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Fixed and Recursive Right-Tailed Dickey–Fuller Tests in the Presence of a Break under the Null

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Abstract: Right-tailed Dickey–Fuller-type unit root tests against the explosive alternative have become popular in economics and finance for detecting asset price bubbles. This paper studies the size properties of fixed sample and recursive right-tailed Dickey–Fuller tests if the relevant series contains a unit root, but a structural break in the drift parameter occurs. It is shown that positive size distortion and therefore spurious rejections of the unit root null hypothesis in favour of the explosive alternative can be a problem for both types of test. Some possible solutions to this problem are briefly discussed.

Keywords: explosive autoregression, structural break, unit root
JEL Classification: C22, C12

1 Introduction

When using Dickey–Fuller unit root tests (Dickey and Fuller 1979) practitioners typically employ left-tailed versions of the tests with the alternative hypothesis of level stationarity or trend stationarity. However, right-tailed Dickey–Fuller-type tests of the unit root null hypothesis against the explosive alternative have recently become popular in economics and finance for detecting asset price bubbles; for example, Phillips et al. (2011) (PWY) propose testing for a rational stock price bubble using the supremum of a series of forward recursive right-tailed augmented Dickey–Fuller tests (sup-DF) applied to the price and dividend series to detect periods of explosive autoregressive behaviour in prices that are not justified by dividends. For the Nasdaq Composite stock price index they find that their sup-DF test rejects the null hypothesis of a unit root in favour of the explosive alternative at conventional significance levels, but that a rejection is not obtained for the associated dividend series which suggests that an explosive rational bubble was

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present. Monte Carlo simulations show that their test has good finite sample power for detecting bubbles, even if the bubbles periodically collapse.

The approach to testing for asset price bubbles proposed by PWY has become popular with researchers in this area and has been applied to a range of financial assets. For example, Phillips and Yu (2011) use a modified version of this approach in their investigation of financial bubbles over the period of the subprime crisis. They find statistically significant evidence of bubbles in house prices, bond prices and oil prices. Homm and Breitung (2012) propose several alternative tests for explosive bubbles that build on the PWY approach and illustrate their tests with empirical applications to stock prices, house prices and commodity prices. Bettendorf and Chen (2013) use the PWY approach to investigate the presence of explosive bubbles in the sterling–US dollar nominal exchange rate. They find statistically significant evidence of explosive behaviour in the nominal exchange rate and this appears to be driven by explosive behaviour in the relevant price index ratio for traded goods. Note that the original PWY test assumes no drift term under the unit root null hypothesis. However, Phillips et al. (2014) (PSY) extend the analysis in PWY and derive the asymptotic distribution of the PWY test for several different drift specifications.

A standard assumption in the literature on testing for asset price bubbles using right-tailed Dickey–Fuller-type tests is that under the unit root null hypothesis there are no structural breaks in the drift parameter. Whilst for many financial assets this assumption will be realistic, for some assets it could be overly restrictive. Indeed in separate research, for various financial assets a stationary autoregressive model with a regime switching mean has been found to be appropriate for the returns series (see, e.g. Schaller and van Norden 1997; Guidolin and Timmermann 2005; Ang and Timmermann 2011), which is consistent with the natural logarithm of the asset price series being a unit root process but with a time-varying drift component. It is important to clarify the extent to which time variation in the drift parameter under the unit root null hypothesis affects the size properties of the right-tailed Dickey–Fuller-type tests used in the literature on testing for asset price bubbles. Hence, as a first step in the analysis of this issue, this paper studies the simple case of a time series that contains a fixed unit root and is not explosive at any point, but a discrete structural break in the drift parameter occurs and this is ignored when computing the relevant tests. Leybourne et al. (1998) studied the performance of a fixed sample left-tailed Dickey–Fuller test against the alternative hypothesis of trend stationarity in the presence of a discrete structural break in the drift under the unit root null hypothesis and found a severe spurious rejection problem, but only for breaks that occur early in the series. Note that Leybourne et al. (1998) include a constant and deterministic trend in the model used for testing. As discussed in PSY, when computing a right-tailed Dickey–Fuller test it is empirically unrealistic to include a deterministic trend in
the model employed. Therefore, in this paper all of the asymptotic and finite sample results are derived for the model including only a constant.

The next section of the paper presents asymptotic results. Section 3 discusses the results from Monte Carlo simulations to investigate the finite sample sizes of the tests. Section 4 briefly discusses some possible solutions to the problem of size distortion when using the tests in the presence of structural breaks under the null hypothesis. Section 5 concludes.

2 Asymptotic Results

2.1 Fixed Sample Tests

Suppose that the true data generating process (DGP) for \( y_t, t = 1, 2, \ldots, T \) is a random walk model with white noise errors and that a structural break in the drift occurs at \( t = \lambda T + 1 \)

\[
y_t = \alpha_1(1 - D_t) + \alpha_2 D_t + y_{t-1} + \varepsilon_t
\]

\[
D_t = 0, \quad t \leq \lambda T
\]

\[
= 1, \quad t > \lambda T
\]

where \( \lambda \in (0, 1), \alpha_1 \neq \alpha_2 \) and \( \varepsilon_t : \text{IID}(0, \sigma^2) \). Assume that using all \( T \) observations on \( y_t \), the practitioner computes a fixed sample right-tailed Dickey–Fuller test against the explosive alternative using the standard model

\[
\Delta y_t = \mu + \rho y_{t-1} + \varepsilon_t
\]

The test statistic and null and alternative hypotheses can be written, \( DF_\mu = \hat{\rho} / \text{se}(\hat{\rho}) \) and \( H_0 : \rho = 0, \; H_1 : \rho > 0 \) (note that the PWY test, which is considered in the next subsection, is computed using this type of model applied recursively).\(^1\) Consider now the asymptotic properties of the right-tailed \( DF_\mu \) test. Using straightforward algebra it can be shown in this case that depending on the direction and location of the break, as \( T \to \infty \) the \( DF_\mu \) test statistic will diverge to either \( +\infty \) or \( -\infty \) and consequently the size of the right-tailed \( DF_\mu \) test will tend to 1 or to 0. Note that since the asymptotic distribution of \( DF_\mu \) in this case will be a point mass distribution and therefore of limited practical interest, for brevity these asymptotic results are omitted.\(^2\) However, they indicate that as

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1 Note that following Leybourne et al. (1998), and without loss of generality, in this paper all of the asymptotic and simulation results assume first-order dynamics in the DGP and the model used to compute the relevant test.
2 The results are available on request.
with the left-tailed Dickey–Fuller test studied by Leybourne et al. (1998), spurious rejections due to the presence of a structural break in the drift under the unit root null hypothesis might also be a problem for the types of right-tailed Dickey–Fuller tests that have become popular in the literature on testing for asset price bubbles.

Rather than work with point mass asymptotic results, ideally we would like to derive a non-degenerate asymptotic distribution for the Dickey–Fuller test statistic $DF_{\mu}$ in the presence of a structural break in the drift under the null. Following Leybourne and Newbold (2000), it is possible to do this by respecifying the break magnitude to be proportional to $T^{-1/2}$ so that asymptotically, the break component and the stochastic component in the DGP are of the same order of magnitude (in probability). Consider, for example, the case of a random walk with an initial drift of $\alpha_1$, and a structural break occurs to a drift of $\alpha_1 + \alpha_2 T^{-1/2}$ at time $t = \lambda T + 1$

$$y_t = d_t + v_t \quad [4]$$

$$v_t = v_{t-1} + \varepsilon_t \quad [5]$$

where

$$d_t = \alpha_1(t - \lambda T), \quad t \leq \lambda T$$

$$= (\alpha_1 + \alpha_2 T^{-1/2})(t - \lambda T), \quad t > \lambda T \quad [6]$$

and $\varepsilon_t$ is defined as before. The non-degenerate asymptotic distribution of the Dickey–Fuller test statistic $DF_{\mu}$ for this DGP is given in Theorem 1 and a proof of Theorem 1 is given in the Appendix. Note that for the purposes of comparison with the orthodox Dickey–Fuller test, which assumes a drift of zero under the null hypothesis, Theorem 1 focuses specifically on the case of $\alpha_1 = 0$ so that as the break magnitude approaches zero ($\alpha_2 \rightarrow 0$), the asymptotic distribution approaches the usual Dickey–Fuller distribution.

**Theorem 1.** Assume that $y_t$ is generated by eqs [4]–[6] with $\alpha_1 = 0$, $\alpha_2 = k\sigma$ and let $W(r)$ denote a standard Wiener process. Then

$$DF_{\mu} \Rightarrow (\sigma^2 + c_2)^{-1/2} \left[ \sigma^2 \int_0^1 \tilde{W}(r)^2 dr + c_3 + 2l_3 \right]^{-1/2}$$

$$\times \left[ \sigma^2 \int_0^1 \tilde{W}(r)dW(r) + c_1 + l_1 + l_2 \right]$$

$$\tilde{W}(r) := W(r) - \int_0^1 W(s)ds$$

where $c_1$, $c_2$, $c_3$ are the limit constants.
\[ c_1 = \sigma^2 k^2 \lambda (1 - \lambda)^2 / 2 \quad \ldots \quad [8] \]
\[ c_2 = 0 \quad \ldots \quad [9] \]
\[ c_3 = \sigma^2 k^2 \left\{ \frac{[(1 - \lambda)^3 / 3] - [(1 - \lambda)^4 / 4]}{1/C_0} \right\} \quad \ldots \quad [10] \]

and \( l_1, l_2, l_3 \) are limit processes defined in the Appendix.

**Remark 1.** Note that as \( k \to 0 \), then \( c_i \to 0 \), \( l_i \to 0 \) (\( i = 1, 2, 3 \)), and therefore

\[ DF_{\mu} \Rightarrow \left[ \int_0^1 \tilde{W}(r)^2 dr \right]^{-1/2} \left[ \int_0^1 \tilde{W}(r) dW(r) \right] \quad \ldots \quad [11] \]

which is the usual Dickey–Fuller distribution.

The effect of a break on the asymptotic distribution of \( DF_{\mu} \) relative to the usual Dickey–Fuller distribution is not immediately clear from eq. [7]. Thus, to help clarify the impact of this type of break, in Figures 1–4 the asymptotic distributions obtained using numerical simulation are plotted for small (\( \alpha_2 = 2.5 \)), medium (\( \alpha_2 = 5 \)), large (\( \alpha_2 = 10 \)) and very large (\( \alpha_2 = 20 \)) break magnitudes and for three different break positions: an early break (\( \lambda = 0.15 \)), a mid-sample break (\( \lambda = 0.50 \)) and a late break (\( \lambda = 0.85 \)). In each case the simulations employ 2,000 replications and the Wiener process is approximated by partial sums of \( NID(0, 1) \) with 5,000 steps. For comparison, also plotted in each graph is the asymptotic distribution of \( DF_{\mu} \) when the drift is always zero (i.e. the usual Dickey–Fuller distribution) and the asymptotic distribution of \( DF_{\mu} \) when the drift is always non-zero (i.e. the standard normal distribution).

It can be seen from Figures 1–4 that, relative to the usual Dickey–Fuller distribution, a break of this type causes the asymptotic distribution of \( DF_{\mu} \) to shift rightwards. This reinforces the message from the point mass asymptotic results obtained for the DGP given by eqs [1]–[2] and suggests that in practice, if a break of this type occurs in a series being tested and critical values from the usual Dickey–Fuller distribution are used, spurious rejections of the unit root null hypothesis in favour of the explosive alternative could be a non-trivial problem. As one might expect, the size of the shift and consequently the extent of the spurious rejection problem is a positive function of the break magnitude. Interestingly, and in contrast to the results in Leybourne et al. (1998) for the left-tailed Dickey–Fuller test against the alternative hypothesis of trend stationarity, as the break increases

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3 These graphs were produced using MATLAB, which is also employed for the simulations in Section 3. Note that the distributions plotted are smoothed versions of the simulated distributions, obtained using kernel density estimation employing an Epanechnikov kernel function.
in size it can be seen that the shift in the distribution is worse for mid-sample breaks than when the break occurs towards the start or end of the sample.

2.2 Recursive Tests

Whilst the asymptotic results above apply to fixed sample Dickey–Fuller tests, they can be used to infer the asymptotic size properties of the right-tailed sup-DF

**Figure 1:** Asymptotic distribution of DFμ in the presence of a small break under the null.

**Figure 2:** Asymptotic distribution of DFμ in the presence of a medium break under the null.
test proposed by PWY in the presence of a structural break in the drift under the unit root null hypothesis. The PWY test is computed using the same model as \(DF_\mu\), but applied recursively. PWY focus specifically on testing the null hypothesis of a unit root with no drift; the test statistic can be written as

\[
SDF_\mu = \sup_{\tau \in [\tau_0, T]} \{DF_\mu, \tau\}
\]

Figure 3: Asymptotic distribution of \(DF_\mu\) in the presence of a large break under the null.

Figure 4: Asymptotic distribution of \(DF_\mu\) in the presence of a very large break under the null.
where DF$_{\mu,\tau T}$ denotes the relevant Dickey–Fuller test statistic computed using sample observations 1, 2, \ldots, $\tau T$. If eqs [1] and [2] are the true DGP and the structural break is such that it causes DF$_{\mu}$ to diverge to $\infty$ as $T \to \infty$ (e.g. if $\alpha_1 = 0$, $\alpha_2 \neq 0$), then it follows straightforwardly that as $T \to \infty$, SDF$_{\mu}$ will be a test statistic computed using a sub-sample that ends after the break, and so the asymptotic size of the right-tailed SDF$_{\mu}$ test will also tend to 1. If the structural break is such that it causes DF$_{\mu}$ to diverge to $-\infty$ as $T \to \infty$ (e.g. if $\alpha_1 \neq 0$, $\alpha_2 = 0$), then as $T \to \infty$, SDF$_{\mu}$ will be a test statistic computed using a sub-sample that ends before the break, and so the asymptotic size of the test will depend on the assumed asymptotic null distribution. Therefore, in contrast to the fixed sample right-tailed DF$_{\mu}$ test, which will be undersized if $\alpha_1 \neq 0$ and $\alpha_2 = 0$, the SDF$_{\mu}$ test will be oversized for this combination of break parameters if the critical values used assume a drift of zero under the null as in PWY, because the true drift for the pre-break period is not zero.

If instead of eqs [1] and [2], we assume that the true DGP is eqs [4]–[6], then Theorem 1 and the limit theory in Section 2 of PWY suggest that

$$
\text{SDF}_\mu \Rightarrow \sup_{r \in [\tau_0, 1]} \left\{ (\sigma^2 + c_{2,\tau})^{-1/2} \left[ \sigma^2 \int_0^r \tilde{W}(r)^2 dr + c_{3,\tau} + 2l_{3,\tau} \right]^{-1/2} \times \left[ \int_0^r \tilde{W}(r) dW(r) + c_{1,\tau} + l_{1,\tau} + l_{2,\tau} \right] \right\}
$$

where $c_{i,\tau}$ and $l_{i,\tau}$ denote the relevant limit constants and limit processes, and

$$
\tilde{W}(r) := W(r) - \frac{1}{r} \int_0^r W(s) ds
$$

If $k \to 0$, then $c_{i,\tau} \to 0$, $l_{i,\tau} \to 0$, and therefore

$$
\text{SDF}_\mu \Rightarrow \sup_{r \in [\tau_0, 1]} \left\{ \left[ \int_0^r \tilde{W}(r)^2 dr \right]^{-1/2} \left[ \int_0^r \tilde{W}(r) dW(r) \right] \right\}
$$

which is the usual PWY distribution.

To help clarify the impact of a break on the asymptotic distribution of SDF$_{\mu}$, and for comparison with Figures 1–4, in Figures 5–8 the asymptotic distributions of SDF$_{\mu}$ are plotted for small ($\alpha_2 = 2.5$), medium ($\alpha_2 = 5$), large ($\alpha_2 = 10$) and very large ($\alpha_2 = 20$) break magnitudes and for three different break positions: an early break ($\lambda = 0.15$), a mid-sample break ($\lambda = 0.50$) and a late break ($\lambda = 0.85$). Also plotted in each graph is the usual PWY distribution which assumes the drift is always zero under the unit root null hypothesis (i.e. the asymptotic distribution of SDF$_{\mu}$ when $k = 0$), and the asymptotic distribution of SDF$_{\mu}$ when the drift is...
always non-zero under the unit root null hypothesis. Similar results are found to those in Figures 1–4 in the sense that a structural break of this type causes the

Figure 5: Asymptotic distribution of $\text{SDF}_\mu$ in the presence of a small break under the null.

Figure 6: Asymptotic distribution of $\text{SDF}_\mu$ in the presence of a medium break under the null.

Note that the algebraic form of the asymptotic distribution for this case is given in Proposition 4.1 in Shi et al. (2011).
asymptotic distribution of SDF, to shift rightwards relative to the usual PWY distribution, suggesting that in practice, spurious rejections of the unit root null hypothesis in favour of the explosive alternative could be a problem if the series being tested is a fixed unit root process with a structural break in the drift parameter. Interestingly it can be seen by comparing the asymptotic distributions

**Figure 7:** Asymptotic distribution of SDF, in the presence of a large break under the null.

**Figure 8:** Asymptotic distribution of SDF, in the presence of a very large break under the null.
in Figures 5–8 with those in Figures 1–4 that for small and medium breaks, ceteris paribus the spurious rejection problem is likely to be less severe in practice for the SDF$_\mu$ test than for the DF$_\mu$ test, in the sense that the rightwards shift relative to the zero drift distribution is in most cases smaller for SDF$_\mu$ than for DF$_\mu$. Note that PSY investigate testing the unit root null hypothesis against the explosive alternative using a right-tailed sup-DF test allowing for a weak, local-to-zero drift under the null

$$y_t = \alpha T^{-\eta} + y_{t-1} + \varepsilon_t$$  \hspace{1cm} [16]$$

where $\eta \geq 0$. As pointed out by a referee, the DGP given by eqs [4]–[6] could be generalized in the manner of PSY as follows, so as to allow for a wider range of break scenarios:

$$y_t = d_t + v_t$$  \hspace{1cm} [17]$$

$$v_t = v_{t-1} + \varepsilon_t$$  \hspace{1cm} [18]$$

$$d_t = \alpha T^{-\eta_1}(t - \lambda T), \quad t \leq \lambda T$$  \hspace{1cm} [19]$$

$$= \alpha T^{-\eta_2}(t - \lambda T), \quad t > \lambda T$$

where $\eta_1 \neq \eta_2$ and $\eta_1, \eta_2 \in [0, \infty)$. Overall, these asymptotic results suggest that in empirical applications, spurious rejections of the unit root null hypothesis in favour of the explosive alternative could be problematic for the original right-tailed DF$_\mu$ and SDF$_\mu$ tests if the relevant series contains a fixed unit root, but with a break in drift. The severity of this problem in finite sample applications of the tests is investigated in more detail with Monte Carlo simulations, which are discussed below.

### 3 Simulation Results

The empirical sizes of the original right-tailed DF$_\mu$ and SDF$_\mu$ tests in the presence of a break in drift under the null are simulated at the 5% nominal size using the DGP given by eqs [4]–[6] with $\varepsilon_t \sim \text{NID}(0, 1)$. Results are reported for small and large sample sizes ($T = 100$ and $T = 500$), for a selection of different break locations and for small, medium, large and very large break sizes. For brevity and for consistency with the asymptotic analysis in Section 2, in all cases we assume that the pre-break drift $\alpha_1$ is zero. When computing SDF$_\mu$ we set $\tau_0 = 0.10$. The finite sample critical values used for DF$_\mu$ are taken from Fuller (1976). The finite sample critical values used for SDF$_\mu$ are obtained by Monte
Carlo simulation assuming a zero drift under the null hypothesis and using standard normal error terms.\(^5\)

Table 1 contains the results for the right-tailed DF\(_{\mu}\) test. Consistent with the asymptotic results in Section 2, the test is oversized for all parameter combinations and the degree of size distortion is a positive function of the break magnitude. Clearly, mid-sample breaks generate more size distortion than late breaks, which is also consistent with the asymptotic results in Section 2 (although for very large breaks the size is close to or equal to 1 irrespective of the break location). Some intuition for this finding is given by noting that for large values of \(k\) in the DGP eqs [4]–[6], then as \(T \to \infty\), it is approximately true that

\[
\text{DF}_{\mu} \to \sigma^{-1}c_3^{-1/2}c_1
\]

Calculus can then be used to show that this function is maximized at \(\lambda = 0.577\). Hence, when the DGP is the model given by eqs [4]–[6], one would expect that the rightwards shift of the asymptotic and finite sample distributions for DF\(_{\mu}\) relative to their orthodox positions will be largest for mid-sample breaks, and therefore positive size distortion will be worse for mid-sample breaks than for late breaks if the orthodox critical values are used. Interestingly the results in Table 1 show that for early breaks, the degree of finite sample size distortion is similar to that for mid-sample breaks and is quite severe even when the break is

<table>
<thead>
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<th>(\lambda)</th>
<th>(\alpha_2 = 2.5)</th>
<th>(\alpha_2 = 5)</th>
<th>(\alpha_2 = 10)</th>
<th>(\alpha_2 = 20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T = 100)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.370</td>
<td>0.746</td>
<td>0.973</td>
<td>1.000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.423</td>
<td>0.862</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
<td>0.50</td>
<td>0.347</td>
<td>0.869</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.75</td>
<td>0.161</td>
<td>0.484</td>
<td>0.949</td>
<td>1.000</td>
</tr>
<tr>
<td>0.85</td>
<td>0.089</td>
<td>0.228</td>
<td>0.664</td>
<td>0.992</td>
</tr>
<tr>
<td>(T = 500)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.339</td>
<td>0.743</td>
<td>0.976</td>
<td>1.000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.375</td>
<td>0.862</td>
<td>0.999</td>
<td>1.000</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.85</td>
<td>0.085</td>
<td>0.216</td>
<td>0.669</td>
<td>0.993</td>
</tr>
</tbody>
</table>

5 10,000 replications are used to simulate the critical values and 1,000 replications are used for the size simulations.
relatively small. Indeed for the smallest break considered, the size distortion is trivial only if the break occurs late in the sample.

Table 2 contains the results for the right-tailed SDF\(_\mu\) test. Again, consistent with the asymptotic distributions in Section 2, the test is oversized for virtually all parameter combinations. Note that relative to the right-tailed DF\(_\mu\) test, for nearly all parameter combinations the SDF\(_\mu\) test suffers from a lower degree of size distortion, which is also suggested by the asymptotic results. When the break is very large then as with the DF\(_\mu\) test, the SDF\(_\mu\) test has finite sample size close to or equal to 1 irrespective of the break location.

4 Possible Solutions

The results given above suggest that for the right-tailed Dickey–Fuller-type tests considered in this paper, in practice if the true DGP is a unit root model with a break in drift then unless the break is relatively small and occurs towards the end of the sample period, the probability of a spurious rejection in favour of the explosive alternative will be non-trivially large. Hence, allowing for the presence of structural breaks under the unit root null hypothesis when using these types of tests is something that practitioners may want to consider. However, in a practical context it is unlikely that the presence or timing of a break would be known a priori; therefore, solutions that endogenize the structural break and treat its location as unknown are required. There is a huge literature on testing for a unit root in the presence...
of endogenous structural breaks using left-tailed Dickey–Fuller tests, and one imagines that these techniques might also be applicable to the right-tailed Dickey–Fuller-type tests used in the literature on testing for asset price bubbles. For example, the additive outlier (AO) approach proposed by Perron (1997) for dealing with endogenous structural breaks in the trend function could perhaps be adapted. When this approach is used the estimated break date is either the date that leads to the strongest support for the alternative hypothesis (for left-tailed tests, the date that minimizes the Dickey–Fuller test statistic), or the date that maximizes the statistical significance of the estimated break parameter.

One possible alternative to the AO approach would be to extend the orthodox specifications employed to compute $DF_\mu$ and $SDF_\mu$ with 1/0 dummy variables defined using a consistent estimate of the break fraction $\hat{\lambda}$ so as to control for the impact of the break. It follows from the literature on testing for structural breaks in stationary series (e.g. Bai 1994) that if $y_t$ is a fixed unit root process with a break in drift, $\Delta y_t$ can be interpreted as a stationary ARMA (autoregressive moving average) process with a break in mean, and a consistent estimate of the break fraction can be obtained using least squares

$$\hat{\lambda} = \arg\min_\lambda \left\{ \min_{\alpha_1, \alpha_2} \sum_{t=2}^{T} [\Delta y_t - \alpha_1(1 - D(\lambda)_t) - \alpha_2 D(\lambda)_t]^2 \right\}$$

where $\lambda \in (0, 1)$. It should be noted, however, that a weakness of this approach if applied to right-tailed Dickey–Fuller-type tests is that it allows for a possible break under both the null and the alternative hypotheses. Consequently if the alternative hypothesis is true and $y_t$ is an explosive process with no break, the extended $DF_\mu$ and $SDF_\mu$ tests will have lower power than the no-break versions. A more detailed analysis of these issues lies outside of the scope of this paper, but would be an interesting topic for future research.

5 Conclusion

Right-tailed Dickey–Fuller-type tests of the unit root null hypothesis against the explosive alternative have recently become popular in economics and finance for detecting asset price bubbles. This paper has studied the asymptotic and finite sample size properties of fixed sample and recursive right-tailed Dickey–
Fuller tests if the relevant series contains a fixed unit root and is not explosive at any point, but a structural break in the drift parameter occurs and this is ignored when computing the tests. It is shown that depending on the magnitude and location of the break, positive size distortion can be non-trivially large, leading to a spurious rejection problem in favour of the explosive alternative. Of course, the empirical relevance of these results for attempts to detect asset price bubbles using the tests considered depends on the likelihood that the particular price series being examined is generated by a fixed unit root model with a structural break (or breaks) in the drift parameter. As discussed in Section 1, for various financial assets previous empirical work has found evidence suggesting that a fixed unit root model with a time-varying drift parameter is appropriate for the natural logarithm of the relevant price series. Thus, further research to clarify the importance of this issue in the context of testing for asset price bubbles would seem to be warranted.

**Appendix**

**Proof of Theorem 1**

This proof is based on the proof of Theorem 2 in Leybourne and Newbold (2000). Start by defining $e_t$ as the residuals from an OLS (ordinary least squares) regression of $y_t$ on an intercept. Thus, we can write

$$e_t = w_t + g_t$$

where

$$g_t = d_t - \bar{d}$$

$$w_t = v_t - \bar{v}$$

The Dickey–Fuller test statistic can then be written as

$$DF_\mu = (\hat{\sigma}^2 f_1^{-1})^{-1/2}(\hat{\rho} - 1)$$

$$= \hat{\sigma}^{-1}(T^{-2}f_1)^{-1/2}T^{-1}(f_2 - f_1)$$

with $\hat{\rho} = f_2 f_1^{-1}$, $f_1 = \sum_{t=2}^{T} e_{t-1}^2$, $f_2 = \sum_{t=2}^{T} e_t e_{t-1}$ and

$$\hat{\sigma}^2 = T^{-1} f_0 + \hat{\rho}^2 T^{-1} f_1 - 2\hat{\rho} T^{-1} f_2$$

where $f_0 = \sum_{t=2}^{T} e_t^2$.

Consider now the scaled numerator term in eq. [26]
\[ T^{-1}(f_2 - f_1) = T^{-1} \sum_{t=2}^{T} e_{t-1} \Delta e_t \]
\[ = T^{-1} \sum_{t=2}^{T} w_{t-1} \Delta w_t + T^{-1} \sum_{t=2}^{T} g_{t-1} \Delta g_t \quad \text{[28]} \]
\[ + T^{-1} \sum_{t=2}^{T} w_{t-1} \Delta g_t + T^{-1} \sum_{t=2}^{T} g_{t-1} \Delta w_t \]

Define \( W(r) \) to be a Wiener process and \( \tilde{W}(r) \) to be a demeaned Wiener process as in Park and Phillips (1988). Using the continuous mapping theorem (see, e.g. Hamilton 1994, Chap. 17) and following Leybourne and Newbold (2000), it can be shown that

\[ T^{-1} \sum_{t=2}^{T} w_{t-1} \Delta w_t \Rightarrow \sigma^2 \int_{0}^{1} \tilde{W}(r) dW(r) \quad \text{[29]} \]

\[ T^{-1} \sum_{t=2}^{T} w_{t-1} \Delta g_t \Rightarrow l_1 \quad \text{[30]} \]

\[ T^{-1} \sum_{t=2}^{T} g_{t-1} \Delta w_t \Rightarrow l_2 \quad \text{[31]} \]

\[ l_1 = \sigma^2 k \int_{\lambda}^{1} \tilde{W}(r) dr \quad \text{[32]} \]

\[ l_2 = \sigma^2 k \left\{ \int_{\lambda}^{1} (r - \lambda) dW(r) - (1 - \lambda)^2 / 2[\tilde{W}(1) - \tilde{W}(0)] \right\} \quad \text{[33]} \]

For the remaining part of eq. [28]

\[ T^{-1} \sum_{t=2}^{T} g_{t-1} \Delta g_t \rightarrow c_1 \quad \text{[34]} \]

where

\[ c_1 = \sigma^2 k^2 \lambda (1 - \lambda)^2 / 2 \quad \text{[35]} \]

Thus, using the results given above we can write

\[ T^{-1}(f_2 - f_1) \Rightarrow \sigma^2 \int_{0}^{1} \tilde{W}(r) dW(r) + c_1 + l_1 + l_2 \quad \text{[36]} \]
Next, consider the denominator term in eq. [26]

\[
T^{-2}f_1 = T^{-2} \sum_{t=2}^{T} w_{t-1}^2 + T^{-2} \sum_{t=2}^{T} g_{t-1}^2 + 2T^{-2} \sum_{t=2}^{T} w_{t-1}g_{t-1}
\] \hspace{1cm} [37]

It follows from an application of the continuous mapping theorem that for the first and third terms on the right-hand-side of eq. [37]

\[
T^{-2} \sum_{t=2}^{T} w_{t-1}^2 \Rightarrow \sigma^2 \int_0^1 \tilde{W}(r)^2 dr \] \hspace{1cm} [38]

\[
T^{-2} \sum_{t=2}^{T} w_{t-1}g_{t-1} \Rightarrow l_3
\] \hspace{1cm} [39]

where

\[
l_3 = \sigma^2 k \left[ \int_0^1 (r - \lambda) \tilde{W}(r) dr \right]
\] \hspace{1cm} [40]

The second term converges as follows:

\[
T^{-2} \sum_{t=2}^{T} g_{t-1}^2 \rightarrow c_3
\] \hspace{1cm} [41]

where

\[
c_3 = \sigma^2 k^2 \left[ [(1 - \lambda)^3 / 3] - [(1 - \lambda)^4 / 4] \right]
\] \hspace{1cm} [42]

Finally, note that

\[
\sigma^2 = T^{-1} \sum_{t=2}^{T} \Delta e_t^2 + o_p(1)
\] \hspace{1cm} [43]

\[
= T^{-1} \sum_{t=2}^{T} \Delta w_t^2 + T^{-1} \sum_{t=2}^{T} \Delta g_t^2 + 2T^{-1} \sum_{t=2}^{T} \Delta w_t \Delta g_t + o_p(1)
\]

and since

\[
T^{-1} \sum_{t=2}^{T} \Delta w_t \Delta g_t = o_p(1)
\] \hspace{1cm} [44]
we can write
\[
\text{plim}(\hat{\sigma}^2) = \sigma^2 + c_2
\] [45]

where \( c_2 = \text{plim} \left( T^{-1} \sum_{t=2}^{T} \Delta g_t^2 \right) = 0.\) Collecting all of the results given above shows that
\[
\text{DF}_\mu \Rightarrow (\sigma^2 + c_2)^{-1/2} \left[ \sigma^2 \int_0^1 \tilde{W}(r)^2 \, dr + c_3 + 2l_3 \right]^{-1/2} \times \left[ \sigma^2 \int_0^1 \tilde{W}(r) dW(r) + c_1 + l_1 + l_2 \right]
\] [46]

References


