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Suggested keywords

TRACE
INDEPENDENCE
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PARTIAL ORDER
TRACE OF STEP SEQUENCES
SIMULTANEITY
SERIALISABILITY
INTERLEAVING
GENERALISED CAUSAL ORDER STRUCTURE

Order Structures for Subclasses of Generalised Traces

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Abstract. Traces are equivalence classes of action sequences which can be represented by partial orders capturing the causality in the behaviour of a concurrent system. Generalised traces, on the other hand, are equivalence classes of step sequences. They are represented by order structures that can describe non-simultaneity and weak causality, phenomena which cannot be expressed by partial orders alone. In this paper, we provide a systematic classification of different subclasses of generalised traces in terms of the order structures representing them. We also show how the original trace model fits into the overall framework.

Keywords: trace, independence, dependence graph, partial order, trace of step sequences, simultaneity, serialisability, interleaving, generalised causal order structure

1 Introduction

Mazurkiewicz traces [14, 15] are a well-established, classical, and basic model for representing and structuring sequential observations of concurrent behaviour; see, e.g., [1, 10]. The fundamental assumption underlying trace theory is that independent events (occurrences of actions) may be observed in any order. Sequences that differ only w.r.t. the ordering of independent events are identified as belonging to the same concurrent run of the system under consideration. Thus a trace is an equivalence class of sequences comprising all (sequential) observations of a single concurrent run. The dependencies between the events of a trace are invariant among (common to) all elements of the trace. They define an acyclic dependence graph which — through its transitive closure — determines the underlying causality structure of the trace as a (labelled) partial order [16]. In fact, this partial order can also be obtained as the intersection of the labelled total orders corresponding to the sequences forming the trace. Moreover, the sequences belonging to the trace correspond exactly to the linearisations (saturations) of this partial order. In [17] the necessary connection between the causal structures (partial orders) and observations (total orders) is provided by showing that each partial order is the intersection of all its linearisations (Szpilrajn's property). Consequently, each trace can also be viewed as a labelled partial order which is unique up to isomorphism, i.e., up to the names of the underlying elements; see, e.g., [1, 3, 10]. Thus, to capture the essence of equivalence between different observations of the same run of a concurrent system, Mazurkiewicz traces bring together two mathematical ideas both based on a notion of independence between actions expressed as a binary independence relation ind. On the one hand, there are equations $ab = ba$ generating the equivalence by expressing the commutativity of occurrences of certain actions as determined by the independence relation. As a result, sequences $wabu$ and $wbau$ of action occurrences are considered equivalent whenever $\langle a, b\rangle \in \mathsf{ind}$, irrespective of what w and u are. On the other hand there is the idea of a common partial order structure that underlies equivalent observations defined by the ordering of the occurrences of dependent actions. However, being based on equating independence and lack of ordering, the concurrency paradigm of Mazurkiewicz traces with the corresponding partial order interpretation of concurrency is rather restricted [6].

In [5], a full generalisation of the theory of Mazurkiewicz traces is presented for the case that actions could occur and may be observed as occurring simultaneously. Thus observations consist of sequences of *steps*, i.e., sets of one or more actions that occur simultaneously. In order to retain the philosophy underlying Mazurkiewicz traces, the extended set-up is based on a few explicit and simple design choices. Instead of the single independence relation ind, now three basic relations between pairs of different actions are distinguished: *simultaneity* indicating that actions may occur together in a step; *serialisability* indicating a possible execution order for potentially simultaneous actions; and *interleaving* indicating that actions can *not* occur simultaneously though no specific ordering is required. These three relations are used to define *fundamental concurrency alphabets* and then applied to identify step sequences as observations of the same concurrent run. In this more general case, the equations are of the form $A_1A_2 = B_1B_2$ where the A_i and B_j are steps, and defined in terms of simultaneity, serialisability, and interleaving. The resulting equivalence classes of step sequences are called *generalised traces*. Actually, in this paper we will work with the, technically more convenient, definition of generalised traces provided by *generalised concurrency alphabets* also introduced in [5]. These concurrency alphabets have only two relations: simultaneity as before and *sequentialisability* which is a combination of serialisability and interleaving.

It is the main aim of this paper to characterise and discuss generalised traces in more detail. As demonstrated in [5], the clear semantical meaning of the three relations — simultaneity, serialisability, interleaving — allows for an intuitive classification of some natural subclasses of fundamental concurrency alphabets.

A hierarchy of interesting families of generalised traces is presented in [5], including new non-trivial classes of traces as well as the original Mazurkiewicz traces, comtraces [7, 13], and g-comtraces [8]. Comtraces are equivalence classes of step sequences derived from equations of the form $AB = A \uplus B$ using the two relations simultaneity and serialisability. Likewise, g-comtraces are equivalence classes of step sequences derived from equations of the form $AB = A \uplus B$ and $AB = BA$ — using simultaneity, serialisability as well as interleaving. Actually, as shown in [11], the equations used in [8] do not model the relevant aspects of concurrent behaviours in a fully adequate way. This has been corrected in the general set-up of [5] with generalised traces and fundamental concurrency alphabet providing the full generalisation of Mazurkiewicz traces to step sequences. There a complete picture is presented including extended dependence graphs and a characterisation of the causal order structures underlying generalised traces as the most general *order structures* from [4].

Modelling concurrency with order structures stems from the results of [2, 6] and [12]. The basic idea is that general concurrent causal behaviour is represented by *a pair* of relations, instead of just one, as in the standard (partial order) approach (see, e.g., [16]). Depending on the assumptions for the chosen model of concurrency, details vary, but basically there are two versions: one in which the two relations are interpreted as standard *causality* (dependence or precedence) and *weak causality* (not later than), respectively (see, e.g., [2, 6, 7]); and an extended, general, version (suggested in [6, 11] but eventually defined in [4]) with the two relations[5] *mutual exclusion* and *weak causality*. The first version has a relatively well developed theory and substantial applications (see, e.g., [2, 6, 7, 9]). The second one, however, is relatively new and as such the starting point for this paper where we identify the order structures that characterise the subfamilies of generalised traces from the classification in [5].

Notation

The inverse of a binary relation R is denoted by R^{-1}, and the symmetric closure by $R^{sym} = R \cup R^{-1}$. Moreover, R is a partial order relation if it is irreflexive and transitive, and a total order relation if it is a partial order relation such that $R^{sym} = (X \times X) \setminus id_X$. Given $R \subseteq X \times X$, $R^0 = id_X$ and $R^n = R^{n-1} \circ R$, for all $n \geq 1$. Then: $R^+ = \bigcup_{i \geq 1} R^i$ and R is acyclic if R^+ is asymmetric; $R^* = \bigcup_{i \geq 0} R^i = R^+ \cup id_X$; $R^\curlywedge = R^+ \setminus id_X = R^* \setminus id_X$ is the irreflexive transitive closure of R; and $R^\circledast = R^* \cap (R^*)^{-1}$ is the largest equivalence relation contained in R^*.

Throughout the paper, $\Sigma \neq \varnothing$ is a finite *alphabet* of actions, $\mathbb{S} = 2^\Sigma \setminus \{\varnothing\}$ is the set of all *steps*, and $\mathbb{S}^*$ is the set of *step sequences*. Let $u = A_1 \dots A_k \in \mathbb{S}^*$ be a step sequence. Then, for every action $a \in \Sigma$, $\#_u(a)$ is the number of occurrences of a within u; $occ(u) = \{\langle a, i\rangle \mid a \in \Sigma \wedge 1 \leq i \leq \#_u(a)\}$ is the set of *action occurrences* of u; and the *position* $pos_u(\alpha)$ within u of an action

[5] with causality as a derived notion

occurrence $\alpha = \langle a, i\rangle \in occ(u)$ is the smallest index $j \leq k$ such that the number of occurrences of a within $A_1 \ldots A_j$ is exactly i.

Let EQ be a finite set of equations on step sequences, each equation being of the form $u = v$, where u and v are nonempty step sequences. This set of equations induces a relation $\approx$ on step sequences comprising all pairs $\langle tuw, tvw\rangle$ such that $t, w \in \mathbb{S}^*$, and $u = v$ or $v = u$ is an equation in EQ. Furthermore, $\equiv$ is the equivalence relation on step sequences defined as $\approx^*$.

2 Generalised traces

The report [5] presents a full generalisation of the theory of Mazurkiewicz traces to the case that the smallest unit of observation is a set of actions (a step) rather than a single action. Thus observation sequences consist of sequences of *steps*, i.e., sets of actions that occur simultaneously. In order to extend the Mazurkiewicz trace approach to this more general situation, [5] proposes *generalised concurrency alphabets* Θ employing two relations defined for a set of atomic actions Σ, namely *simultaneity* sim defining legal steps, and *sequentialisation* seq specifying actions which can be swapped, or actions whose simultaneous occurrence means that they can also occur one after another. Together sim and seq define a set of equations and then an equivalence relation for step sequences over Σ.

A *generalised concurrency alphabet* is a triple $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle \in \Theta$, where Σ is a finite nonempty set, and sim and seq are two irreflexive relations over Σ such that sim and $\mathsf{seq} \setminus \mathsf{sim}$ are symmetric. The sets of *steps* and *step sequences* defined by θ are given by $\mathbb{S}_\theta = \{A \subseteq \Sigma \mid A \neq \varnothing \wedge (A \times A) \setminus id_\Sigma \subseteq \mathsf{sim}\}$ and $\mathsf{SSEQ}_\theta = \mathbb{S}_\theta^*$; and the induced *equations* are as follows, where $A, B \in \mathbb{S}_\theta$:

$$\begin{array}{lll} AB = BA & \text{if } A \times B \subseteq \mathsf{seq} \cap \mathsf{seq}^{-1} & (\textit{interleaving}) \\ AB = A \cup B & \text{if } A \times B \subseteq \mathsf{seq} \cap \mathsf{sim} & (\textit{serialisability}) \end{array} \quad (1)$$

Note that if $A, B \in \mathbb{S}_\theta$ and $A \times B \subseteq \mathsf{seq} \cap \mathsf{sim}$ then $A \cap B = \varnothing$ and $A \cup B \in \mathbb{S}_\theta$, and so the above equations (1) can never transform a step sequence in $\mathbb{S}_\theta^*$ into a sequence of sets outside $\mathbb{S}_\theta^*$.

Similarly as in the case of Mazurkiewicz traces, the equations (1) induce an equivalence relation $\equiv$ on the step sequences SSEQ_θ defined by θ. The equivalence classes TSSEQ_θ of the relation $\equiv$ are called (*generalised*) *traces*, and the generalised trace containing a step sequence $u \in \mathsf{SSEQ}_\theta$ is denoted by $[\![u]\!]_\theta$.

There are six semantically meaningful relationships between pairs of actions which together form a partition of $\Sigma \times \Sigma$:

(*i*) $\mathsf{con} = \mathsf{seq} \cap \mathsf{seq}^{-1} \cap \mathsf{sim}$ is *concurrency* identifying actions which can be executed simultaneously as well as in any order;

(*ii*) $\mathsf{inl} = (\mathsf{seq} \cap \mathsf{seq}^{-1}) \setminus \mathsf{sim}$ is *interleaving* allowing a pair of actions to be swapped, but disallowing simultaneous execution;

(*iii*) $\mathsf{ssi} = \mathsf{sim} \setminus (\mathsf{seq} \cup \mathsf{seq}^{-1})$ is *strong simultaneity* allowing a pair of actions to be executed simultaneously, but disallowing serialisation and interleaving;

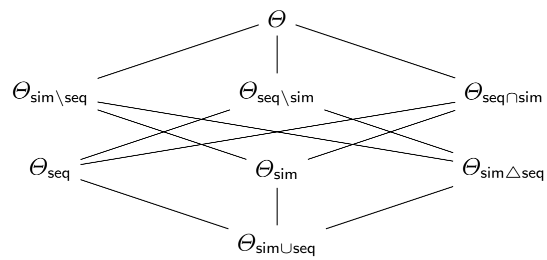

Fig. 1. Subclasses of generalised concurrency alphabets.

(*iv*) $\mathsf{sse} = (\mathsf{seq} \setminus \mathsf{seq}^{-1}) \cap \mathsf{sim}$ is *semi-serialisability* allowing a pair of simultaneously executed actions to be executed in the order given, but not in the reverse order;

(*v*) $\mathsf{wdp} = (\mathsf{seq}^{-1} \setminus \mathsf{seq}) \cap \mathsf{sim}$ is *weak dependence*, the inverse of semi-serialisability; and

(*vi*) $\mathsf{rig} = (\Sigma \times \Sigma) \setminus (\mathsf{sim} \cup (\mathsf{seq} \cap \mathsf{seq}^{-1}))$ is *rigid order* allowing neither simultaneity nor changing of the order of actions.

The Venn diagram of the relations sim and seq consists of three components, namely $\mathsf{sim} \setminus \mathsf{seq}$, $\mathsf{seq} \setminus \mathsf{sim}$, and $\mathsf{sim} \cap \mathsf{seq}$. Hence, one can distinguish in a natural way eight classes of generalised concurrency alphabets, as shown in Figure 1, where the subscripts indicate which relations are empty. Out of the seven proper subclasses of Θ, there is little to be gained from studying $\Theta_{\mathsf{sim}\cup\mathsf{seq}}$ and Θ_{seq} as for these each trace consists of only one step sequence. We will therefore concentrate in this paper on the remaining five types of generalised concurrency alphabets, viz.

$$\Theta_{\mathsf{seq}\setminus\mathsf{sim}}, \Theta_{\mathsf{sim}}, \Theta_{\mathsf{sim}\setminus\mathsf{seq}}, \Theta_{\mathsf{seq}\cap\mathsf{sim}}, \text{ and } \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$$

where $\mathsf{sim}\triangle\mathsf{seq} = (\mathsf{sim} \setminus \mathsf{seq}) \cup (\mathsf{seq} \setminus \mathsf{sim})$.

3 Order structures for generalised traces

The order theoretic treatment of generalised traces is based on *relational structures* $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ comprising a finite *domain* Δ, two binary relations $\rightleftharpoons$ and $\sqsubset$ on Δ, and a domain labelling $\Delta \xrightarrow{\ell} \Sigma$. To represent observational and causal relationships in the behaviours of concurrent systems we use OS, the *order structures* from [4] which are an extension of an idea first proposed in [2, 6, 12]. Individual observations (step sequences) are represented by *saturated* order structures, or SO-structures for short, and causal relationships are represented by *invariant* order structures (IO-structures). Formal definitions follow below.

An *order structure* is a relational structure $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ with a symmetric and irreflexive *mutex* relation $\rightleftharpoons$ and an irreflexive *weak causality* relation $\sqsubset$. Intuitively, Δ is the set of events that have happened during some execution of a concurrent system; $x \rightleftharpoons y$ means that x occurred *not simultaneously* with y, and $x \sqsubset y$ that x occurred *not later* than y, i.e., *before or simultaneously* with y. Hence if $x \sqsubset y$ and $x \rightleftharpoons y$, then x must have occurred *before* y. We will therefore refer to the intersection $\sqsubset \cap \rightleftharpoons$ as *causality* (or *precedence*), denoting it by $\prec$. Note that $x \sqsubset y \sqsubset x$ intuitively means that x and y were observed as *simultaneous*. It is assumed that os is *separable* meaning that $\rightleftharpoons \cap \sqsubset^{\circledast} = \varnothing$. Separability excludes situations where events forming a weak causality cycle in $\sqsubset^{\circledast}$ are also involved in the mutex relationship. Furthermore, it is assumed that os is *label-linear* meaning that $\rightleftharpoons \cap \sqsubset$ is a total order relation when restricted to the domain elements labelled by the same action. Referring to the set-up of Mazurkiewicz traces, order structures correspond to (labelled) acyclic relations.

An *extension* of the order structure os is any order structure $\langle \Delta, \rightleftharpoons', \sqsubset', \ell \rangle$ such that $\rightleftharpoons \subseteq \rightleftharpoons'$ and $\sqsubset \subseteq \sqsubset'$.

An SO-*structure* is a relational structure $sos = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ satisfying

$$\begin{array}{rcl} x \neq y \;\wedge\; x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y \\ x \rightleftharpoons y & \Longrightarrow & x \sqsubset^{sym} y \\ x \neq y \;\wedge\; x \not\rightleftharpoons y & \Longleftrightarrow & x \sqsubset y \sqsubset x \\ x \neq y \;\wedge\; \ell(x) = \ell(y) & \Longrightarrow & x \rightleftharpoons y \end{array}$$

One can see that saturated order structures are the only order structures without proper extensions. Referring to the set-up of Mazurkiewicz traces, SO-structures correspond to total order relations, i.e., the only acyclic relations which cannot be extended without violating their acyclicity. We denote by $\mathsf{satext}(os)$ the set of all saturated extensions of $os \in \mathsf{OS}$.

An IO-*structure* is a relational structure $ios = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ satisfying

$$\begin{array}{rcll} & & x \not\sqsubset x & \textit{(I1)} \\ x \neq y \;\wedge\; x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(I2)} \\ x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y & \textit{(I3)} \\ x \prec z \sqsubset y \;\vee\; x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y & \textit{(I4)} \\ z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z & \Longrightarrow & x \rightleftharpoons y & \textit{(I5)} \\ z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y & \Longrightarrow & x \rightleftharpoons y & \textit{(I6)} \\ x \neq y \;\wedge\; \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y & \textit{(I7)} \end{array}$$

Invariant order structures are the only order structures which cannot be extended without making the set of their saturated extensions smaller (follows from the results of [5]). Referring to the set-up of Mazurkiewicz traces, IO-structures correspond to partial order relations, the only acyclic relations which cannot be extended without making the set of their total order extensions smaller. Crucially, IOS are exactly those order structures os for which $\mathsf{satext}(os) \neq \varnothing$ and $os = \bigcap \mathsf{satext}(os)$. In other words, IO-structures are exactly those order structures which can be represented by their saturated extensions. This fundamental property is a counterpart of Szpilrajn's Theorem [17] which implies that partial orders are exactly those acyclic relations which can be represented by their total order extensions.

The *order structure closure* $\mathsf{OS} \xrightarrow{\mathsf{os2ios}} \mathsf{IOS}$ corresponds to the transitive closure for acyclic relations, and is given by:

$$\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \xmapsto{\mathsf{os2ios}} \langle \Delta, \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \,\cup\, \sqsubset^{\circledast} \circ \nabla^{sym} \circ \sqsubset^{\circledast}, \sqsubset^{\curlywedge}, \ell \rangle$$

where $\nabla = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons w \wedge x \sqsubset^{*} z \sqsubset^{*} y \wedge x \sqsubset^{*} w \sqsubset^{*} y\}$.

Order structure closure is the unique mapping $\mathsf{OS} \xrightarrow{\mathsf{f}} \mathsf{IOS}$ such that $\mathsf{f}(ios) = ios$, for every $ios \in \mathsf{IOS}$, and $\mathsf{satext}(os) = \mathsf{satext} \circ \mathsf{f}(os)$, for every $os \in \mathsf{OS}$ (see [5]). This corresponds to the fact that transitive closure is the unique mapping from acyclic relations to partial orders which preserves the total order extensions.

4 Relating generalised traces and order structures

In this section we will identify the order structures corresponding to the five subclasses of generalised concurrency alphabets identified in Section 2, but first we recall from [5] the main results established for the general case.

Let $\theta = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$ be a generalised concurrency alphabet. An *event domain* (for θ) is a set $\Delta \subseteq \Sigma \times \mathbb{N}$ for which there is a mapping $\Sigma \xrightarrow{\epsilon} \mathbb{N}$ such that $\Delta = \{\langle a, i \rangle \mid a \in \Sigma \wedge 1 \leq i \leq \epsilon(a)\}$.

An SO-structure $sos = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ is *consistent* with θ if Δ is an event domain for θ, $\langle a, i \rangle \overset{\ell}{\mapsto} a$ is the default labelling of Δ, and, for all distinct $\langle a, i \rangle, \langle a, j \rangle, \langle b, k \rangle \in \Delta$, we have:

$$\langle a, i \rangle \prec \langle a, j \rangle \iff i < j \quad \textit{and} \quad \langle a, i \rangle \sqsubset^{\circledast} \langle b, k \rangle \implies \langle a, b \rangle \in \mathsf{sim}\ .$$

We let SOS_θ denote the set of all SO-structures *consistent* with θ. Step sequences defined by θ correspond to SO-structures in SOS_θ via the bijection $\mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2sos}} \mathsf{SOS}_\theta$ such that $\mathsf{sseq2sos}(u) = \langle occ(u), \rightleftharpoons, \sqsubset, \ell \rangle$, where, for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$ we have:

$$k \neq m \implies \alpha \rightleftharpoons \beta \quad \textit{and} \quad k \leq m \wedge \alpha \neq \beta \implies \alpha \sqsubset \beta\ .$$

Dependencies between events are captured by the map $\mathsf{SSEQ}_\theta \xrightarrow{\mathsf{sseq2os}_\theta} \mathsf{OS}$ such that $\mathsf{sseq2os}_\theta(u) = \langle occ(u), \rightleftharpoons, \sqsubset, \ell \rangle$, where, for all $\alpha, \beta \in occ(u)$ with $pos_u(\alpha) = k$ and $pos_u(\beta) = m$:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq} \quad \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{sim} \cap \mathsf{seq}^{-1} \ \wedge \ k > m \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \cap \mathsf{seq}^{-1} \ \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sim} \setminus \mathsf{seq}^{-1} \ \wedge \ k = m \end{array} \tag{2}$$

or, alternatively:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{wdp} \cup \mathsf{rig} \cup \mathsf{inl} \ \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{sse} \cup \mathsf{rig} \cup \mathsf{inl} \ \wedge \ k > m \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{sse} \cup \mathsf{wdp} \cup \mathsf{rig} \ \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{sse} \ \wedge \ k = m \end{array} \tag{3}$$

We refer to $\mathsf{sseq2os}_\theta(u)$ as the *dependence graph* of u. Crucially, if $u \equiv w$, then $\mathsf{sseq2os}_\theta(u) = \mathsf{sseq2os}_\theta(w)$, and so dependence graphs can be lifted to the level of generalised traces via $\mathsf{sseq2os}_\theta([\![u]\!]) = \mathsf{sseq2os}_\theta(u)$. Hence there are two kinds of order structures capturing causal dependencies in the step sequences of SSEQ_θ and traces in TSSEQ_θ, namely dependence graphs and their closures, i.e., $\mathsf{OS}_\theta = \mathsf{sseq2os}_\theta(\mathsf{SSEQ}_\theta)$ and $\mathsf{IOS}_\theta = \mathsf{os2ios}(\mathsf{OS}_\theta)$.
In what follows, for every set $\varPhi$ of generalised concurrency alphabets, we will denote $\mathsf{OS}_\varPhi = \bigcup_{\theta\in\varPhi} \mathsf{OS}_\theta$ and $\mathsf{IOS}_\varPhi = \bigcup_{\theta\in\varPhi} \mathsf{IOS}_\theta$.

Generalised traces in TSSEQ_θ can be identified with the invariant order structures in IOS_θ and a suitable correspondence is established by the pair of inverse bijections

$$\mathsf{TSSEQ}_\theta \xrightarrow{\mathsf{os2ios}\,\circ\,\mathsf{sseq2os}_\theta} \mathsf{IOS}_\theta \xrightarrow{\mathsf{sseq2sos}^{-1}\,\circ\,\mathsf{satext}} \mathsf{TSSEQ}_\theta\ .$$

Moreover, if an order structure os has injective labelling, then there is a generalised concurrency alphabet θ and a step sequence $u \in \mathsf{SSEQ}_\theta$ such that os is isomorphic to $\mathsf{sseq2os}_\theta(u)$. Thus generalised concurrency alphabets can generate all the complex patterns involving causal relationships captured by IO-structures.

An example system model for which generalised traces and invariant order structures provide a suitable semantical treatment are the elementary net systems with inhibitor and mutex arcs [11]. Note that every complex pattern (without labels) can be obtained as a closure of dependence graph for a computation enabled in an elementary net system with inhibitor and mutex arcs.

The restriction to subclasses of generalised concurrency alphabets can lead to striking simplifications in the order structures involved and the corresponding order structure closure. In particular, such simplifications enable a more concise and efficient treatment of the computational aspects involving generalised traces and their corresponding order structures. In what follows, we will consider the five non-trivial subclasses of generalised concurrency alphabets, aiming at as simple as possible descriptions of the order structures capturing the corresponding invariant order structures.

4.1 Order structures for the alphabets in $\varTheta_{\mathsf{sim}}$

A generalised concurrency alphabet $\mu = \langle \varSigma, \mathsf{sim}, \mathsf{seq}\rangle \in \varTheta_{\mathsf{sim}}$ has $\mathsf{sim} = \varnothing$ and so does not allow for true step sequences and there are no serialisability equations as in (1). Moreover, $\mathsf{con} = \mathsf{ssi} = \mathsf{sse} = \mathsf{wdp} = \varnothing$, $\mathsf{seq} = \mathsf{seq}^{-1} = \mathsf{inl}$ and $\mathsf{rig} = (\varSigma \times \varSigma) \setminus \mathsf{inl}$. As a result, one can simplify the definition of the dependence graph of a step sequence $u \in \mathsf{SSEQ}_\mu$, by replacing (3) with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & k \neq m \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{rig} \ \wedge\ k < m\ . \end{array} \qquad (4)$$

It is possible to treat μ as a Mazurkiewicz concurrency alphabet $\langle \varSigma, \mathsf{seq}\rangle$ with seq and rig playing the roles of the standard independence and dependence relations, respectively. As all step sequences in SSEQ_μ consist of singleton steps, they correspond one-to-one to the sequences in Σ^*. Moreover, the saturated order structures in SOS_μ correspond one-to-one to the sequences in Σ^*. Indeed, since $\mathsf{sim} = \varnothing$, we have that for every $sos = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{SOS}_\mu$ it is the case that $\sqsubset^{\circledast} = id_\Delta$, and so $\prec$ is a total order relation.

The order structures $\mathsf{OS}_{\mathsf{sim}}$ reflecting the causal dependencies in the generalised traces over the alphabets of Θ_{sim} are those $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}$ for which $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$. The corresponding invariant order structures can then be provided with a simpler definition.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IOS}_{\mathsf{sim}}$ if

$$
\begin{array}{rcrl}
 & & x \not\sqsubset x & (A1) \\
x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & (A2) \\
x \neq y & \Longleftrightarrow & x \rightleftharpoons y & (A3) \\
x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & (A4)
\end{array}
$$

The simplified order closure $\mathsf{OS}_{\mathsf{sim}} \xrightarrow{\mathsf{os2ios}_{\mathsf{sim}}} \mathsf{IOS}_{\mathsf{sim}}$ is such that:

$$
\mathsf{os2ios}_{\mathsf{sim}}(os) = \langle \Delta, \rightleftharpoons, \sqsubset^+, \ell \rangle \ ,
$$

for every $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}}$. Hence it corresponds to the transitive closure of an acyclic relation. The justification of these definitions is provided by the following results.

Theorem 1.

$$
\begin{array}{ccccc}
\mathsf{OS}_{\Theta_{\mathsf{sim}}} & \subset & \mathsf{OS}_{\mathsf{sim}} & \subset & \mathsf{OS} \\
\cup & & \cup & & \cup \\
\mathsf{IOS}_{\Theta_{\mathsf{sim}}} & \subset & \mathsf{IOS}_{\mathsf{sim}} & \subset & \mathsf{IOS}
\end{array}
$$

Theorem 2. $\mathsf{os2ios}_{\mathsf{sim}}$ *is a surjection with* $\mathsf{os2ios}_{\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}}}$.

Theorem 3. *If $os \in \mathsf{OS}_{\mathsf{sim}}$ has an injective labelling, then there are $\mu \in \Theta_{\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_\mu$ such that os is isomorphic to $\mathsf{sseq2os}_\mu(u)$.*

Following Mazurkiewicz [15], the classical example of a system model for which the generalised concurrency alphabets in Θ_{sim} and invariant order structures $\mathsf{IOS}_{\mathsf{sim}}$ provide a suitable semantical treatment are the elementary net systems with sequential execution semantics. Note that every complex pattern (without labels) can be obtained as a closure of dependence graph for a computation enabled in an elementary net system with sequential execution semantics.

4.2 Order structures for the alphabets in $\Theta_{\mathsf{seq} \setminus \mathsf{sim}}$

A generalised concurrency alphabet $\sigma = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle \in \Theta_{\mathsf{seq} \setminus \mathsf{sim}}$ is the one satisfying $\mathsf{seq} \setminus \mathsf{sim} = \varnothing$ and therefore we have $\mathsf{seq} \subseteq \mathsf{sim}$, $\mathsf{rig} = (\Sigma \times \Sigma) \setminus \mathsf{sim}$, and $\mathsf{inl} = \varnothing$. As a result, one can simplify the definition of the dependence graph of a step sequence $u \in \mathsf{SSEQ}_\sigma$, by replacing (3) with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \quad \wedge \ k \neq m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{sse} \quad \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{wdp} \quad \wedge \ k > m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{rig} \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{rig} \cup \mathsf{wdp} \ \wedge \ k < m \\ & \textit{or} & \langle \ell(\alpha), \ell(\beta) \rangle \in \mathsf{ssi} \cup \mathsf{sse} \ \wedge \ k \leq m \end{array} \tag{5}$$

Alphabets in $\Theta_{\mathsf{seq}\backslash\mathsf{sim}}$ do not allow true interleaving, and swapping of steps can be achieved by splitting and combining. In [6], such alphabets are referred to as *comtrace alphabets*.

The order structures $\mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}}$ needed to reflect causal dependencies in the generalised traces over the concurrent alphabets of $\Theta_{\mathsf{seq}\backslash\mathsf{sim}}$ are all those order structures $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}$ for which $x \rightleftharpoons y \Longrightarrow x \sqsubset^{sym} y$. The corresponding invariant order structures can then be provided with a simpler definition.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IOS}_{\mathsf{seq}\backslash\mathsf{sim}}$ if

$$\begin{array}{rcl} & & x \not\sqsubset x \qquad (B1) \\ x \neq y \ \wedge \ \ x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y \qquad (B2) \\ x \rightleftharpoons y & \Longrightarrow & x \sqsubset^{sym} y \ \wedge \ y \rightleftharpoons x \qquad (B3) \\ x \prec z \sqsubset y \ \vee \ \ x \sqsubset z \prec y & \Longrightarrow & x \rightleftharpoons y \qquad (B4) \\ x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \rightleftharpoons y \qquad (B5) \end{array}$$

The simplified order closure $\mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}} \xrightarrow{\mathsf{os2ios}_{\mathsf{seq}\backslash\mathsf{sim}}} \mathsf{IOS}_{\mathsf{seq}\backslash\mathsf{sim}}$ is such that:

$$\mathsf{os2ios}_{\mathsf{seq}\backslash\mathsf{sim}}(os) = \langle \Delta, (\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym}, \sqsubset^{\curlywedge}, \ell \rangle \ ,$$

for every $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}}$. The justification of these definitions is provided by the following results.

Theorem 4.

$$\begin{array}{ccccc} \mathsf{OS}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} & \subset & \mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}} & \subset & \mathsf{OS} \\ \cup & & \cup & & \cup \\ \mathsf{IOS}_{\Theta_{\mathsf{seq}\backslash\mathsf{sim}}} & \subset & \mathsf{IOS}_{\mathsf{seq}\backslash\mathsf{sim}} & \subset & \mathsf{IOS} \end{array}$$

Theorem 5. $\mathsf{os2ios}_{\mathsf{seq}\backslash\mathsf{sim}}$ *is a surjection with* $\mathsf{os2ios}_{\mathsf{seq}\backslash\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}}}$.

Theorem 6. *If $os \in \mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}}$ has an injective labelling $\ell : \Delta \to \Sigma$, then there are $\sigma \in \Theta_{\mathsf{seq}\backslash\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_\sigma$ such that os is isomorphic to $\mathsf{sseq2os}_\sigma(u)$.*

An example of a system model for which the generalised concurrency alphabets in $\Theta_{\mathsf{seq}\backslash\mathsf{sim}}$ and invariant order structures $\mathsf{IOS}_{\mathsf{seq}\backslash\mathsf{sim}}$ provide a suitable semantical treatment are the elementary net systems with inhibitor arcs [7]. Note that every complex pattern (without labels) can be obtained as a closure of dependence graph for a computation in an elementary net system with inhibitor arcs.

Finally, as shown below, traces generated by the alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ are histories satisfying the concurrency paradigm π_3 of [6] by which actions that can be executed in any order can also be executed simultaneously (but not necessarily vice versa).

Proposition 1. *Let* α *and* β *be two action occurrences of a generalised trace* τ *generated by* $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$. *Then*

$$(\exists u \in \tau:\ pos_u(\alpha) < pos_u(\beta)) \ \wedge\ (\exists w \in \tau:\ pos_w(\alpha) > pos_w(\beta)) \\ \Longrightarrow \\ (\exists v \in \tau:\ pos_v(\alpha) = pos_v(\beta))$$

4.3 Order structures for the alphabets in $\Theta_{\mathsf{sim}\setminus\mathsf{seq}}$

A generalised concurrency alphabet $\kappa = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle \in \Theta_{\mathsf{sim}\setminus\mathsf{seq}}$ is the one satisfying $\mathsf{sim} \setminus \mathsf{seq} = \varnothing$ and therefore we have $\mathsf{ssi} = \mathsf{sse} = \mathsf{wdp} = \varnothing$ and $\mathsf{rig} = (\Sigma \times \Sigma) \setminus \mathsf{seq}$. As a result, one can simplify the definition of the dependence graph of a step sequence $u \in \mathsf{SSEQ}_\mu$, by replacing (2) with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta)\rangle \notin \mathsf{sim} \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta)\rangle \notin \mathsf{seq} \ \wedge\ k < m \end{array} \tag{6}$$

For the alphabets in $\Theta_{\mathsf{sim}\setminus\mathsf{seq}}$ the serialisability equations are rich enough to split any step in every possible way.

The order structures $\mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}}$ are all those $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}$ for which $x \sqsubset^{sym} y \Longrightarrow x \rightleftharpoons y$. The corresponding invariant order structures can also be provided with a simpler definition.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$ belongs to $\mathsf{IOS}_{\mathsf{sim}\setminus\mathsf{seq}}$ if:

$$\begin{array}{rcll} x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(C1)} \\ x \sqsubset^{sym} y & \Longrightarrow & x \rightleftharpoons y & \textit{(C2)} \\ x \rightleftharpoons y & \Longrightarrow & y \rightleftharpoons x \neq y & \textit{(C3)} \\ x \neq y \ \wedge\ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & \textit{(C4)} \end{array}$$

The simplified order closure $\mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}} \xrightarrow{\mathsf{os2ios}_{\mathsf{sim}\setminus\mathsf{seq}}} \mathsf{IOS}_{\mathsf{sim}\setminus\mathsf{seq}}$ is such that:

$$\mathsf{os2ios}_{\mathsf{sim}\setminus\mathsf{seq}}(os) = \langle \Delta, \rightleftharpoons \cup (\sqsubset^+)^{sym}, \sqsubset^+, \ell\rangle\ ,$$

for every $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}}$. The justification of these definitions is provided by the following results.

Theorem 7.

$$\begin{array}{ccccc} \mathsf{OS}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} & \subset & \mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}} & \subset & \mathsf{OS} \\ \cup & & \cup & & \cup \\ \mathsf{IOS}_{\Theta_{\mathsf{sim}\setminus\mathsf{seq}}} & \subset & \mathsf{IOS}_{\mathsf{sim}\setminus\mathsf{seq}} & \subset & \mathsf{IOS} \end{array}$$

Theorem 8. $\mathsf{os2ios}_{\mathsf{sim}\setminus\mathsf{seq}}$ *is a surjection with* $\mathsf{os2ios}_{\mathsf{sim}\setminus\mathsf{seq}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}}}$.

Theorem 9. *If* $os \in \mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}}$ *has an injective labelling* $\ell : \Delta \to \Sigma$*, then there are* $\kappa \in \Theta_{\mathsf{sim}\setminus\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\kappa$ *such that os is isomorphic to* $\mathsf{sseq2os}_\kappa(u)$.

Finally, as shown below, traces generated by the alphabets in $\Theta_{\mathsf{sim}\setminus\mathsf{seq}}$ are histories satisfying the concurrency paradigm π_2 of [6].

Proposition 2. *Let* α *and* β *be distinct action occurrences* α *and* β *of a generalised trace* τ *generated by* $\kappa \in \Theta_{\mathsf{sim}\setminus\mathsf{seq}}$. *Then*

$$(\exists v \in \tau : \ pos_v(\alpha) = pos_v(\beta)) \\ \Longrightarrow \\ (\exists u \in \tau : \ pos_u(\alpha) < pos_u(\beta)) \ \wedge \ (\exists w \in \tau : \ pos_w(\alpha) > pos_w(\beta))$$

4.4 Order structures for the alphabets in $\Theta_{\mathsf{seq}\cap\mathsf{sim}}$

A generalised concurrency alphabet $\nu \in \Theta_{\mathsf{sim}\cap\mathsf{seq}}$ is the one satisfying $\mathsf{sim} \cap \mathsf{seq} = \varnothing$, and so we have $\mathsf{ssi} = \mathsf{sim}$, $\mathsf{sse} = \mathsf{wdp} = \mathsf{con} = \varnothing$, and $\mathsf{rig} = (\Sigma \times \Sigma) \setminus (\mathsf{sim} \uplus \mathsf{seq})$. As a result, one can simplify the definition of the dependence graph of a step sequence $u \in \mathsf{SSEQ}_\mu$, by replacing (2) with:

$$\begin{aligned} &\alpha \rightleftharpoons \beta \quad \textit{if} \ \ k \neq m \\ &\alpha \sqsubset \beta \quad \textit{if} \ \ \langle \ell(\alpha), \ell(\beta) \rangle \notin \mathsf{seq} \ \ \wedge \ \ k \leq m \ \wedge \ \alpha \neq \beta \end{aligned} \tag{7}$$

For the alphabets in $\Theta_{\mathsf{sim}\cap\mathsf{seq}}$ steps can be only manipulated through the interleaving equations.

The order structures $\mathsf{OS}_{\mathsf{sim}\cap\mathsf{seq}}$ are all those $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}$ for which $x \neq y \Longrightarrow x \rightleftharpoons y \ \vee \ x \sqsubset y \sqsubset x$, and the axiomatisation of the corresponding invariant order structures becomes simpler.

A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$ belongs to $\mathsf{IOS}_{\mathsf{sim}\cap\mathsf{seq}}$ if:

$$\begin{array}{rcll} & & x \not\rightleftharpoons x & \textit{(D1)} \\ x \neq y \wedge x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(D2)} \\ x \not\rightleftharpoons y \wedge x \neq y & \Longleftrightarrow & x \sqsubset y \sqsubset x & \textit{(D3)} \\ x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \prec^{sym} y & \textit{(D4)} \end{array}$$

The simplified order closure $\mathsf{OS}_{\mathsf{sim}\cap\mathsf{seq}} \xrightarrow{\mathsf{os2ios}_{\mathsf{sim}\cap\mathsf{seq}}} \mathsf{IOS}_{\mathsf{sim}\cap\mathsf{seq}}$ is such that:

$$\mathsf{os2ios}_{\mathsf{sim}\cap\mathsf{seq}}(os) = \langle \Delta, \rightleftharpoons, \sqsubset^{\curlywedge}, \ell \rangle \ ,$$

for every $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}\cap\mathsf{seq}}$. The justification of these definitions is provided by the following results.

Theorem 10.

$$\begin{array}{ccccc} \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} & \subset & \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}} & \subset & \mathsf{OS} \\ \cup & & \cup & & \cup \\ \mathsf{IOS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} & \subset & \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} & \subset & \mathsf{IOS} \end{array}$$

Theorem 11. $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}$ *is a surjection with* $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}}$.

Theorem 12. *If* $os \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ *has an injective labelling* $\ell : \Delta \to \Sigma$*, then there are* $\nu \in \Theta_{\mathsf{seq}\cap\mathsf{sim}}$ *and* $u \in \mathsf{SSEQ}_\nu$ *such that os is isomorphic to* $\mathsf{sseq2os}_\nu(u)$.

4.5 Order structures for the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$

A generalised concurrency alphabet $\omega = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ is the one satisfying $\mathsf{sim}\triangle\mathsf{seq} = \varnothing$ and therefore we have $\mathsf{sim} = \mathsf{seq} = \mathsf{con}$, $\mathsf{ssi} = \mathsf{sse} = \mathsf{wdp} = \mathsf{inl} = \varnothing$ and $\mathsf{rig} = (\Sigma \times \Sigma) \setminus \mathsf{con}$. As a result, one can simplify the definition of the dependence graph of a step sequence $u \in \mathsf{SSEQ}_\mu$, by replacing (3) with:

$$\begin{array}{lll} \alpha \rightleftharpoons \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{rig} \\ \alpha \sqsubset \beta & \textit{if} & \langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{rig} \ \wedge \ k < m \end{array} \tag{8}$$

For the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ the interleaving equations are not really needed, and the serialisability equations are rich enough to split and reorder steps in every possible way. As a result, all steps can be completely sequentialised.

The order structures $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ are all those $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}$ for which $x \rightleftharpoons y \iff x \sqsubset^{sym} y$. The corresponding invariant order structures can also be provided with a simpler definition. A relational structure $\langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$ belongs to $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ if

$$\begin{array}{rcll} & & x \not\sqsubset x & \textit{(E1)} \\ x \sqsubset z \sqsubset y & \Longrightarrow & x \sqsubset y & \textit{(E2)} \\ x \rightleftharpoons y & \Longleftrightarrow & x \sqsubset^{sym} y & \textit{(E3)} \\ x \neq y \ \wedge \ \ell(x) = \ell(y) & \Longrightarrow & x \sqsubset^{sym} y & \textit{(E4)} \end{array}$$

The simplified order closure $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} \xrightarrow{\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}} \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ is such that:

$$\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(os) = \langle \Delta, (\sqsubset^+)^{sym}, \sqsubset^+, \ell\rangle \ ,$$

for every $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$. The justification of these definitions is provided by the following results.

Theorem 13.

$$\begin{array}{ccccc} \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} & \subset & \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} & \subset & \mathsf{OS} \\ \cup & & \cup & & \cup \\ \mathsf{IOS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} & \subset & \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} & \subset & \mathsf{IOS} \end{array}$$

Theorem 14. $\mathsf{os2ios}_{\mathsf{seq}\triangle\mathsf{sim}}$ *is a surjection with* $\mathsf{os2ios}_{\mathsf{seq}\triangle\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\triangle\mathsf{sim}}}$.

Theorem 15. *If* $os \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ *has an injective labelling* $\ell : \Delta \to \Sigma$*, then there are* $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ *and* $u \in \mathsf{SSEQ}_\omega$ *such that os is isomorphic to* $\mathsf{sseq2os}_\omega(u)$.

It may come as a surprise that although the structures $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ are in a one-to-one correspondence with partial orders, similarly as for $\mathsf{IOS}_{\mathsf{sim}}$, the actual definition of the two classes of order structures is different.

Finally, as shown below, the generalised traces generated by the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ are histories satisfying the true concurrency paradigm π_8 of [6] and a system model for which this subclass provides a suitable semantical treatment are the elementary net systems with step sequence semantics. Note that every complex pattern (without labels) can be obtained as a closure of dependence graph for a computation in an elementary net system with step sequence semantics.

Proposition 3. *Let α and β be distinct action occurrences α and β of a generalised trace τ generated by $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$. Then*

$$(\exists v \in \tau : \ pos_v(\alpha) = pos_v(\beta)) \\ \Longleftrightarrow \\ (\exists u \in \tau : \ pos_u(\alpha) < pos_u(\beta)) \ \wedge \ (\exists w \in \tau : \ pos_w(\alpha) > pos_w(\beta))$$

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A Proofs for the alphabets in Θ_{sim}

Lemma 1. $\mathsf{IOS}_{\mathsf{sim}} \subseteq \mathsf{IOS}$.

Proof. We first note that *(I1)* is simply *(A1)*. To show *(I2)* we observe that:

$$x \neq y \ \wedge \ x \sqsubset z \sqsubset y \ \Longrightarrow_{(A2)} \ x \sqsubset y \ .$$

To show *(I3)* we observe that:

$$x \rightleftharpoons y \ \Longrightarrow_{(A3)} \ x \neq y \ \Longrightarrow \ x \neq y \ \wedge \ y \neq x \ \Longrightarrow_{(A3)} \ x \neq y \ \wedge \ y \rightleftharpoons x \ .$$

To show *(I4)* we observe that:

$$\begin{aligned} x = y \ \wedge \ (x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y) \ &\Longrightarrow \quad x \prec z \sqsubset x \ \vee \ x \sqsubset z \prec x \\ &\Longrightarrow_{(A2)} \ x \sqsubset x \\ &\Longrightarrow_{(A1)} \ \mathit{false} \end{aligned}$$

and so we have:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \ \Longrightarrow \ x \neq y \ \Longrightarrow_{(A3)} \ x \rightleftharpoons y \ .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \Longrightarrow_{(A2)} \ z \sqsubset z \ \Longrightarrow_{(A1)} \ \mathit{false} \ .$$

To show *(I6)* we observe that:

$$x = y \ \wedge \ x \sqsubset z \sqsubset y \ \Longrightarrow \ x \sqsubset z \sqsubset x \ \Longrightarrow_{(A2,A1)} \ \mathit{false}$$

and so we have:

$$z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \Longrightarrow \ x \neq y \ \Longrightarrow_{(A3)} \ x \rightleftharpoons y \ .$$

We finally note that *(I7)* follows from *(A3)* and *(A4)*. □

Lemma 2. $\mathsf{IOS}_{\mathsf{sim}} \subseteq \mathsf{OS}_{\mathsf{sim}}$.

Proof. Follows from Lemma 1, $\mathsf{IOS} \subseteq \mathsf{OS}$, and *(A3)*. □

Lemma 3. $\mathsf{os2ios}_{\mathsf{sim}}(\mathsf{OS}_{\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{sim}}$.

Proof. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}}$ and $ios = \mathsf{os2ios}_{\mathsf{sim}}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(A1)* suppose that $x \mathrel{\widehat{\sqsubset}} x$ which means $x \sqsubset^+ x$. Since $\sqsubset$ is irreflexive, there is $y \neq x$ satisfying $x \sqsubset^* y \sqsubset^* x$. Hence, by the separability of os, $x \neq y$, contradicting the definition of $\mathsf{OS}_{\mathsf{sim}}$.
To show *(A2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \ \Longrightarrow \ x \sqsubset^+ z \sqsubset^+ y \ \Longrightarrow \ x \sqsubset^+ y \ \Longrightarrow \ x \mathrel{\widehat{\sqsubset}} y \ .$$

We then observe that *(A3)* follows from $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$. Finally, *(A4)* follows from the libel-linearity of os, as shown below:

$$x \neq y \ \wedge \ \ell(x) = \ell(y) \ \Longrightarrow \ x \prec^{sym} y \ \Longrightarrow \ x \mathrel{\widehat{\sqsubset}^{sym}} y \ .$$

Hence $ios \in \mathsf{IOS}_{\mathsf{sim}}$. □

Proof of Theorem 1

Let us consider one by one all the inclusions:

- $\mathsf{IOS} \subset \mathsf{OS}$ follows from the general results proven in [5] and
$$os = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OS} \setminus \mathsf{IOS} \ .$$
- $\mathsf{IOS}_{\mathsf{sim}} \subset \mathsf{OS}_{\mathsf{sim}}$ follows from $os \in \mathsf{OS}_{\mathsf{sim}} \setminus \mathsf{IOS}_{\mathsf{sim}}$ and Lemma 2.
- $\mathsf{IOS}_{\Theta_{\mathsf{sim}}} \subset \mathsf{OS}_{\Theta_{\mathsf{sim}}}$ follows from $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}}}$ and the general results proven in [5].
- $\mathsf{OS}_{\mathsf{sim}} \subset \mathsf{OS}$ follows from the definition of $\mathsf{OS}_{\mathsf{sim}}$ and
$$os' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OS} \setminus \mathsf{OS}_{\mathsf{sim}} \ .$$
- $\mathsf{IOS}_{\mathsf{sim}} \subset \mathsf{IOS}$ follows from $os' \in \mathsf{IOS} \setminus \mathsf{IOS}_{\mathsf{sim}}$ and Lemma 1.
- $\mathsf{OS}_{\Theta_{\mathsf{sim}}} \subset \mathsf{OS}_{\mathsf{sim}}$ can be proven by taking $\mu \in \Theta_{\mathsf{sim}}$, $u \in \mathsf{SSEQ}_\mu$, and $os = \mathsf{sseq2os}_\mu(u)$. We know that $os \in \mathsf{OS}$. Suppose that $\alpha, \beta \in occ(u)$ and $\alpha \neq \beta$. Then, by $\mathsf{sim} = \varnothing$, $pos_u(\alpha) \neq pos_u(\beta)$. Hence, by (4), we have $\alpha \rightleftharpoons_{os} \beta$, and so $os \in \mathsf{OS}_{\mathsf{sim}}$. Moreover, we note that
$$os'' = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OS}_{\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}}}.$$
- $\mathsf{IOS}_{\Theta_{\mathsf{sim}}} \subset \mathsf{IOS}_{\mathsf{sim}}$ follows from $os'' \in \mathsf{IOS}_{\mathsf{sim}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}}}$, $\mathsf{OS}_{\Theta_{\mathsf{sim}}} \subseteq \mathsf{OS}_{\mathsf{sim}}$ and Lemma 3.

Moreover, note that $os \in \mathsf{OS}_{\mathsf{sim}} \setminus \mathsf{IOS}$ and $os' \in \mathsf{IOS} \setminus \mathsf{OS}_{\mathsf{sim}}$ which justifies that IOS and $\mathsf{OS}_{\mathsf{sim}}$ are not related. Similarly, there is no inclusion between $\mathsf{IOS}_{\mathsf{sim}}$ and $\mathsf{OS}_{\Theta_{\mathsf{sim}}}$ since $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}}} \setminus \mathsf{IOS}_{\mathsf{sim}}$ and $os'' \in \mathsf{IOS}_{\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}}}$. □

Proof of Theorem 2

We first show that $\mathsf{os2ios}_{\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}}}$. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{sim}}$ and $ios = \mathsf{os2ios}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell\rangle$. In this case $\sqsubset^{\circledast} = id_\Delta$ which follows directly from $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$ and the separability of os. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^+$. Hence
$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons \cup \nabla^{sym}, \sqsubset^+, \ell\rangle \ ,$$
where $\nabla = \{\langle x,y\rangle \mid \exists z,w : z \rightleftharpoons w \wedge x \sqsubset^* z \sqsubset^* y \wedge x \sqsubset^* w \sqsubset^* y\}$. Moreover, ∇ is irreflexive (as $\widehat{\rightleftharpoons}$ is irreflexive) and $\rightleftharpoons = (\Delta \times \Delta) \setminus id_\Delta$. We therefore obtain:
$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons, \sqsubset^+, \ell\rangle \ .$$

We then observe that $\mathsf{os2ios}_{\mathsf{sim}}(\mathsf{OS}_{\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{sim}}$ follows from Lemma 1, Lemma 2, Lemma 3, $\mathsf{os2ios}_{\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}}}$, and the fact that $\mathsf{os2ios}$ is the identity on IOS, as then we obtain
$$\mathsf{os2ios}_{\mathsf{sim}}(\mathsf{OS}_{\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{sim}}$$
and
$$\mathsf{os2ios}_{\mathsf{sim}}(\mathsf{OS}_{\mathsf{sim}}) \supseteq \mathsf{os2ios}_{\mathsf{sim}}(\mathsf{IOS}_{\mathsf{sim}}) = \mathsf{os2ios}(\mathsf{IOS}_{\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{sim}}.$$
□

Proof of Theorem 3

Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [5] it follows that there exists $sos \in \mathsf{satext}(os)$ which, by the definition of $\mathsf{OS}_{\mathsf{sim}}$ satisfies $\rightleftharpoons_{sos} = (\Delta \times \Delta) \setminus id_\Delta$. Hence $u = \mathsf{sseq2sos}^{-1}(sos)$ is a sequence of singleton steps. Let $\mu = \langle \Sigma, \varnothing, \mathsf{seq} \rangle$, where:

$$\mathsf{seq} = \left\{ \langle a, b \rangle \in \Sigma \times \Sigma \;\middle|\; \begin{array}{l} pos_u(\langle a,1 \rangle) < pos_u(\langle b,1 \rangle) \wedge \langle a,1 \rangle \not\sqsubset \langle b,1 \rangle \;\; \vee \\ pos_u(\langle b,1 \rangle) < pos_u(\langle a,1 \rangle) \wedge \langle b,1 \rangle \not\sqsubset \langle a,1 \rangle \end{array} \right\} .$$

Clearly, $\mu \in \Theta_{\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_\mu$. It is easy to check that $os = \mathsf{sseq2os}_\mu(u)$. □

B Proofs for the alphabets in $\Theta_{\mathsf{seq}\setminus\mathsf{sim}}$

Lemma 4. $\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \subseteq \mathsf{IOS}$.

Proof. We first note that *(I1)*, *(I2)* and *(I4)* are respectively *(B1)*, *(B2)* and *(B4)*. To show *(I3)* we observe that:

$$x \rightleftharpoons y \implies_{(B3)} x \sqsubset^{sym} y \;\wedge\; y \rightleftharpoons x \implies_{(B1)} x \neq y \;\wedge\; y \rightleftharpoons x \;.$$

To show *(I5)* we observe that:

$$\begin{aligned} z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z &\implies_{(B3)} && z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z \;\wedge\; z \sqsubset^{sym} y \;\wedge\; y \rightleftharpoons z \\ &\implies && x \sqsubset z \prec y \;\vee\; y \prec z \sqsubset x \\ &\implies_{(B4)} && x \rightleftharpoons y \;\vee\; y \rightleftharpoons x \\ &\implies_{(B3)} && x \rightleftharpoons y \;. \end{aligned}$$

To show *(I6)* we observe that:

$$\begin{aligned} & z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \\ & \implies_{(B3)} \; z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\wedge\; z' \sqsubset^{sym} z \;\wedge\; z' \rightleftharpoons z \\ & \implies_{(B1)} \; (x \sqsubset z \prec z' \sqsubset y \;\vee\; x \sqsubset z' \prec z \sqsubset y) \;\wedge\; x \neq z \;\wedge\; y \neq z \\ & \implies_{(B2,B4)} \; x \sqsubset z \prec y \;\vee\; x \prec z \sqsubset y \\ & \implies_{(B4)} \; x \rightleftharpoons y \;. \end{aligned}$$

We finally note that *(I7)* follows from *(B3)* and *(B5)*. □

Lemma 5. $\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \subseteq \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$.

Proof. Follows from Lemma 4, $\mathsf{IOS} \subseteq \mathsf{OS}$, and *(B3)*. □

Lemma 6. $\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$.

Proof.
Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $ios = \mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(B1)*, we observe that:

$$x \,\widehat{\sqsubset}\, x \implies x \sqsubset^{\curlywedge} x \implies \mathit{false} \;.$$

To show *(B2)*, we observe that:

$$x \neq y \;\wedge\; x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \implies x \neq y \;\wedge\; x \sqsubset^{\curlywedge} z \sqsubset^{\curlywedge} y \implies x \sqsubset^{\curlywedge} y \implies x \mathrel{\widehat{\sqsubset}} y\,.$$

To show *(B3)* we observe that all we need is to prove that $x \mathrel{\widehat{\rightleftharpoons}} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y$, in the following way:

$$\begin{aligned} x \mathrel{\widehat{\rightleftharpoons}} y &\implies x(\sqsubset^* \circ \prec \circ \sqsubset^*)^{sym} y \implies x \neq y \;\wedge\; x(\sqsubset^+)^{sym} y \\ &\implies x(\sqsubset^{\curlywedge})^{sym} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y\,, \end{aligned}$$

where $x \mathrel{\widehat{\rightleftharpoons}} y \implies x \neq y$ follows from Lemma 4 and *(I3)*. Finally, *(B5)* follows from the libel-linearity of os, as shown below:

$$x \neq y \;\wedge\; \ell(x) = \ell(y) \implies x \mathrel{\widehat{\prec}}^{sym} y \implies x \mathrel{\widehat{\rightleftharpoons}} y\,.$$

Hence $ios \in \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$. □

Proof of Theorem 4

Let us consider one by one all the inclusions:

- $\mathsf{IOS} \subset \mathsf{OS}$ was already justified in the proof of Theorem 1. Note, however, that we also have

$$os = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle\}, \{\langle x,y\rangle, \langle y,z\rangle\}, \\ \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OS} \setminus \mathsf{IOS}\,.$$

- $\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ follows from $os \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and Lemma 5.
- $\mathsf{IOS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \subset \mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$ follows from $os \in \mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$ and the general results proven in [5].
- $\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{OS}$ follows from the definition of $\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and

$$os' = \langle \{x,y\}, \{\langle x,y\rangle\}, \varnothing, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OS} \setminus \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}\,.$$

- $\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \subset \mathsf{IOS}$ follows from $os' \in \mathsf{IOS} \setminus \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and Lemma 4.
- $\mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \subset \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ can be proven by taking $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$, $u \in \mathsf{SSEQ}_\sigma$ and $os = \mathsf{sseq2os}_\sigma(u) = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since we know that $os \in \mathsf{OS}$, we only need to show that $\rightleftharpoons \subseteq \sqsubset^{sym}$. This, however, follows from (5) and the fact that in this case $\mathsf{rig} = (\Sigma \times \Sigma) \setminus \mathsf{sim}$. Hence $os \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$. Moreover, we note that

$$os'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}\,.$$

- $\mathsf{IOS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \subseteq \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ follows from Lemma 6, $os'' \in \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$ and $\mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \subseteq \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$.

Moreover, note that $os \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{IOS}$ and $os' \in \mathsf{IOS} \setminus \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ which justifies that IOS and $\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ are not related. Similarly, $os \in \mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}} \setminus \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $os'' \in \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$, hence there is no inclusion between $\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $\mathsf{OS}_{\Theta_{\mathsf{seq}\setminus\mathsf{sim}}}$. □

Proof of Theorem 5

We first show that $\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}}$. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$ and $ios = \mathsf{os2ios}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell\rangle$. We first observe that

$$\sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} = \sqsubset^{\circledast} \circ \prec^{sym} \circ \sqsubset^{\circledast} \quad \mathit{and} \quad \nabla = \sqsubset^{*} \circ \prec \circ \sqsubset^{*}$$

which follows from $x \rightleftharpoons y \Longrightarrow x \sqsubset^{sym} y$. Hence

$$\widehat{\rightleftharpoons} = \sqsubset^{\circledast} \circ (\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym} \circ \sqsubset^{\circledast} = (\sqsubset^{*} \circ \prec \circ \sqsubset^{*})^{sym} .$$

We then observe that $\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$ follows directly from Lemma 4, Lemma 5, Lemma 6, $\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}}$, and the fact that $\mathsf{os2ios}$ is the identity on IOS, as then we obtain

$$\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}$$

and

$$\begin{aligned}\mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}) &\supseteq \mathsf{os2ios}_{\mathsf{seq}\setminus\mathsf{sim}}(\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}) \\ &= \mathsf{os2ios}(\mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{seq}\setminus\mathsf{sim}}.\end{aligned}$$

□

Proof of Theorem 6

Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [5] it follows that there exists $sos \in \mathsf{satext}(os)$. Let $u = \mathsf{sseq2sos}^{-1}(sos)$, and $\sigma = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle$, where:

$$\begin{aligned}\mathsf{sim} = \{\langle a,b\rangle \in \Sigma\times\Sigma \mid\ & (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \ \wedge\ a \neq b) \vee \\ & (pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle) \ \wedge\ \langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle)\} \\ \mathsf{seq} = \{\langle a,b\rangle \in \Sigma\times\Sigma \mid\ & (pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle) \ \wedge\ a \neq b \ \wedge\ \langle b,1\rangle \not\sqsubset \langle a,1\rangle) \\ & \vee(pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle) \ \wedge\ \langle a,1\rangle \not\sqsubset \langle b,1\rangle) \\ & \vee(pos_u(\langle b,1\rangle) < pos_u(\langle a,1\rangle) \ \wedge\ \langle b,1\rangle \not\sqsubset \langle a,1\rangle)\} .\end{aligned}$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric, and $\mathsf{seq} \setminus \mathsf{sim}$ is symmetric because it is empty (it follows from $\mathsf{seq} \subseteq \mathsf{sim}$, as we show below). Hence σ is a generalised concurrency alphabet. To show $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$ we need to show that $\mathsf{seq} \subseteq \mathsf{sim}$.

Let $\langle a,b\rangle \in \mathsf{seq}$. If $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ then, clearly, $\langle a,b\rangle \in \mathsf{sim}$. If $pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle)$ and $\langle a,1\rangle \not\sqsubset \langle b,1\rangle$ then, by $os \in \mathsf{OS}_{\mathsf{seq}\setminus\mathsf{sim}}$, we obtain $\langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle$ or $\langle a,1\rangle \rightleftharpoons \langle b,1\rangle \ \wedge\ \langle b,1\rangle \sqsubset \langle a,1\rangle$.
Moreover, by $pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle)$, we obtain $\langle b,1\rangle \not\sqsubset \langle a,1\rangle$ and so we have $\langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle$. Hence $\langle a,b\rangle \in \mathsf{sim}$, and so $\sigma \in \Theta_{\mathsf{seq}\setminus\mathsf{sim}}$.

We then observe that $u \in \mathsf{SSEQ}_\sigma$ as $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ and $a \neq b$ together imply $\langle a,b\rangle \in \mathsf{sim}$, and it is easy to check that $os = \mathsf{sseq2os}_\sigma(u)$.

Proof of Proposition 1

Let $ios = \mathsf{os2ios} \circ \mathsf{sseq2os}_\kappa(u) = \mathsf{os2ios} \circ \mathsf{sseq2os}_\kappa(w)$. From $pos_u(\alpha) < pos_u(\beta)$ it follows that there is $sos_u \in \mathsf{satext}(ios)$ such that $\alpha \prec_{sos_u} \beta$. Similarly, from $pos_w(\alpha) > pos_w(\beta)$ it follows that there is $sos_w \in \mathsf{satext}(ios)$ such that $\beta \prec_{sos_w} \alpha$. Hence, $\alpha \not\sqsubset_{ios} \beta \not\sqsubset_{ios} \alpha$. Moreover, by $ios \in \mathsf{OS}_{\mathsf{seq}\backslash\mathsf{sim}}$, $\alpha \not\rightleftharpoons_{ios} \beta$. This, by the general results proved in [5], there is $sos_v \in \mathsf{satext}(ios)$ such that $\alpha \sqsubset_{sos_v} \beta \sqsubset_{sos_v} \alpha$. Then the conclusion holds by taking $v = \mathsf{sseq2os}_\sigma^{-1}(sos_v)$. □

C Proofs for the alphabets in $\Theta_{\mathsf{sim}\backslash\mathsf{seq}}$

Lemma 7. $\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \subseteq \mathsf{IOS}$.

Proof. To show *(I1)* we observe that:

$$x \sqsubset x \implies_{(C2)} x \rightleftharpoons x \implies_{(C3)} x \neq x \implies \textit{false} .$$

To show *(I2)* we observe that:

$$x \neq y \ \wedge \ x \sqsubset z \sqsubset y \implies_{(C1)} x \sqsubset y .$$

We then note that *(I3)* is simply *(C3)*, and to show *(I4)* we observe that:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \implies_{(C1)} x \sqsubset y \implies_{(C2)} x \rightleftharpoons y .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \implies_{(C1)} z \sqsubset z \implies_{(C2,C3)} \textit{false} .$$

To show *(I6)* we observe that:

$$z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \implies_{(C1)} x \sqsubset y \implies_{(C2)} x \rightleftharpoons y .$$

We finally note that *(I7)* follows from *(C2)* and *(C4)*. □

Lemma 8. $\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \subseteq \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Proof. Follows from Lemma 7, $\mathsf{IOS} \subseteq \mathsf{OS}$, and *(C2)*. □

Lemma 9. $\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}) \subseteq \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Proof.
Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $ios = \mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
$ios \in \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$. To show *(C1)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \implies x \sqsubset^+ z \sqsubset^+ y \implies x \sqsubset^+ y \implies x \mathrel{\widehat{\sqsubset}} y .$$

To show *(C2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} y \implies x \sqsubset^+ y \implies x \mathrel{\widehat{\rightleftharpoons}} y .$$

To show *(C3)* we observe that:

$$x \mathrel{\widehat{\rightleftharpoons}} y \implies x \rightleftharpoons y \;\vee\; x(\sqsubset^{+})^{sym} y \implies y \rightleftharpoons x \;\vee\; y(\sqsubset^{+})^{sym} x \implies y \mathrel{\widehat{\rightleftharpoons}} x \,.$$

Moreover, $x \widehat{\rightleftharpoons} y \implies x \neq y$ follows from the general results proved in [5]. Finally, *(C4)* follows from the libel-linearity of os, as shown below:

$$x \neq y \;\wedge\; \ell(x) = \ell(y) \implies x \mathrel{\widehat{\prec}}^{sym} y \implies x \mathrel{\widehat{\sqsubset}}^{sym} y \,.$$

Hence $ios \in \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$. □

Proof of Theorem 7

Let us consider one by one all the inclusions:

- $\mathsf{IOS} \subset \mathsf{OS}$ was already justified in the proof of Theorem 1. Note, however, that we also have

$$os = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \{\langle x,y\rangle, \langle y,z\rangle\}, \\ \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OS} \setminus \mathsf{IOS} \,.$$

- $\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \subset \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ follows from $os \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and Lemma 8.
- $\mathsf{IOS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \subset \mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$ follows from $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$ and the general results proven in [5].
- $\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \subset \mathsf{OS}$ follows from the definition of $\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and

$$os' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OS} \setminus \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \,.$$

- $\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \subset \mathsf{IOS}$ follows from $os' \in \mathsf{IOS} \setminus \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and Lemma 7.
- $\mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \subset \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ can be shown by taking $\kappa \in \Theta_{\mathsf{sim}\backslash\mathsf{seq}}$, $u \in \mathsf{SSEQ}_\kappa$, and $os = \mathsf{sseq2os}_\kappa(u)$. Since we know from the general theory that $os \in \mathsf{OS}$, we only need to show that $\sqsubset_{os}^{sym} \subseteq \rightleftharpoons_{os}$. This, however, follows from (6). Hence $os \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$. Moreover, we note that

$$os'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \,.$$

- $\mathsf{IOS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \subsetneq \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ follows from Lemma 9 $os'' \in \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$ and $\mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \subsetneq \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$.

Moreover, note that $os \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{IOS}$ and $os' \in \mathsf{IOS} \setminus \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ which justifies that IOS and $\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ are not related. Similarly, $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}} \setminus \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $os'' \in \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$, hence there is no inclusion between $\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $\mathsf{OS}_{\Theta_{\mathsf{sim}\backslash\mathsf{seq}}}$.

□

Proof of Theorem 8

We first show that $\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}}$. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$ and $ios = \mathsf{os2ios}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$. We first observe that in such a case $\sqsubset^{\circledast} = id_\Delta$ which follows from $x \sqsubset^{sym} y \Longrightarrow x \rightleftharpoons y$ and the separability of os. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^+$. Hence

$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons \cup \nabla^{sym}, \sqsubset^+, \ell \rangle \ ,$$

where $\nabla = \{\langle x, y \rangle \mid \exists z, w : z \rightleftharpoons w \wedge x \sqsubset^* z \sqsubset^* y \wedge x \sqsubset^* w \sqsubset^* y\}$. We will now show that $(\rightleftharpoons \cup \nabla^{sym}) = (\rightleftharpoons \cup (\sqsubset^+)^{sym})$.
Suppose first that $x \nabla y$ which means that $x \neq y$ (which follows from the general theory), and there is z such that $x \sqsubset^* z \sqsubset^* y$. Hence $x \sqsubset^+ y$ showing that the $(\subseteq)$ inclusion holds. To show the reverse inclusion, suppose that $x \sqsubset^+ y$. If $x \sqsubset y$ then, by the definition of $\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$, we have $x \rightleftharpoons y$. Otherwise, there is z such that $x \sqsubset z \sqsubset^* y$. Then, again by the definition of $\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}$, $z \rightleftharpoons x$. We therefore obtain that $\langle x, y \rangle \in \nabla$, after taking $w = x$. Hence

$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons \cup (\sqsubset^+)^{sym}, \sqsubset^+, \ell \rangle \ .$$

We then observe that $\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}) = \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$ follows from Lemma 7, Lemma 8, Lemma 9, $\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}}$, and the fact that $\mathsf{os2ios}$ is the identity on IOS, as then we obtain

$$\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}) \subseteq \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}$$

and

$$\begin{aligned}\mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}}) &\supseteq \mathsf{os2ios}_{\mathsf{sim}\backslash\mathsf{seq}}(\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}) \\ &= \mathsf{os2ios}(\mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}) = \mathsf{IOS}_{\mathsf{sim}\backslash\mathsf{seq}}.\end{aligned}$$

□

Proof of Theorem 9

Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [5] it follows that there exists $sos \in \mathsf{satext}(os)$. Let $u = \mathsf{sseq2sos}^{-1}(sos)$, and $\kappa = \langle \Sigma, \mathsf{sim}, \mathsf{seq} \rangle$, where:

$$\begin{aligned}\mathsf{sim} = \{\langle a, b \rangle \in \Sigma \times \Sigma \mid\ & (pos_u(\langle a, 1 \rangle) = pos_u(\langle b, 1 \rangle) \ \wedge \ a \neq b) \vee \\ & (pos_u(\langle a, 1 \rangle) \neq pos_u(\langle b, 1 \rangle) \ \wedge \ \langle a, 1 \rangle \not\rightleftharpoons \langle b, 1 \rangle)\} \\ \mathsf{seq} = \{\langle a, b \rangle \in \Sigma \times \Sigma \mid\ & (pos_u(\langle a, 1 \rangle) = pos_u(\langle b, 1 \rangle) \ \wedge \ a \neq b) \\ & \vee (pos_u(\langle a, 1 \rangle) < pos_u(\langle b, 1 \rangle) \ \wedge \ \langle a, 1 \rangle \not\sqsubset \langle b, 1 \rangle) \\ & \vee (pos_u(\langle b, 1 \rangle) < pos_u(\langle a, 1 \rangle) \ \wedge \ \langle b, 1 \rangle \not\sqsubset \langle a, 1 \rangle)\} \ .\end{aligned}$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric, and $\mathsf{seq} \backslash \mathsf{sim}$ is symmetric because sim and seq are symmetric. Hence κ is a generalised concurrency alphabet. To show $\kappa \in \Theta_{\mathsf{sim}\backslash\mathsf{seq}}$ we need to show that $\mathsf{sim} \subseteq \mathsf{seq}$.

Let $\langle a,b\rangle \in \mathsf{sim}$. If $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ and $a \neq b$ then clearly we have $\langle a,b\rangle \in \mathsf{seq}$. Moreover, if $pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle)$ and $\langle a,1\rangle \not\rightleftharpoons \langle b,1\rangle$ then, by $os \in \mathsf{OS}_{\mathsf{sim}\setminus\mathsf{seq}}$, $pos_u(\langle a,1\rangle) \neq pos_u(\langle b,1\rangle)$ and $\langle a,1\rangle \not\sqsubset^{sym} \langle b,1\rangle$. Hence $\langle a,b\rangle \in \mathsf{seq}$, and so $\kappa \in \Theta_{\mathsf{sim}\setminus\mathsf{seq}}$.

We then observe that $u \in \mathsf{SSEQ}_\kappa$ as $pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)$ and $a \neq b$ together imply $\langle a,b\rangle \in \mathsf{sim}$, and it is easy to check that $os = \mathsf{sseq2os}_\kappa(u)$. □

Proof of Proposition 2

Let $ios = \mathsf{os2ios} \circ \mathsf{sseq2os}_\kappa(v)$. From $pos_v(\alpha) = pos_v(\beta)$ it follows directly that $\langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{sim}$ and there is $sos \in \mathsf{satext}(ios)$ such that $\alpha \sqsubset_{sos} \beta \sqsubset_{sos} \alpha$. Hence, $\alpha \not\rightleftharpoons_{ios} \beta$. Moreover, by the simplified form of the $\mathsf{sseq2os}_\kappa$ mapping and the order closure, $\alpha \not\sqsubset_{ios} \beta$ and $\beta \not\sqsubset_{ios} \alpha$. This, by the general results proved in [5], means that there are $sos', sos'' \in \mathsf{satext}(ios)$ such that $\alpha \prec_{sos'} \beta$ and $\beta \prec_{sos''} \alpha$. Then the conclusion holds by taking $u = \mathsf{sseq2os}_\kappa^{-1}(sos')$ and $w = \mathsf{sseq2os}_\kappa^{-1}(sos'')$. □

D Proofs for the alphabets in $\Theta_{\mathsf{seq}\cap\mathsf{sim}}$

Lemma 10. $\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \subseteq \mathsf{IOS}$.

Proof. We first note that:

$$x \sqsubset y \sqsubset x \wedge x \rightleftharpoons y \implies_{(D3)} x \not\rightleftharpoons y \wedge x \neq y \wedge x \rightleftharpoons y \implies false \quad (*)$$

Hence, by *(D1)*,

$$x \rightleftharpoons y \iff x \neq y \wedge \neg(x \sqsubset y \sqsubset x) \quad (**)$$

To show *(I1)* we observe that:

$$x \sqsubset x \implies x \sqsubset x \sqsubset x \implies_{(D3)} x \not\rightleftharpoons x \wedge x \neq x \implies false.$$

Then we note that *(I2)* is simply *(D2)*. To show *(I3)* we observe that:

$$\begin{aligned} x \rightleftharpoons y &\implies_{(**)} x \neq y \wedge \neg(x \sqsubset y \sqsubset x) \\ &\implies x \neq y \wedge (y \neq x \wedge \neg(y \sqsubset x \sqsubset y)) \\ &\implies_{(**)} x \neq y \wedge y \rightleftharpoons x\,. \end{aligned}$$

To show *(I4)* we observe that:

$$\begin{aligned} x \not\rightleftharpoons y \wedge x \prec z \sqsubset y &\implies_{(**)} (x = y \vee x \sqsubset y \sqsubset x) \wedge x \prec z \sqsubset x \\ &\implies_{(D1)} (x = y \vee x \sqsubset y \sqsubset x) \wedge \\ &\qquad x \sqsubset z \sqsubset y \wedge x \rightleftharpoons z \wedge z \neq x \\ &\implies x \sqsubset z \sqsubset x \wedge x \rightleftharpoons z \vee \\ &\qquad x \sqsubset z \sqsubset y \sqsubset x \wedge x \rightleftharpoons z \wedge z \neq x \\ &\implies_{(D2)} x \sqsubset z \sqsubset x \wedge x \rightleftharpoons z \vee x \sqsubset z \sqsubset x \wedge x \rightleftharpoons z \\ &\implies x \sqsubset z \sqsubset x \wedge x \rightleftharpoons z \\ &\implies_{(D3)} \mathit{false}\,. \end{aligned}$$

Similarly, $x \neq y \ \wedge \ x \sqsubset z \prec y \implies \mathit{false}$. Hence we have:

$$x \prec z \sqsubset y \ \vee \ x \sqsubset z \prec y \implies x \rightleftharpoons y \ .$$

To show *(I5)* we first observe that:

$$\begin{array}{ll} z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \wedge \ x \sqsubset y \sqsubset x & \\ \qquad \implies_{(D1)} & z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset y \sqsubset x \sqsubset z \ \wedge \ z \neq y \\ \qquad \implies_{(D2)} & z \rightleftharpoons y \ \wedge \ z \sqsubset y \sqsubset z \\ \qquad \implies_{(*)} & \mathit{false} \ , \end{array}$$

$$\begin{array}{ll} z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \ \wedge \ x = y & \\ \qquad \implies & z \rightleftharpoons y \ \wedge \ z \sqsubset y \sqsubset z \\ \qquad \implies_{(*)} & \mathit{false} \ . \end{array}$$

Hence we have:

$$z \rightleftharpoons y \ \wedge \ z \sqsubset x \sqsubset z \implies \neg(y \sqsubset x \sqsubset y) \ \wedge \ x \neq y \implies_{(**)} x \rightleftharpoons y \ .$$

To show *(I6)* we observe that:

$$\begin{array}{ll} z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \wedge \ x \sqsubset y \sqsubset x & \\ \qquad \implies_{(D1)} & z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \wedge \ x \sqsubset y \sqsubset x \ \wedge \\ & z \neq z' \ \wedge \ z \neq x \ \wedge \ y \neq z \\ \qquad \implies & z \rightleftharpoons z' \ \wedge \ z \sqsubset y \sqsubset x \sqsubset z' \sqsubset y \sqsubset x \sqsubset z \ \wedge \\ & z \neq z' \ \wedge \ z \neq x \ \wedge \ y \neq z \\ \qquad \implies_{(D2)} & z \rightleftharpoons z' \ \wedge \ z \sqsubset x \sqsubset z' \sqsubset y \sqsubset z \ \wedge \ z \neq z' \\ \qquad \implies_{(D2)} & z \rightleftharpoons z' \ \wedge \ z \sqsubset z' \sqsubset z \\ \qquad \implies_{(*)} & \mathit{false} \end{array}$$

$$\begin{array}{ll} z \rightleftharpoons z' \ \wedge \ x \sqsubset z \sqsubset y \ \wedge \ x \sqsubset z' \sqsubset y \ \wedge \ x = y & \\ \qquad \implies_{(D1)} & z \rightleftharpoons z' \ \wedge \ z \sqsubset x \sqsubset z' \sqsubset x \sqsubset z \ \wedge \ z \neq z' \\ \qquad \implies_{(D2)} & z \rightleftharpoons z' \ \wedge \ z \sqsubset z' \sqsubset z \\ \qquad \implies_{(*)} & \mathit{false} \ . \end{array}$$

Hence we have:

$$z \rightleftharpoons z' \wedge x \sqsubset z \sqsubset y \wedge x \sqsubset z' \sqsubset y \implies \neg(y \sqsubset x \sqsubset y) \wedge x \neq y \implies_{(**)} x \rightleftharpoons y \ .$$

We finally note that *(I7)* is simply *(D4)*. □

Lemma 11. $\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \subseteq \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$.

Proof. Follows from Lemma 10, $\mathsf{IOS} \subseteq \mathsf{OS}$, and *(D3)*. □

Lemma 12. $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$.

Proof.
Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ and $ios = \mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(D1)* we observe that $\widehat{\rightleftharpoons} \ = \rightleftharpoons$, and to show *(D2)*, we observe that:

$$x \neq y \ \wedge \ x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \implies x \neq y \ \wedge \ x \sqsubset^{\curlywedge} z \sqsubset^{\curlywedge} y \implies x \sqsubset^{\curlywedge} y \implies x \mathrel{\widehat{\sqsubset}} y \ .$$

To show *(D3)* we observe that:

$$\sqsubset^{\circledast} = \sqsubset^{*} \cap (\sqsubset^{*})^{-1} = (\sqsubset^{\curlywedge} \uplus id_{\Delta}) \cap (\sqsubset^{\curlywedge} \uplus id_{\Delta})^{-1} = (\sqsubset^{\curlywedge} \cap (\sqsubset^{\curlywedge})^{-1}) \uplus id_{\Delta},$$

hence

$$\widehat{\rightleftharpoons} = \rightleftharpoons = (\Delta \times \Delta) \setminus \sqsubset^{\circledast} = (\Delta \times \Delta) \setminus (\sqsubset^{\curlywedge} \cap (\sqsubset^{\curlywedge})^{-1} \uplus id_{\Delta}),$$

and so

$$x \mathrel{\widehat{\not\rightleftharpoons}} y \ \wedge\ x \neq y \iff x \mathrel{\widehat{\sqsubset}} y \mathrel{\widehat{\sqsubset}} x.$$

Finally, *(D4)* follows from the libel-linearity of os, as shown below:

$$x \neq y \ \wedge\ \ell(x) = \ell(y) \implies x \prec^{sym} y \implies x \mathrel{\widehat{\precsim}}^{sym} y\ .$$

Hence $ios \in \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ □

Proof of Theorem 10

Let us consider one by one all the inclusions:

– $\mathsf{IOS} \subset \mathsf{OS}$ was already justified in the proof of Theorem 1. Note, however, that we also have

$$os = \left\langle \begin{array}{r} \{x,y,z\}, \{\langle y,z\rangle, \langle z,y\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OS} \setminus \mathsf{IOS}\ .$$

– $\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \subset \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ follows from $os \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}} \setminus \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ and Lemma 11.
– $\mathsf{IOS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \subset \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}$ follows from $os \in \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}$ and the general results proven in [5].
– $\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}} \subset \mathsf{OS}$ follows from the definition of $\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ and

$$os' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OS} \setminus \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}\ .$$

– $\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \subset \mathsf{IOS}$ follows from $os' \in \mathsf{IOS} \setminus \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ and Lemma 10.
– $\mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \subset \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ can be shown by taking $\nu \in \Theta_{\mathsf{seq}\cap\mathsf{sim}}$, $u \in \mathsf{SSEQ}_{\nu}$, and $os = \mathsf{sseq2os}_{\nu}(u) = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since we know that $os \in \mathsf{OS}$, we only need to demonstrate that:

$$(\Delta \times \Delta) \setminus id_{\Delta} \subseteq \rightleftharpoons \cup (\sqsubset \cap \sqsubset^{-1})\ .$$

The above holds since, by (7), $pos_u(\alpha) = pos_u(\beta) \wedge \alpha \neq \beta$ implies $\alpha \sqsubset \beta \sqsubset \alpha$, and $pos_u(\alpha) \neq pos_u(\beta)$ implies $\alpha \rightleftharpoons \beta$. Hence $os \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$. Moreover, we note that

$$os'' = \left\langle \begin{array}{r} \{x,y,z\}, \\ \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}.$$

– $\mathsf{IOS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \subset \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ follows from Lemma 12, $os'' \in \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}$ and $\mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \subseteq \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$.

Moreover, note that $os \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}} \setminus \mathsf{IOS}$ and $os' \in \mathsf{IOS} \setminus \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ which justifies that IOS and $\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ are not related. Similarly, $os \in \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}} \setminus \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ and $os'' \in \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}} \setminus \mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}$, hence there is no inclusion between $\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ and $\mathsf{OS}_{\Theta_{\mathsf{seq}\cap\mathsf{sim}}}$. □

Proof of Theorem 11

We show that $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}}$. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ and $ios = \mathsf{os2ios}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell\rangle$. We first observe that in such a case we have $\rightleftharpoons = (\Delta \times \Delta)\setminus \sqsubset^{\circledast}$, which follows from $x \neq y \Rightarrow x \rightleftharpoons y \vee x \sqsubset y \sqsubset z$ and the separability of os. By the general theory we know that

$$(\sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast} \cup \sqsubset^{\circledast} \circ \nabla^{sym} \circ \sqsubset^{\circledast}) \cap \sqsubset^{\circledast} = \varnothing.$$

and since $\rightleftharpoons \subseteq \sqsubset^{\circledast} \circ \rightleftharpoons \circ \sqsubset^{\circledast}$ we obtain

$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons, \sqsubset^{\curlywedge}, \ell\rangle \ .$$

We observe that $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$ follows from Lemma 10, Lemma 11, Lemma 12 $\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}}$, and the fact that $\mathsf{os2ios}$ is the identity on IOS, as then we obtain

$$\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}) \subseteq \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}$$

and

$$\begin{aligned}\mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}) &\supseteq \mathsf{os2ios}_{\mathsf{seq}\cap\mathsf{sim}}(\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}) \\ &= \mathsf{os2ios}(\mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}) = \mathsf{IOS}_{\mathsf{seq}\cap\mathsf{sim}}.\end{aligned}$$

□

Proof of Theorem 12

Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [5] it follows that there exists $sos \in \mathsf{satext}(os)$ which, by the definition of $\mathsf{OS}_{\mathsf{seq}\cap\mathsf{sim}}$ and separability of OS satisfies

$$(\Delta \times \Delta) = id_{\Delta} \uplus \rightleftharpoons_{sos} \uplus (\sqsubset_{sos} \cap \sqsubset_{sos}^{-1}) \ .$$

Let $\nu = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle$, where:

$$\begin{aligned}\mathsf{sim} &= \{\langle a,b\rangle \in \Sigma \times \Sigma \mid pos_u(\langle a,1\rangle) = pos_u(\langle b,1\rangle)\} \\ \mathsf{seq} &= \{\langle a,b\rangle \in \Sigma \times \Sigma \mid (pos_u(\langle a,1\rangle) < pos_u(\langle b,1\rangle) \ \wedge \ \langle a,1\rangle \not\sqsubset \langle b,1\rangle) \\ &\qquad \vee (pos_u(\langle b,1\rangle) < pos_u(\langle a,1\rangle) \ \wedge \ \langle b,1\rangle \not\sqsubset \langle a,1\rangle)\} \ .\end{aligned}$$

Clearly, $\nu \in \Theta_{\mathsf{seq}\cap\mathsf{sim}}$ and $u \in \mathsf{SSEQ}_{\nu}$. It is easy to check that $os = \mathsf{sseq2os}_{\nu}(u)$. □

E Proofs for the alphabets in $\Theta_{\mathsf{sim}\triangle\mathsf{seq}}$

Lemma 13. $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \subseteq \mathsf{IOS}$.

Proof. We first note that *(I1)* is simply *(E1)*. To show *(I2)* we observe that

$$x \neq y \;\wedge\; x \sqsubset z \sqsubset y \;\Longrightarrow_{(E2)}\; x \sqsubset y\ .$$

To show *(I3)* we observe that

$$x \rightleftharpoons y \;\Longrightarrow_{(E3)}\; x \sqsubset^{sym} y \;\Longrightarrow_{(E3)}\; y \rightleftharpoons x\ .$$

and we observe that if $x \rightleftharpoons x$ then we obtain a contradiction as follows:

$$x \rightleftharpoons x \;\Longrightarrow_{(E3)}\; x \sqsubset^{sym} x \;\Longrightarrow\; x \sqsubset x \;\Longrightarrow_{(E1)}\; x \neq x\ .$$

To show *(I4)* we observe that:

$$x \prec z \sqsubset y \;\vee\; x \sqsubset z \prec y \;\Longrightarrow_{(E2)}\; x \sqsubset y \;\Longrightarrow_{(E3)}\; x \rightleftharpoons y\ .$$

To show *(I5)* we observe that:

$$z \rightleftharpoons y \;\wedge\; z \sqsubset x \sqsubset z \;\Longrightarrow_{(E2)}\; z \sqsubset z \;\Longrightarrow_{(E1)}\; \mathit{false}\ .$$

To show *(I6)* we observe that:

$$z \rightleftharpoons z' \;\wedge\; x \sqsubset z \sqsubset y \;\wedge\; x \sqsubset z' \sqsubset y \;\Longrightarrow_{(E2)}\; x \sqsubset y \;\Longrightarrow_{(E3)}\; x \rightleftharpoons y\ .$$

We finally note that *(I7)* follows from *(E3)* and *(E4)*. □

Lemma 14. $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \subseteq \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Proof. Follows from Lemma 13, $\mathsf{IOS} \subseteq \mathsf{OS}$, and *(E3)*. □

Lemma 15. $\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}) \subseteq \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Proof.
Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell \rangle \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $ios = \mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell \rangle$.
To show *(E1)* we observe that $x \mathrel{\widehat{\sqsubset}} x$ together with $x \not\sqsubset x$ imply that there are y, z such that $x \sqsubset^{*} y \sqsubset z \sqsubset^{*} x$. Hence, by the definition of $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$, $y \rightleftharpoons z$, contradicting the separability of os.
To show *(E2)* we observe that:

$$x \mathrel{\widehat{\sqsubset}} z \mathrel{\widehat{\sqsubset}} y \;\Longrightarrow\; x \sqsubset^{+} z \sqsubset^{+} y \;\Longrightarrow\; x \sqsubset^{+} y \;\Longrightarrow\; x \mathrel{\widehat{\sqsubset}} y\ .$$

To show *(E3)* we observe that:

$$x \mathrel{\widehat{\sqsubset}^{sym}} y \;\Longleftrightarrow\; x (\sqsubset^{+})^{sym} y \;\Longleftrightarrow\; x \mathrel{\widehat{\rightleftharpoons}} y\ .$$

Finally, *(E4)* follows from the libel-linearity of os, as shown below:

$$x \neq y \;\wedge\; \ell(x) = \ell(y) \;\Longrightarrow\; x \mathrel{\widehat{\prec}^{sym}} y \;\Longrightarrow\; x \mathrel{\widehat{\sqsubset}^{sym}} y\ .$$

Hence $ios \in \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$. □

Proof of Theorem 13

Let us consider one by one all the inclusions:

- $\mathsf{IOS} \subset \mathsf{OS}$ was already justified in the proof of Theorem 1. Note, however, that we also have

$$os = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle y,z\rangle, \langle z,y\rangle\}, \\ \{\langle x,y\rangle, \langle y,z\rangle\}, \{x \mapsto a, y \mapsto b, z \mapsto c\} \end{array} \right\rangle \in \mathsf{OS} \setminus \mathsf{IOS} \ .$$

- $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from $os \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and Lemma 14.
- $\mathsf{IOS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subset \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ follows from $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ and the general results proven in [5].
- $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{OS}$ follows from the definition of $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and

$$os' = \langle \{x,y\}, \varnothing, \{\langle x,y\rangle\}, \{x \mapsto a, y \mapsto b\}\rangle \in \mathsf{OS} \setminus \mathsf{OS}_{\mathsf{sim}\backslash\mathsf{seq}} \ .$$

- $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \subset \mathsf{IOS}$ follows from $os' \in \mathsf{IOS} \setminus \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and Lemma 13.
- $\mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subset \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ can be proven by taking $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$, $u \in \mathsf{SSEQ}_\omega$, and $os = \mathsf{sseq2os}_\omega(u)$. Since $os \in \mathsf{OS}$, we only need to show that $\sqsubset_{os}^{sym} = \rightleftharpoons_{os}$. This, however, follows from (8). Moreover, we note that

$$os'' = \left\langle \begin{array}{c} \{x,y,z\}, \{\langle x,y\rangle, \langle y,x\rangle, \langle x,z\rangle, \langle z,x\rangle\}, \\ \{\langle x,y\rangle, \langle x,z\rangle\}, \{x \mapsto a, y \mapsto a, z \mapsto b\} \end{array} \right\rangle \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \ .$$

- $\mathsf{IOS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subsetneq \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from Lemma 15, $os'' \in \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{IOS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$ and $\mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \subsetneq \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$.

Moreover, note that $os \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{IOS}$ and $os' \in \mathsf{IOS} \setminus \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ which justifies that IOS and $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ are not related. Similarly, $os \in \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}} \setminus \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $os'' \in \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} \setminus \mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$, hence there is no inclusion between $\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $\mathsf{OS}_{\Theta_{\mathsf{sim}\triangle\mathsf{seq}}}$. □

Proof of Theorem 14

We show that $\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}}$. Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle \in \mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$ and $ios = \mathsf{os2ios}(os) = \langle \Delta, \widehat{\rightleftharpoons}, \widehat{\sqsubset}, \ell\rangle$. We first observe that in such a case we have $\sqsubset^{\circledast} = id_\Delta$ which follows from $x \sqsubset^{sym} y \iff x \rightleftharpoons y$ and the separability of os. As a result, we also have $\sqsubset^{\curlywedge} = \sqsubset^{+}$. Hence

$$\mathsf{os2ios}(os) = \langle \Delta, \rightleftharpoons \cup \nabla^{sym}, \sqsubset^{+}, \ell\rangle \ ,$$

where $\nabla = \{\langle x,y\rangle \mid \exists z,w : z \rightleftharpoons w \wedge x \sqsubset^* z \sqsubset^* y \wedge x \sqsubset^* w \sqsubset^* y\}$. We will now show that $(\rightleftharpoons \cup \nabla^{sym}) = (\sqsubset^{+})^{sym}$.
Suppose first that $x \nabla y$ which means that $x \neq y$ (which follows from the general theory), and there is z such that $x \sqsubset^* z \sqsubset^* y$. Hence $x \sqsubset^{+} y$ showing that the $(\subseteq)$ inclusion holds. To show the reverse inclusion, suppose that $x \sqsubset^{+} y$. If $x \sqsubset y$ then, by the definition of $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$, we have $x \rightleftharpoons y$. Otherwise, there is z such

that $x \sqsubset z \sqsubset^* y$. Then, again by the definition of $\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}$, $z \rightleftharpoons x$. We therefore obtain that $\langle x, y\rangle \in \nabla$, after taking $w = x$. Hence

$$\mathsf{os2ios}(os) = \langle \Delta, (\sqsubset^+)^{sym}, \sqsubset^+, \ell\rangle \ .$$

We observe that $\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}) = \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$ follows from Lemma 13, Lemma 14, Lemma 15, $\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}} = \mathsf{os2ios}|_{\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}}$, and the fact that $\mathsf{os2ios}$ is the identity on IOS, as then we obtain

$$\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}) \subseteq \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}$$

and

$$\begin{aligned}\mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{OS}_{\mathsf{sim}\triangle\mathsf{seq}}) &\supseteq \mathsf{os2ios}_{\mathsf{sim}\triangle\mathsf{seq}}(\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}) \\ &= \mathsf{os2ios}(\mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}}) = \mathsf{IOS}_{\mathsf{sim}\triangle\mathsf{seq}} .\end{aligned}$$

□

Proof of Theorem 13

Let $os = \langle \Delta, \rightleftharpoons, \sqsubset, \ell\rangle$. Since the labelling ℓ is injective, we may assume that $\Delta = \Sigma \times \{1\}$. Then, from the general results proved in [5] it follows that there exists $sos \in \mathsf{satext}(os)$. Let $u = \mathsf{sseq2sos}^{-1}(sos)$, and $\omega = \langle \Sigma, \mathsf{sim}, \mathsf{seq}\rangle$, where:

$$\mathsf{seq} = \mathsf{sim} = \{\langle a, b\rangle \in \Sigma \times \Sigma \mid (pos_u(\langle a, 1\rangle) \neq pos_u(\langle b, 1\rangle) \ \wedge \ \langle a, 1\rangle \not\rightleftharpoons \langle b, 1\rangle)\} \ .$$

We then observe that sim is symmetric since $\rightleftharpoons$ is symmetric. Hence ω is a generalised concurrency alphabet. Clearly, $\omega \in \Theta_{\mathsf{sim}\triangle\mathsf{seq}}$ and $u \in \mathsf{SSEQ}_\omega$. It is easy to check that $os = \mathsf{sseq2os}_\kappa(u)$. □

Proof of Proposition 3

Let $ios = \mathsf{os2ios} \circ \mathsf{sseq2os}_\omega(v)$. By $pos_v(\alpha) = pos_v(\beta)$, we obtain $\langle \ell(\alpha), \ell(\beta)\rangle \in \mathsf{sim}$ and there is $sos \in \mathsf{satext}(ios)$ such that $\alpha \sqsubset_{sos} \beta \sqsubset_{sos} \alpha$. Hence, $\alpha \not\rightleftharpoons_{ios} \beta$. Moreover, by the order closure, $\alpha \not\sqsubset_{ios} \beta$ and $\beta \not\sqsubset_{ios} \alpha$. This, by the general results proved in [5], means that there are $sos', sos'' \in \mathsf{satext}(ios)$ such that $\alpha \prec_{sos'} \beta$ and $\beta \prec_{sos''} \alpha$. Then the first implication holds by taking $u = \mathsf{sseq2os}_\omega^{-1}(sos')$ and $w = \mathsf{sseq2os}_\omega^{-1}(sos'')$.

On the other hand, let $ios = \mathsf{os2ios} \circ \mathsf{sseq2os}_\omega(u) = \mathsf{os2ios} \circ \mathsf{sseq2os}_\omega(w)$. Then there exist $sos_u, sos_w \in \mathsf{satext}(ios)$ such that $\alpha \prec_{sos_u} \beta$ and $\beta \prec_{sos_w} \alpha$, and so, by the order closure, $\alpha \not\rightleftharpoons_{ios} \beta$. This, by the general results proved in [5], means that there exists $sos \in \mathsf{satext}(ios)$ such that $\alpha \sqsubset_{sos} \beta \sqsubset_{sos} \alpha$. Hence the second implication holds by taking $v = \mathsf{sseq2os}_\omega^{-1}(sos)$, which ends the proof. □