ are obtained as

$$
h_k = \tilde{B}_k - \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} + \bar{B}_{k-1} - \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} \tag{41}
$$

and

$$
g_k = p_1 \psi_{2,k-1} + \alpha_k \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} + \bar{B}_{k-1} \tilde{\zeta}_{2,k-1} \tilde{\zeta}_{1,k-1} \tag{42}
$$

Finally, the results (39), (40), (41) and (42) can be substituted in (38) for the computation of $\alpha_k$.

As can be seen from Example 1, the procedure for calculating $\alpha_k$ is straightforward and it is possible to extend it to a higher number of uncertain parameters.

Lemma 3: If $\alpha_k$ is computed from the lower bound in (38) such that

$$
\alpha_k = \frac{f_k}{d_1 h_k - l_k} \tag{43}
$$

and that $\tilde{\zeta}_{k-1} \tilde{\zeta}_k$ are bounded, then there exists an upper-bound $\alpha_{\text{max}}$ such that $\alpha_{\text{max}} \leq \alpha_k$.

Proof: Consider the expression (43), using the results (39), (40), (41) and (42) from Example 1, then it is obtained that

$$
\alpha_k = \frac{f_k - d_1 g_k}{d_1 h_k - l_k} = \frac{\mu_{1,k}}{d_1 h_k - l_k} + \frac{\mu_{2,k}}{d_1 h_k - l_k} \tag{44}
$$

where $\nu_k$, $\mu_{1,k}$, $\mu_{2,k}$, $\ldots$ are functions of $p_1$, $p_2$, $d_1$, $\tilde{\zeta}_k$, $\tilde{\zeta}_k$, $\tilde{\zeta}_k$, $\tilde{\zeta}_k$. Furthermore, the system (44) is augmented to the form

$$
\begin{align*}
\alpha_k &= \nu_k + \frac{\mu_{1,k}}{d_1 h_k - l_k} \alpha_k + \frac{\mu_{2,k}}{d_1 h_k - l_k} \alpha_{k-2p} + \cdots \\
\alpha_{k-p} &= \alpha_{k-p} \\
\alpha_{k-2p} &= \alpha_{k-2p} \\
&\vdots \\
\end{align*}
$$

which can be written in a vector form as

$$
\begin{align*}
\tilde{\alpha}_k &= \begin{bmatrix} \mu_{1,k} \alpha_k & \mu_{2,k} \alpha_{k-2p} & \cdots \end{bmatrix} \\
&\vdots \\
\alpha_{k-p} &= 0 \\
\end{align*}
$$

Finally, the results (39), (40), (41) and (42) can be substituted in (38) for the computation of $\alpha_k$.

Remark 4: Even though the results from Example 1 are used, an expression similar to (44) can be obtained for a system with any number of uncertain parameters.

Remark 5: From Lemma 3 it is seen that a constant $\alpha_{\text{max}}$ exists that will satisfy (38). Thus, $\alpha_k = \alpha_{\text{max}}$ can be tuned rather than using (43) to compute $\alpha_k$.

C. STABILITY ANALYSIS

Stability is given by the following Theorem:

Theorem 1: The closed-loop system, consisting of the system (1) with uncertain parameters $\varphi$ and $b$, the controller (13) with adaptive laws (16) and (17), is stable if and only if

$$
|\alpha_{\text{max}}| + \kappa_3 (1 + \alpha_{\text{max}} \delta_d)^2 |x_k - \bar{x}_k| < 1
$$

Furthermore, the tracking error, $e_k = x_k - x_{\text{ref}}$, converges asymptotically to a bound $\bar{E}$.

Proof: The first part of Theorem 1 discusses the boundedness of the signals in the closed loop system while the second part discusses the asymptotic convergence of the tracking error. However, note that the boundedness of $\alpha_k$ can only be considered after the convergence of the tracking error is established.
It was shown in Lemma 1 and Lemma 2 that the adaptive parameters \( \varphi_k \) and \( \beta_k \) are bounded. Now consider the condition (a) of Lemma 1 given as
\[
\lim_{k \to \infty} \frac{\alpha_k + \beta_k \gamma_k}{1 + \alpha_k + \beta_k \gamma_k} \left| \tilde{e} \right|^2 = 0 \tag{48}
\]
which is true for \( \left| \tilde{e} \right| + 1 \geq (1 + \alpha_{\text{max}} \beta_k) \frac{1}{\omega_k} \). To guarantee that \( \lim_{k \to \infty} \left| \tilde{e} \right| + 1 \leq (1 + \alpha_{\text{max}} \beta_k) \frac{1}{\omega_k} \) it must be guaranteed that \( I_k \left| \tilde{e} \right| \to \infty \) is bounded. From Lemma 2 it is shown that this is indeed the case. Therefore, since \( \alpha_k + 1, \beta_k \) and \( \gamma_k \) are positive in addition to \( I_k \left| \tilde{e} \right| \to \infty \), \( \tilde{e} \) is positive constant, then (48) implies that
\[
\lim_{k \to \infty} \frac{\alpha_k + \beta_k \gamma_k}{1 + \alpha_k + \beta_k \gamma_k} \left| \tilde{e} \right|^2 = 0 \tag{49}
\]
and ultimately, \( \lim_{k \to \infty} \left| \tilde{e} \right| + 1 \leq (1 + \alpha_{\text{max}} \beta_k) \frac{1}{\omega_k} \). Consider now the error dynamics given by
\[
e_{k+1} = a_m e_k - p + e_{k+1} \tag{50}
\]
and
\[
|e_{k+1}| \leq |a_m| + \kappa_2 \left( 1 + \alpha_{\text{max}} \beta_k \right) \left| \tilde{e} \right| \left| e_k \right| + |\kappa_2| \left| \tilde{e} \right|^2 \tag{51}
\]
Consider that \( \kappa_2 \) is small enough and \( \tilde{e} \) lies in a neighborhood such that
\[
\left| a_m \right| + \kappa_2 \left( 1 + \alpha_{\text{max}} \beta_k \right) \left| \tilde{e} \right| \left| e_k \right| + |\kappa_2| \left| \tilde{e} \right|^2 < \tilde{a}_m < 1 \text{ and } \left| \kappa_2 \right| \left| \tilde{e} \right|^2 \leq \kappa_2 \left( 1 + \alpha_{\text{max}} \beta_k \right) \left| \tilde{e} \right| \left| e_k \right| + |\kappa_2| \left| \tilde{e} \right|^2\]
for some positive constants \( \tilde{a}_m \) and \( \kappa_2 \). Then (51) has a solution that satisfies
\[
|e_k| \leq \tilde{a}_m \frac{|e_0|}{1 + \kappa_2} + \frac{1}{1 + \kappa_2} \sum_{i=1}^{k-1} \tilde{a}_m \max (52)
\]
Also, since \( \tilde{a}_m \) is in the unit disk, it follows that
\[
\lim_{k \to \infty} \tilde{a}_m \max (53)
\]
which establishes the boundedness of \( |e_k| \). Since it has been established that \( |e_k| \) is bounded, then \( \tilde{e} \) and \( \tilde{e} \) are also bounded. Therefore, using Lemma 3 it is concluded that \( \alpha_k \) is bounded.

III. EXTENSION TO MULTIVARIABLE SYSTEMS

In this section the proposed discrete-time adaptive controller is extended to multivariable nonlinear systems with time-delay.

Consider the \( n \)th order feedback linearizable nonlinear system of the form
\[
\dot{x}_k = C^T x_k + \phi_k \tag{54}
\]
where \( x_k \in \mathbb{R}^n \) is the state vector, \( u_k \in \mathbb{R}^m \) is the control input vector, \( y_k \in \mathbb{R}^n \) is the output vector, \( \phi_k \in \mathbb{R}^n \times \mathbb{R}^m \) is a matrix of uncertain parameters, \( r \in \mathbb{R}^n \times \mathbb{R}^m \) is the uncertain input gain, \( C \in \mathbb{R}^{n \times m} \) is the output matrix and \( \delta_k \in \mathbb{R}^{m \times m} \) is a smooth time-varying disturbance vector such that \( \delta_k I \in O(1) \).

The function \( \xi(x_k) \in \mathbb{R}^n \) is a vector of known polynomial functions \( x_k \). For the system (54), the following assumptions are made:

Assumption 4: The delay \( p \) is known a priori.
Assumption 5: \( C^T r \) is non-singular.
Assumption 6: The norm of the function vector \( \|\xi(x_k)\| \) is bounded for a bounded \( \|x_k\| \) further. Furthermore, \( \|\xi(x_k)\| \leq c_0 + c_1 x_k \) for some positive constants \( c_0, c_1 \) and \( x \in Z^+ \).

Assumption 7: There exists a \( \delta_k \in \mathbb{R}^{m \times m} \) and a positive definite \( Q \in \mathbb{R}^{m \times m} \) such that \( \delta = \delta_k \delta_k^T \) and \( r = r_n Q \) for an augmented parameter matrix and \( r_n \) is a known nominal input gain matrix.

Consider now the sampled-data reference model
\[
x_{m,k+1} = c_m x_{m,k} + r_m r_{k-p} \tag{55}
\]
where \( x_{m,k} \in \mathbb{R}^n \) is the reference model state vector, \( r_k \in \mathbb{R}^n \) is the reference input vector, \( r_{m,k} \in \mathbb{R}^m \) is the reference model output vector, \( c_m x_{m,k} + r_m r_{k-p} \) is a known Hurwitz matrix and \( r_{m,k} \in \mathbb{R}^n \) is a known matrix. The control objective is to force the system (54) to follow the reference model (55).

Before proceeding with the controller design, consider the system (54). Using Assumption 5 and the fact that \( \delta_k \in \mathbb{R}^{n \times n} \) is a known Hurwitz matrix and \( r_{m,k} \in \mathbb{R}^n \) is a known matrix, the control objective is to force the system (54) to follow the reference model (55).

By considering the controller design, consider the system (54). Using Assumption 5 and the fact that \( \delta_k \in \mathbb{R}^{n \times n} \) is a known Hurwitz matrix and \( r_{m,k} \in \mathbb{R}^n \) is a known matrix, the control objective is to force the system (54) to follow the reference model (55).

By considering the controller design, a p-steps ahead reference model (55) is subtracted from (57) and using Assumption 7 results in the error dynamics of the form
\[
e_{k+1} = \frac{1}{1 + \delta_k \phi_k} + \frac{1}{u_{k-p} + r_n Q} \tag{58}
\]
state \( z_k \in \mathbb{R}^m \) such that
\[
y(e) = <l(e) > .
\] (59)
\[
z_{k+p+1} = C^T x_{k+p+1} + m \kappa
\]
\[
\text{Note that since it is assumed that } C^T r \text{ and } Q \text{ are non-singular, then } C^T r_n \text{ is non-singular. Substitution of (58) in (59), gives}
\]
\[
z_{k+p+1} = \Theta \zeta_k - \delta x - \delta r + Qu
\]
\[
+ C^T Y^T \phi_k + \xi_k + u_{k+p} \tag{60}
\]
where \( ru_k = r_0 \). Similar to the scalar case, it can be shown that \( \sum_{k=1}^{\infty} \phi_{k-1} I \leq \kappa_0 + \kappa_1 \max_{i \in \{0, k\}} k_i \Theta_1 \)
where \( \kappa_0 \) and \( \kappa_1 \) are positive constants. To achieve stability, the controller is formulated as
\[
u_k = - Q^{-1} \delta^T \zeta_k - \delta m x_k - \delta r x_k .
\] (61)
However, since the parameters \( \delta \) and \( Q \) are assumed to be uncertain the controller is modified to the form
\[
u_k = - Q^{1} \delta \zeta_k - \delta m x_k - \delta r x_k .
\] (62)
Substitution of (62) in (60) results in
\[
x_k \zeta_k + \tilde{Q} \nu_k + C^T Y^T \phi_k + \tilde{u} \tag{63}
\]
\[
z_{k+p+1} = \tilde{\Theta} \zeta_k - \tilde{\delta} \tilde{x}_k - \tilde{\delta} \tilde{r}_k + \tilde{u} \tag{64}
\]
where \( \tilde{\Theta} \), \( \tilde{\delta} \), \( \tilde{\delta} \), and \( \tilde{\delta} \) are associated parameters. Rewriting (63) and delaying by \( p \)-time steps it is obtained that
\[
z_{k+1} = \tilde{\Theta} \zeta_k + \tilde{\tilde{Q}} \zeta_k u_{k-p} + C^T Y^T \phi_k + \tilde{u}_k
\]
\[
= \tilde{\tilde{\zeta}}_k \zeta_{k-p} + C^T Y^T \phi_k + \tilde{u}_k \tag{66}
\]
where \( \tilde{\tilde{\zeta}}_k \) is the augmented parameter estimate error vector and \( \zeta_k \in \mathbb{R}^{n+m} \) is the augmented vector of known functions. Using (64), it is possible to formulate the adaptation law as follows
\[
\alpha \beta \gamma \delta \epsilon \zeta_k \zeta_k = \tilde{\tilde{\zeta}}_k \zeta_{k-p} + C^T Y^T \phi_k + \tilde{u}_k
\]
\[
\text{Substitution of (65) and (66) into (69) and following the same procedures as in Lemma 1 it is obtained that}
\]
\[
\alpha \beta \gamma \delta \epsilon \zeta_k \zeta_k = \tilde{\tilde{\zeta}}_k \zeta_{k-p} + C^T Y^T \phi_k + \tilde{u}_k
\]
\[
\text{which is true for } I \tilde{\zeta}_k \zeta_k = (1 + \alpha \beta \gamma \delta \epsilon \zeta) \zeta k \omega_k . \text{ To guarantee that}
\]
\[
\text{it must be guaranteed that}
\]
\[
\text{Theorem 2: The closed loop system (64) with adaptive}
\]
\[
\text{laws (65) and (66), is stable if and only if}
\]
\[
\text{Furthermore, the tracking error, } I \tilde{\zeta}_k \zeta_k - x, \text{ converges asymptotically to a bound } E .
\]
\[
\text{Proof: Consider the positive function}
\]
\[
V_k = \langle \nu \rangle \tag{67}
\]
\[
\text{where } \nu = \tilde{\tilde{\zeta}}_k \zeta_{k-p} + C^T Y^T \phi_k + \tilde{u}_k
\]
\[
\text{which is true for } I \tilde{\zeta}_k \zeta_k = (1 + \alpha \beta \gamma \delta \epsilon \zeta) \zeta k \omega_k . \text{ To guarantee that}
\]
\[
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\]
\[
\text{it must be guaranteed that}
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\text{Theorem 2: The closed loop system (64) with adaptive}
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\[
\text{laws (65) and (66), is stable if and only if}
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\[
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\text{Proof: Consider the positive function}
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\[
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\[
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\]
\[
\text{which is true for } I \tilde{\zeta}_k \zeta_k = (1 + \alpha \beta \gamma \delta \epsilon \zeta) \zeta k \omega_k . \text{ To guarantee that}
\]
\[
\text{it must be guaranteed that}
\]
\[
\text{Theorem 2: The closed loop system (64) with adaptive}
\]
\[
\text{laws (65) and (66), is stable if and only if}
\]
\[
\text{Furthermore, the tracking error, } I \tilde{\zeta}_k \zeta_k - x, \text{ converges asymptotically to a bound } E .
\]
\[
\text{Proof: Consider the positive function}
\]
\[
V_k = \langle \nu \rangle \tag{67}
\]
\[
\text{where } \nu = \tilde{\tilde{\zeta}}_k \zeta_{k-p} + C^T Y^T \phi_k + \tilde{u}_k
\]
where \( P_k \in \mathbb{R}^{(q+m)\times(q+m)} \) is a symmetric positive-definite covariance matrix, \( \alpha_k \) is a positive coefficient and \( \beta_k > 0 \) is a scalar coefficient used to prevent a singular \( Q_k \). The coefficient \( \gamma_k \) is defined similar to (18) and is given as

\[
\gamma_k = \begin{cases} 
\frac{1}{\omega_k^2} \left( 1 + \alpha_{\text{max}} d_0 \right) \omega_k^2, & \text{if } l_{z_k} I \geq (1 + \alpha_{\text{max}} d_0) \frac{\omega_k}{\alpha_{\text{max}} d_0} \\
0, & \text{if } l_{z_k} I < (1 + \alpha_{\text{max}} d_0) < \omega_k \end{cases}
\]

where \( \omega_k = I C_k I + \kappa_0 I x_{(g+1-g)} \), \( d \geq \frac{\gamma_k}{1 + \alpha_{\text{max}} d_0} \), and \( \alpha_{\text{max}} \geq \alpha_k \).

\[
\begin{bmatrix} P_{k_0} > 0 \\
\forall k \in [0, k_0) \end{bmatrix}
\]

and a \( p \) time-steps delayed (71) satisfies

\[
(1 + \alpha_{\text{max}} d_0) I x_{k-p} I \leq I_{m} I + \kappa_1 \left( 1 + 2 I C_k I + \left( \alpha_{\text{max}} d_0 \right)^T \right) \times I_{x_{k-p} I} + \kappa_1 + 2 I C_{\gamma I} I_{x_{k-p} I} \times (1 + \alpha_{\text{max}} d_0) I x_{k-p} I + \kappa_1 + 2 I C_{\gamma I} I_{x_{k-p} I} + \kappa_0 I_{x_{k-p} I} \leq I_{m} I + \kappa_1 \left( 1 + 2 I C_{\gamma I} I_{x_{k-p} I} \times (g+1-g-1) \right) \times (g+1-g-1) \times (g+1-g-1)
\]
that $\kappa_1 + 2 I C^T P r_0 I > 0$ for some hurwitz $I \neq 0$ and a positive $\phi_{\max}$, then the expression (72) satisfies

$$I e_k \leq I \tilde{m}_l^{1-\frac{r}{\kappa_1}} + \sum_{i=1}^{\frac{r}{\kappa_1}} I \tilde{m}_l^{1-\frac{r}{\kappa_1}} I e_0 I + \frac{1}{\kappa_1} I \tilde{m}_l^{1-\frac{r}{\kappa_1}} \phi_{\max}. \quad (73)$$

Since $\tilde{m}_l$ is Hurwitz, then $\lim_{k \to \infty} I \tilde{m}_l^{1-\frac{r}{\kappa_1}} I e_0 I = 0$ and $\lim_{k \to \infty} \sum_{i=1}^{\frac{r}{\kappa_1}} I \tilde{m}_l^{1-\frac{r}{\kappa_1}} \phi_{\max} = E$ for some positive constant $E$. Therefore,

$$\lim_{k \to \infty} I e_k I \leq E \quad (74)$$

which establishes the boundedness of $I e_k I$.

**IV. SIMULATION EXAMPLE**

In this section, a scalar system and a multivariable system will be used to demonstrate the performance of the controller. The scalar system example will compare the proposed approach to that in [2].

**A. SCALAR SYSTEM WITHOUT TIME-DELAY**

Consider the nonlinear discrete-time system presented in [2]

$$x_{k+1} = -3 x_k^2 + u_k + \sin \left( \frac{50 \pi}{\xi} \right) \quad (75)$$

with $x_0 = 0$. To attenuate the influence of the disturbance the system is written in the form

$$x_{k+1} = 2 x_k - x_{k-1} - 3 x_{k-2} = 2 x_k - x_{k-1} + x_{k-2} + u_k - 2 u_{k-1} + u_{k-2} + u_k. \quad (76)$$

The design objective is to track the reference model

$$x_{m,k+1} = 0.9 x_{m,k} + 0.25 r_k \quad (77)$$

where $r_k = 1$. Using (76) and (77), the control law is derived as

$$u_k = 2 u_{k-1} - u_{k-2} + 0.9 x_k + 0.25 r_k - 2 x_k + x_{k-1} - \phi_k x_k^2 - 2 x_{k-1}^2 + x_{k-2}^2. \quad (78)$$

As in [2], the parameter uncertainty is assumed to be 90% and the initial value of the adaptive parameter is selected as $\phi_0 = -0.3$. After a number of trials, the remaining parameters are selected as $P_0 = 100$, $\alpha_k = 1$ and $d_k \alpha_{\max} = 0.1$. The value $P_0 = 100$ is the same as in [2]. The system is simulated using both control approaches and the tracking performance is shown in Fig. 1. From the results, it can be seen that both approaches result in stable performance, however, the approach proposed in this work can attenuate the effects of external disturbances leading to better tracking performance. In Fig. 2 the parameter convergence for both approaches is

**B. SCALAR SYSTEM WITH TIME-DELAY**

Consider the system (75) with a control input time-delay of $p = 1$ given as

$$x_{k+1} = -3 x_k^2 + u_{k-1} + \sin \left( \frac{50 \pi}{\xi} \right). \quad (79)$$

Using successive substitutions, it is obtained that

$$x_{k+1} = -27 x_k^4 - 3 u_{k+1}^2 + 18 x_k^2 u_{k-1}^2 + u_{k-1}^2 + 18 \delta_k - 6 \delta_{k-1} u_{k-2} - 3 \delta_{k-1} + \delta_k \quad (80)$$

where $\phi^T \phi = \frac{\rho_{k-1}}{\delta_k} + u_{k-1} \phi_{k-1} + u_k \phi_k$ and $\delta_k = \sin \left( \frac{\rho_{k-1}}{\delta_k} \right) \left( x_k \right)^2 \left( x_{k+1} \right)^2 + 1$. The terms $\rho_{k-1} \phi_{k-1}$ and $u_k$ are the augmented disturbance terms as a result of successive substitutions. The system (80) is now written in the disturbance attenuating form as

\[ \text{FIGURE 1. Tracking performance of the proposed controller and the approach in [2].} \]

\[ \text{FIGURE 2. Parameter convergence of the proposed controller and the approach in [2].} \]
shown. It can be seen that in both approaches the adaptive parameter converges to the true value which is to be expected for the case of a single uncertain parameter.

\[
x_{k+1} = 2x_k - x_{k-1} + \phi^T \xi_{k-1} - 2 \xi_{k-2} + \xi_{k-3} + u_{k-1} \\
- 2u_{k-2} + u_{k-3} + \bar{\rho}^T \phi_{k-1} + \bar{\nu}_k.
\] (81)
The goal is for the system (81) to track reference model given as
\[ x_{m,k+1} = 0.9 x_{m,k-1} + 0.25 r_{k-1} \] (82)
resulting in the control law of the form
\[ u_k = 2 u_{k-1} - u_{k-2} + 0.9 x_k + 0.25 r_k - 2 x_k + 2 x_{k-1} - \varphi_k^T \hat{\xi}_k - 2 \bar{\xi}_{k-1} + \bar{\xi}_{k-2} \] (83)

After a number of trials, the controller parameters are selected as \( \varphi_0 = [-21 \ 0 \ 12] \), \( P_0 = 200 I \), \( \alpha_k = 70 \) and \( d_0 \alpha_{\text{max}} = 0.2 \). For the approach in [2], the parameters are similarly selected as \( \varphi_0^T = [-21 \ 0 \ 12] \) and \( P_0 = 200 I \). The system is simulated using both control approaches and the tracking performance is shown in Fig.3. The results show that the proposed approach can produce stable performance while minimizing the effects of the external disturbance. On the other hand, the approach in [2] is unable to handle the effects even though the nonlinear function does not satisfy the sector bound condition that is required for the classical discrete-time adaptive control approach.

C. MULTIVARIABLE SYSTEMS
Consider a nonlinear discrete-time system with matched disturbance of the form
\[ x_{k+1} = \begin{bmatrix} 1.5 & 0 & 0 & 2 & 1.2 & 0 \\ 0 & 1.5 & 0 & 0 & 0 & 1.3 \end{bmatrix} u_{k-p} \]
\[ + \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0.1 \end{bmatrix} \sin \left( \frac{k \pi}{50} \right) \]
\[ y_{k+1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_k \] (84)
with delay \( p = 4 \). The reference model is selected as
\[ x_{m,k+1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_{m,k-1} + \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} r_{k-p} \] (85)
The reference is selected as \( r = \begin{bmatrix} 0 & 15 & 0 & 15 \end{bmatrix} \). The nominal gain matrix and controller parameters are set as
\[ k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ r_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ \delta_m = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]
\[ \dot{\delta}_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0 \end{bmatrix} \]
\[ d_0 \alpha_{\text{max}} = 0.2 \alpha_k = 10 \) and \( P_0 = 5 \times 10^4 I_{16 \times 16} \). The matrix \( \delta x_0 \) is computed using a nominal \( \delta I = I_{3 \times 3} \).
of the time-delay coupled with the external disturbance and results in unstable performance. In Fig.4 the convergence of the adaptive parameters is shown while Fig.5 shows the non-linear growth of $\dot{\xi}_k$ with respect to $|x_k|$. As it can be seen from the results, the proposed approach guarantees convergence system is simulated and the results can be seen in Fig.6. It can be seen that the system output converges to the desired trajectory. To demonstrate the ability to handle unknown control directions, the matrix $\hat{Q}_0$ is set as $\hat{Q}_0 = -0.5I_{2 \times 2}$ while some of the controller parameters are retuned as $\alpha_k = 20$ and $P_0 = 1 \times 10^2I_{16 \times 16}$. It can be seen from Fig.7 that the system output converges to the desired trajectory. In Fig.8 the
The convergence of the elements of $\tilde{Q}_k$ is shown. It can be seen in the results that the adaptive law is capable of correcting $\tilde{Q}_k$ to match the actual system control direction.

Finally, the system is simulated with different values of the input delay $p$ using the controller parameters that led to the results in Fig. 7. The average of the norm of the tracking error $\text{avg} \|e_k\|$ is computed over an interval of 1000 steps and plotted in Fig. 9. The results show that the tracking performance may degrade with the increase in input delay $p$. This is due to the fact that the delay $p$ influences the transient performance of the system (see Remark 1).

V. CONCLUSION

In this paper, a discrete-time adaptive controller for nonlinear systems with non-sector bounded nonlinearities is proposed. Although numerous approaches have been proposed in other works that can handle non-sector bounded nonlinearities, stability proofs are shown only for single parameter adaptive laws. This paper presents stability proofs for systems with multiple parameters while at the same time demonstrating the difficulty of addressing systems with non-sector bounded nonlinearities. Simulation results are given to demonstrate the effectiveness of the controller for a nonlinear system.

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KHALID ABIDI received the Ph.D. degree in electrical and computer engineering, specializing in control engineering, from the National University of Singapore, in 2009. He was an Assistant Professor of mechatronics engineering at Bahcesehir University, Istanbul, Turkey, from 2009 to 2014. He is currently the Degree Program Director and an Assistant Professor of electrical power engineering at Newcastle University in Singapore. He has published over 30 papers in the area of systems and control theory.

IAN POSTLETHWAITE received the Ph.D. degree from Cambridge University and studied electrical engineering at Imperial College London. He has held visiting positions at The Australian National University and the University of California at Berkeley, and has made many shorter visits to major research groups, such as MIT, UC Santa Barbara, Tsinghua, and Trondheim. He was a Vice-Chancellor at Oxford University. He was a Deputy Vice-Chancellor at Northumbria University. He held academic posts at Oxford and Cambridge Universities and was the Head of the Engineering Department, Leicester University. He is currently the CEO of NU International Singapore Pte. Ltd., the CEO of Newcastle Research & Innovation Institute Pte. Ltd., and the Dean (Singapore) of the Newcastle University, UK. In 2019, he returns to Newcastle University as the deputy head of the School of Engineering. He has co-authored the highly cited textbook Multivariable Feedback Control, with S. Skogestad. His research, for more than 40 years, has involved theoretical contributions to the field of robust multivariable control and the application of advanced control system design to engineering systems. He is a fellow of the Royal Academy of Engineering, The Institution of Engineering and Technology, and the Institute of Measurement and Control. He has received three best paper prizes: the IFAC Automatica Prize Paper Award, Applications category, in 2008; the IFAC Control Engineering Practice Prize Paper Award, Applications category, in 2002; and the Institution of Electrical Engineers (now IET) FC Williams Premium in 1991.

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