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Oscillatory convection and limitations of the Boussinesq approximation

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We determine the asymptotic conditions under which the Boussinesq approximation is valid for oscillatory convection in a rapidly rotating fluid. In the astrophysically relevant parameter regime of small Prandtl number, we show that the Boussinesq prediction for the onset of convection is valid only under much more restrictive conditions than those that are usually assumed. In the case of an ideal gas, we recover the Boussinesq results only if the ratio of the domain height to a typical scale height is much smaller than the Prandtl number. This requires an extremely shallow domain in the astrophysical parameter regime. Other commonly-used "sound-proof" approximations generally perform no better than the Boussinesq approximation. The exception is a particular implementation of the pseudo-incompressible approximation, which predicts the correct instability threshold beyond the range of validity of the Boussinesq approximation.

1. Introduction

Most astrophysical objects contain regions in which heat is transported by convection. The numerical modelling of these convective flows (which are usually turbulent) is difficult because of the stiffness of the governing equations caused by the presence of acoustic waves. Almost all convection models therefore filter out these waves by using a "sound-proof" set of equations, such as the Boussinesq, anelastic or pseudo-incompressible equations. Each of these sound-proof approximations is founded on certain physical assumptions, which may not be valid in all cases of interest. Specifically, the Boussinesq equations (e.g. Spiegel & Veronis 1960) are valid only for small perturbations to the thermodynamic variables in systems with small vertical lengthscales (in particular, the domain height must be much smaller than all of the thermodynamic scale heights). The anelastic equations require less restrictive assumptions, but do require the fluid to be nearly adiabatically stratified (e.g. Ogura & Phillips 1962; Lipps & Hemler 1982). The pseudo-incompressible equations were introduced by Durran (1989) as an improvement to the anelastic equations. Although they are formally valid only under the same conditions as the anelastic equations, albeit for stratifications stronger than anticipated by standard asymptotics (Klein *et al.* 2010), in some cases they are found to better approximate the true dynamics (e.g. Achatz *et al.* 2010). This situation is further complicated by the fact that there are several different versions of both the anelastic and pseudo-incompressible equations currently in use, with no general consensus on which is "best" (e.g. Brown *et al.* 2012; Vasil *et al.* 2013).

The interiors of stars and gaseous planets are characterised by rapid rotation and low viscosity. In this parameter regime, convection is often oscillatory close to onset (e.g. Jones *et al.* 2009). The first studies of oscillatory convection were performed under the Boussinesq approximation (Chandrasekhar 1953), but it has never been determined in precisely what asymptotic limit the Boussinesq and fully compressible results agree.

Perhaps surprisingly, some implementations of the anelastic approximation exhibit un-physical "negative Rayleigh number" convection in this parameter regime (Drew *et al.* 1995; Calkins *et al.* 2015). As other implementations do not exhibit this unphysical behaviour (Jones *et al.* 2009), it seems that oscillatory convection is an important test case for comparing different sound-proof models.

In this paper, we perform a careful analysis of the onset of oscillatory convection in the fully compressible system, in order to determine the precise conditions under which the Boussinesq results are valid. We find that these conditions are much more stringent than anticipated from the usual heuristic arguments. In particular, it is *not* sufficient that the vertical scale of the domain is much smaller than the thermodynamic scale heights. This analysis is then extended to a very general set of sound-proof equations, which includes the anelastic and pseudo-incompressible approximations as special cases. In doing so, we introduce a simple but powerful technique that can be used to standardise the anelastic and pseudo-incompressible equations, building on an observation of Ogura & Phillips (1962) about energy conservation, thus removing any ambiguity in their formulation. Our standardised anelastic and pseudo-incompressible equations are both free from the unphysical behaviour noted by Calkins *et al.* (2015). However, the standardised pseudo-incompressible equations are the only "sound-proof" system that correctly predicts the threshold for oscillatory convection on larger vertical scales than the Boussinesq approximation.

2. Fully compressible vs. Boussinesq

2.1. *The governing equations*

In order to determine the true convective stability threshold, we linearise the fully compressible equations about a hydrostatic background in a reference frame rotating with angular velocity $\boldsymbol{\Omega}$. For simplicity we neglect self-gravity and the centrifugal force, so the gravitational acceleration $\boldsymbol{g}$ points directly downward. We adopt Cartesian coordinates in which $\boldsymbol{g} = -g\boldsymbol{e}_z$ and $\boldsymbol{\Omega} = \Omega\boldsymbol{e}_z$, and write the linearised equations in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -s_1\boldsymbol{\nabla}T_0 + T_1\boldsymbol{\nabla}s_0 - \boldsymbol{\nabla}(p_1/\rho_0) + \boldsymbol{f}_1 \tag{2.1}$$

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0\boldsymbol{u}) = 0 \tag{2.2}$$

$$\frac{\partial s_1}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}s_0 = \frac{q_1}{T_0}, \tag{2.3}$$

where subscripts 0 and 1 refer to unperturbed quantities (which are functions of z only) and their linear perturbations, respectively. The quantities $\boldsymbol{f}_1$ and q_1 are the viscous force and heating rate per unit mass, whose precise form we specify later. In order to close these equations we require relations between the thermodynamic perturbations. For convenience we introduce the specific enthalpy, $H(p, s)$, which is a function of pressure, p, and specific entropy, s (e.g. Landau & Lifshitz 1980, §14). We can then define the density, $\rho = 1/H_p$, and temperature, $T = H_s$, where the subscripts on H represent partial derivatives. The linear perturbations to ρ and T are then given by

$$-\frac{\rho_1}{\rho_0^2} = H_{ps}s_1 + H_{pp}p_1 \tag{2.4}$$

$$T_1 = H_{ss}s_1 + H_{sp}p_1, \tag{2.5}$$

with similar expressions applying to the spatial variations of the background state:

$$-\frac{\nabla \rho_0}{\rho_0^2} = H_{ps}\nabla s_0 + H_{pp}\nabla p_0 \qquad (2.6)$$

$$\nabla T_0 = H_{ss}\nabla s_0 + H_{sp}\nabla p_0. \qquad (2.7)$$

(Note that $H_{ps} = H_{sp}$.) The second derivatives in equations (2.4)–(2.7) can also be expressed in terms of the sound speed, c, and specific heats, c_p and c_v, as follows

$$H_{pp} = -\frac{1}{\rho_0^2 c^2}, \qquad H_{ss} = \frac{T_0}{c_p}, \qquad \frac{H_{ps}^2}{H_{pp}H_{ss}} = 1 - \frac{c_p}{c_v}. \qquad (2.8)$$

Finally, the buoyancy frequency, N, is defined by the formula $N^2 = -H_{ps}\dfrac{\mathrm{d}p_0}{\mathrm{d}z}\dfrac{\mathrm{d}s_0}{\mathrm{d}z}$. We are concerned here with convective instability, which requires that $N^2 < 0$. For notational convenience, we therefore define the parameter $\mathcal{N} = -N^2$ as a measure of the degree of superadiabaticity. In what follows we will assume that $\mathcal{N}$ is positive throughout the domain, but we will make no assumption about its magnitude. Note that the right-hand side of the momentum equation (2.1) can be written in many different forms, by using the relations (2.4)–(2.7) and the hydrostatic condition $\nabla p_0 = \rho_0 \boldsymbol{g}$; the form used here has been chosen for later convenience.

The linearised Boussinesq equations, for comparison, are

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = \frac{\rho_1}{\rho_0}\boldsymbol{g} - \frac{1}{\rho_0}\nabla p_1 + \boldsymbol{f}_1 \qquad (2.9)$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad (2.10)$$

$$\frac{\partial s_1}{\partial t} + \boldsymbol{u} \cdot \nabla s_0 = \frac{q_1}{T_0} \qquad (2.11)$$

$$-\frac{\rho_1}{\rho_0^2} = H_{ps}s_1 \qquad (2.12)$$

$$T_1 = H_{ss}s_1, \qquad (2.13)$$

where ρ_0 is approximated as a constant, but s_0 retains its dependence on z. In the Boussinesq approximation, the viscous force and the heating term take the form

$$\boldsymbol{f}_1 = \nabla \cdot (\nu \nabla \boldsymbol{u}) \qquad \text{and} \qquad \frac{q_1}{T_0} = \nabla \cdot (\kappa \nabla s_1) \qquad (2.14)$$

where ν is the kinematic viscosity and κ is the thermal diffusivity. The derivation of these Boussinesq equations relies upon the following assumptions (e.g. Spiegel & Veronis 1960; Mihaljan 1962):

(i) the domain height, L say, is much smaller than the scale height of each thermodynamic variable;

(ii) perturbations to the thermodynamic variables are even smaller than the variations in their background values;

(iii) fluid motions are slow in comparison with the sound speed, c;

(iv) any timescale of the flow is much longer than the acoustic timescale, L/c.

Often these assumptions are not stated explicitly, but are implicit in the scalings assumed for the various physical quantities. Many derivations incorporate additional assumptions in order to simplify the Boussinesq equations still further, for example by neglecting viscous heating (Veronis 1962; Mihaljan 1962; Gray & Giorgini 1976). In what follows we examine the onset of oscillatory convective instability in the Boussinesq and compressible systems, which renders such additional assumptions unnecessary, because viscous heating

vanishes for linear perturbations to a hydrostatic state. Moreover, conditions (ii) and (iii) are automatically satisfied in this case, because near onset the perturbations to the hydrostatic background state are infinitesimally small. Our aim is therefore to determine whether the Boussinesq and fully compressible systems have the same convective stability threshold in the regime described by conditions (i) and (iv).

2.2. *The linear stability of the Boussinesq system*

We first derive the stability threshold for the Boussinesq equations, summarising the results of Chandrasekhar (1953). Suppose, for simplicity, that ν, κ, and $\mathcal{N}$ are positive and constant throughout the domain. We can then seek solutions of equations (2.9)–(2.13) in which the perturbations are $\propto \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t)$, and thus obtain a dispersion relation

$$\frac{\mathcal{N}k_{\mathrm{h}}^2}{(\kappa k^2 - \mathrm{i}\omega)} = (\nu k^2 - \mathrm{i}\omega)k^2 + \frac{4\Omega^2 k_z^2}{(\nu k^2 - \mathrm{i}\omega)}, \qquad (2.15)$$

where $k = |\boldsymbol{k}|$ and $k_{\mathrm{h}} = \sqrt{k_x^2 + k_y^2}$. The onset of instability can be either direct (i.e. $\omega = 0$) or oscillatory (i.e. $\omega^2 > 0$) depending on parameter values. In the former case, a perturbation with given $\boldsymbol{k}$ is convectively unstable if

$$\mathcal{N} > \nu\kappa\left(\frac{k^6}{k_{\mathrm{h}}^2}\right) + \frac{\kappa}{\nu}\left(\frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2}\right). \qquad (2.16)$$

In the latter case, such a perturbation is unstable if

$$\tfrac{1}{2}\mathcal{N} > \nu(\kappa + \nu)\left(\frac{k^6}{k_{\mathrm{h}}^2}\right) + \frac{\nu}{\kappa + \nu}\left(\frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2}\right), \qquad (2.17)$$

and, at onset, it oscillates with a frequency given by

$$\omega^2 = \frac{\kappa - \nu}{\kappa + \nu}\left(\frac{4\Omega^2 k_z^2}{k^2}\right) - \nu^2 k^4. \qquad (2.18)$$

In the simplest case of Rayleigh–Bénard convection between horizontal plates separated by distance L, with fixed-temperature and stress-free boundary conditions, the most unstable mode can be found by simply setting $k_z = \pi/L$ and finding the minimum unstable value of $\mathcal{N}$ over all k_{h}. From an examination of equations (2.16) and (2.17) it can be seen that oscillatory instability is favoured in a rapidly rotating fluid with low viscosity, and that in such cases the instability broadly resembles a growing inertial wave. In the double asymptotic limit with $\nu \ll \kappa \ll \Omega L^2$ the unstable modes are quasi-geostrophic ($k_{\mathrm{h}}^2 \gg k_z^2$), and the instability boundary is approximately described by

$$\tfrac{1}{2}\mathcal{N} \simeq \nu\kappa k_{\mathrm{h}}^4 + \frac{\nu}{\kappa}\left(\frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2}\right) \qquad (2.19)$$

and

$$\omega^2 \simeq \frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2}. \qquad (2.20)$$

Minimising over k_h, we obtain the asymptotic scalings for the instability at onset:

$$k_{\mathrm{h}} \simeq \left(\sqrt{2}\Omega k_z/\kappa\right)^{1/3} \qquad (2.21)$$

$$\omega \simeq \left(4\sqrt{2}\Omega^2 \kappa k_z^2\right)^{1/3} \qquad (2.22)$$

$$\mathcal{N} \simeq 6\left(2\Omega^2 \kappa k_z^2\right)^{2/3}\nu/\kappa. \qquad (2.23)$$

It is sometimes helpful to rewrite these results in terms of dimensionless quantities. For Boussinesq convection, the standard dimensionless numbers are the Rayleigh, Taylor, and Prandtl numbers, defined respectively as

$$Ra \equiv \frac{\mathcal{N}L^4}{\nu\kappa}, \qquad Ta \equiv \frac{4\Omega^2 L^4}{\nu^2}, \qquad Pr \equiv \frac{\nu}{\kappa}. \tag{2.24}$$

After putting $k_z = \pi/L$, equation (2.23) can be rewritten as

$$Ra \simeq \tfrac{3}{2}(4\pi^2 Pr^2 Ta)^{2/3} \tag{2.25}$$

(Chandrasekhar 1953) and the double asymptotic limit in which this result applies can be expressed as $1 \ll Pr^{-1} \ll Ta^{1/2}$. Equations (2.21) and (2.22) imply that $\omega \simeq \sqrt{2}\kappa k_{\mathrm{h}}^2$, so the horizontal lengthscale of the instability is such that heat can diffuse between the upflows and downflows on the same timescale as their oscillation period. The instability can be understood physically as an inertial wave that is damped by viscosity but amplified by thermal diffusion, which feeds heat into the upflows, making them buoyant, thereby extracting potential energy from the background stratification.

2.3. *The linear stability of the fully compressible system*

We now consider the stability properties of the fully compressible equations (2.1)–(2.5). We recall that these equations are valid for an arbitrary equation of state, and an arbitrary hydrostatic background, so in general their solutions will have complex, non-sinusoidal dependence on z. Nevertheless we can still seek solutions with a given horizontal wavenumber k_{h} and frequency ω. Our aim is to determine in what asymptotic limit (if any) the fully compressible equations have the same instability threshold as the Boussinesq equations. Assuming that such a limit does exist, we anticipate that the unstable modes will follow the Boussinesq scaling laws (2.21)–(2.23), and in particular that $k_{\mathrm{h}} \gg 1/L$. In that case, we can neglect vertical diffusion, and approximate the diffusive terms in equations (2.1) and (2.3) as

$$\boldsymbol{f}_1 \simeq -\nu k_{\mathrm{h}}^2 \boldsymbol{u} \qquad \text{and} \qquad q_1 \simeq -c_p \kappa k_{\mathrm{h}}^2 T_1. \tag{2.26}$$

(Note that we use the correct definition of κ given by Spiegel & Veronis (1960), rather than that of Chandrasekhar (1961).) The validity of this approximation will be confirmed later, by solving the full set of linear equations numerically. With this simplification, the linear equations (2.1)–(2.5) can be reduced to a pair of coupled ordinary differential equations for the perturbation quantities p_1 and u_z:

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{g}{c^2}\left(\frac{\frac{c_p}{c_v}\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}\right)\right] p_1 = \left[\frac{\mathcal{N}}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega} - (\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega)\right] \rho_0 u_z \tag{2.27}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\mathcal{N}}{g}\left(\frac{\mathrm{i}\omega}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}\right)\right] \rho_0 u_z = \left[\frac{\mathrm{i}\omega}{c^2}\left(\frac{\frac{c_p}{c_v}\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}\right) - \frac{(\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega)k_{\mathrm{h}}^2}{4\Omega^2 + (\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega)^2}\right] p_1. \tag{2.28}$$

In order to make contact with the Boussinesq results, we must now impose the restrictions built into the Boussinesq approximation. First, we take the limit in which the height of the domain, L, is much smaller than any lengthscale on which the background state varies. This corresponds to condition (i) given earlier. In this limit, and on the assumption that the scaling $\omega \sim \kappa k_{\mathrm{h}}^2$ is still valid, the left-hand sides of equations (2.27) and (2.28) are dominated by the derivative terms, and the parenthetical terms on the right-hand sides can be approximated as constant. Under condition (i), it is reasonable to expect growing modes near onset to have a sinusoidal dependence on z, which means

that we can replace $\frac{\mathrm{d}}{\mathrm{d}z} \to \mathrm{i}k_z$, thus obtaining an approximate dispersion relation

$$0 \simeq k_z^2 + \left[\frac{\mathcal{N}}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega} - (\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega) \right] \left[\frac{\mathrm{i}\omega}{c^2} \left(\frac{\frac{c_p}{c_v} \kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega} \right) - \frac{(\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega) k_{\mathrm{h}}^2}{4\Omega^2 + (\nu k_{\mathrm{h}}^2 - \mathrm{i}\omega)^2} \right]. \quad (2.29)$$

Next, we assume that the oscillation frequency of the instability is much smaller than the resonant acoustic frequency (i.e. $\omega \ll c/L$), in accordance with condition (iv). Provided that the scalings (2.21)–(2.23) are still correct in order of magnitude, the real and imaginary parts of the dispersion relation (2.29) imply that

$$\tfrac{1}{2}\mathcal{N} \simeq \nu\kappa k_{\mathrm{h}}^4 + \frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2} \left[\frac{\nu}{\kappa} + \left(\frac{c_p}{c_v} - 1 \right) \left(\frac{2\Omega^2}{c^2 k_{\mathrm{h}}^2} \right) \right] \qquad (2.30)$$

and
$$\omega^2 \simeq \frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2} \qquad\qquad\qquad (2.31)$$

at the onset of instability. We note that the imaginary part of equation (2.29) is smaller than the real part by a factor of $Pr = \nu/\kappa \ll 1$. Therefore in order to obtain equations (2.30)–(2.31) it is not sufficient to consider only the leading-order terms in equation (2.29).

Comparing equations (2.19) and (2.30), we see that the Boussinesq results are valid only if $(\nu/\kappa)^{1/2} c \gg \Omega/k_{\mathrm{h}} \sim \omega L$, which is much more restrictive than the condition $c \gg \omega L$ assumed above. Conditions (i)–(iv) are therefore *not* sufficient to guarantee the validity of the Boussinesq approximation. The additional term in equation (2.30), compared with equation (2.19), renders the fully compressible system subject to conditions (i)–(iv) more stable than the equivalent Boussinesq system, because each wavenumber becomes marginally stable at a larger value of $\mathcal{N}$.

The discrepancy between the true instability threshold and that predicted by the Boussinesq equations suggests that some effect of compressibility remains significant even under conditions (i)–(iv). In the compressible system, equations (2.3) and (2.26) describe the advection of *entropy* and the diffusion of *temperature*. If we rewrite equation (2.3) in terms of the density and pressure perturbations, then we obtain

$$\frac{\rho_1}{\rho_0} = -\frac{\mathcal{N} u_z/g}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega} + \left(\frac{\frac{c_p}{c_v} \kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega}{\kappa k_{\mathrm{h}}^2 - \mathrm{i}\omega} \right) \frac{p_1}{\rho_0 c^2}. \qquad (2.32)$$

The term in parentheses is complex, and its imaginary part quantifies the density perturbations arising from pressure perturbations that are *out of phase*. This is the source of the extra term in equation (2.30) compared with equation (2.19). In the Boussinesq equations, the density, temperature and entropy perturbations are all proportional to one another (equations 2.12 and 2.13), so this effect is absent. We note that this discrepancy does not arise in direct (i.e. non-oscillatory) convection, for which all perturbations are necessarily in phase with one another, so our findings apply only to oscillatory instabilities.

2.4. *A specific example: The case of an ideal gas*

To more precisely illustrate the discrepancy between the Boussinesq and compressible results, we now consider the particular case of an ideal gas. With this simplification, the compressible system can be defined in terms of six dimensionless parameters. It is convenient to choose three of these to be Ra, Ta and Pr, as in the Boussinesq system. For the remaining three, we will choose the ratio of specific heats, c_p/c_v, the ratio of

the temperature gradient to the adiabatic temperature gradient, $\dfrac{\nabla}{\nabla_{\mathrm{ad}}} = 1 + \dfrac{H_{ss}}{H_{sp}}\dfrac{\mathrm{d}s_0}{\mathrm{d}p_0}$, and the ratio of the domain height to the temperature scale height, $\theta = L/h_T$, where $h_T \equiv -(\mathrm{d}\ln T_0/\mathrm{d}z)^{-1}$. Note that, in a domain of finite height, all of these parameters except c_p/c_v will generally depend on z. However, if the domain is sufficiently shallow, as we will shortly assume, then they may be approximated as constant. For simplicity we will fix $c_p/c_v = 5/3$ and $\nabla/\nabla_{\mathrm{ad}} = 2$, so that θ is the only additional variable in our compressible system. We anticipate that the system will become Boussinesq in the limit $\theta \to 0$, but we need to determine exactly how small θ must be for the Boussinesq results to hold. Given our choices for the other two parameters, the scale heights of pressure and density are $h_p \equiv -(\mathrm{d}\ln p_0/\mathrm{d}z)^{-1} = \frac{4}{5}L/\theta$ and $h_\rho \equiv -(\mathrm{d}\ln\rho_0/\mathrm{d}z)^{-1} = 4L/\theta$, and we have

$$\frac{\mathcal{N}}{c^2} = \frac{3}{8}(\theta/L)^2. \tag{2.33}$$

Because all of the thermodynamic scale heights are of order L/θ, condition (i) is simply $\theta \ll 1$. Condition (iv), which requires that $\omega \ll c/L$, turns out to be more strict. Combining equation (2.33) with (2.22) and (2.23), this condition becomes $\theta \ll (8\,Pr)^{1/2}$. But even when this condition is satisfied, the Boussinesq results may not be valid. In fact, the marginal stability criterion (2.30) then becomes, in terms of our chosen dimensionless parameters,

$$Ra \simeq 2k_{\mathrm{h}}^4 + Pr^2\,Ta\frac{2\pi^2}{k_{\mathrm{h}}^2} + \frac{Pr^2\,Ta^2}{Ra}\frac{\pi^2\theta^2}{4k_{\mathrm{h}}^4}. \tag{2.34}$$

Here k_{h} and k_z are measured in units of $1/L$, and we have put $k_z = \pi$, assuming stress-free, fixed-temperature boundary conditions. This expression can be solved for Ra, and the critical Rayleigh number is then found by minimising over all k_{h}. The result is

$$Ra \simeq (1 + X^{-1})(2\pi^2 X\,Pr^2\,Ta)^{2/3}, \tag{2.35}$$

where

$$X = \frac{1}{2} + \sqrt{\frac{9}{4} + \frac{\theta^2}{2\pi^2\,Pr^2}}. \tag{2.36}$$

We recover the Boussinesq result, which corresponds to $X = 2$, only if $\theta \ll Pr$. Under astrophysical conditions, this is a rather extreme restriction. In the solar interior, for example, the Prandtl number is of order $Pr \simeq 10^{-6}$, and the temperature scale height is of order 10^5 km. So our result implies that the Boussinesq approximation is only valid on vertical scales smaller than 1 km! Of course, convection in the solar interior is well developed, and so the linear stability analysis presented here is not directly applicable. However, there is no obvious reason why the Boussinesq equations should be any more valid in the strongly nonlinear regime than they are in the linear one. Moreover, numerical simulations of the Sun (and, indeed, other convective systems) are never performed very far above the convective threshold, owing to computational constraints, and so our results may well be directly applicable to those simulations.

In order to confirm the validity of these results, we have used a linear eigensolver to compute the critical Rayleigh number as a function of Taylor number in an ideal gas with $\theta = 0.02$ for three different values of Pr, with other parameters as given above. The solver, which uses the algorithm originally developed by Gough *et al.* (1976), solves the exact linearised equations for an ideal gas with constant dynamic viscosity and thermal conductivity. We consider the simplest case of stress-free, fixed-temperature boundary conditions, so that the exact stability threshold can be directly compared with the asymptotic scalings obtained from the Boussinesq (2.25) and fully compressible (2.35) equations. This comparison is shown in Figure 1. In the rapidly rotating regime, with
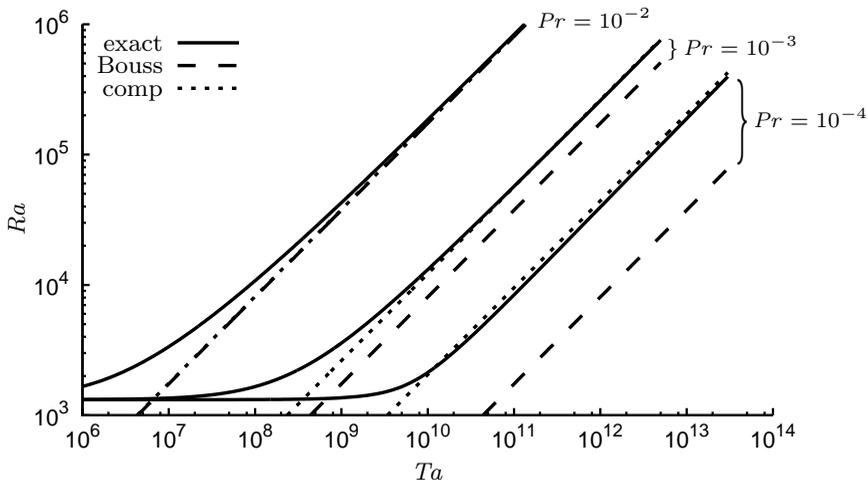
FIGURE 1. The critical Rayleigh number for onset of oscillatory convection as a function of Taylor number in a compressible fluid (solid curves) for three different Prandtl numbers. The dashed and dotted lines show the asymptotic scalings given by equations (2.25) and (2.35) respectively.

$Ta \gg 1/Pr^2$, the stability threshold closely matches that predicted by equation (2.35) in each case. As the Prandtl number is decreased, the Boussinesq prediction becomes less accurate, and always underestimates the true stability threshold.

3. Other sound-proof models

We have shown that the Boussinesq equations generally fail to predict the true onset of oscillatory convection, unless the height of the domain is *much* smaller than a typical thermodynamic scale height, by a factor of less than $Pr \ll 1$. However, there are alternatives to the Boussinesq equations that purport to be valid even on scales larger than a typical scale height. We might then wonder whether these other "sound-proof" approximations, such as the anelastic and pseudo-incompressible approximations, more accurately predict the onset of oscillatory convection. In fact, some implementations of the anelastic approximation certainly do not perform better: the anelastic model of Drew *et al.* (1995) produces spurious convective instability in some cases with a negative Rayleigh number. Calkins *et al.* (2015), using a similar anelastic model, showed that the discrepancy between the anelastic and fully compressible equations becomes increasingly serious at smaller Prandtl numbers, which is reminiscent of the results shown in figure 1. Before proceeding, it is worth reviewing some important points about the different sets of sound-proof equations, and the origins of their multiplicity.

As originally shown by Eckart & Ferris (1956), the linearized, fully compressible equations (2.1)–(2.5) have an energy principle

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (p_1 \boldsymbol{u}) = \rho_0 \boldsymbol{u} \cdot \boldsymbol{f}_1 + \left(T_1 - \frac{\mathrm{d}T_0}{\mathrm{d}s_0} s_1 \right) \frac{\rho_0 q_1}{T_0}, \tag{3.1}$$

where E is the "available" or "external" energy

$$E = \frac{1}{2} \rho_0 \left(u^2 + \frac{N^2 s_1^2}{(\mathrm{d}s_0/\mathrm{d}z)^2} + \frac{p_1^2}{\rho_0^2 c^2} \right). \tag{3.2}$$

The three contributions to E are generally referred to as "kinetic", "thermobaric", and "elastic", respectively.

It can be shown that the Boussinesq equations also satisfy equation (3.1), provided that the available energy is redefined as

$$E = \frac{1}{2}\rho_0 \left(u^2 + \frac{N^2 s_1^2}{(\mathrm{d}s_0/\mathrm{d}z)^2} \right), \tag{3.3}$$

indicating that these equations do not support elastic motions (i.e. sound waves). However, the Boussinesq equations assume that the background density ρ_0 is constant. Ogura & Phillips (1962) sought to extend the Boussinesq equations to more general background states, and they recognised the importance of preserving equation (3.1), with E defined as in equation (3.3). They referred to this as the "anelastic" approximation. Subsequently, however, the meaning of this term has become distorted, and most sets of equations in use today that are referred to as anelastic do not actually satisfy equation (3.1) for *any* definition of E (e.g. see review by Brown *et al.* 2012). Instead, the term "anelastic" is now used exclusively to describe sets of equations that include the velocity constraint

$$\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{u}) = 0 \tag{3.4}$$

(e.g. Braginsky & Roberts 2007). Rather than appealing to energy conservation, most derivations of the anelastic equations have relied entirely on formal asymptotics, writing each term in the equations as a truncated expansion in powers of one or more hypothetically small parameters. This approach has some potential pitfalls. First, the resulting set of equations may not be unique: for example, by expressing the original, exact equations in slightly different forms, different sets of anelastic equations can be obtained as first-order truncations under the same asymptotic conditions (e.g. Berkoff *et al.* 2010). Second, as demonstrated by the unphysical "negative Rayleigh number" convection seen in some implementations of the anelastic equations, there is no guarantee that equations obtained from a truncated asymptotic expansion will exhibit sensible behaviour. In some problems, important information is contained in the higher-order terms, though this may not be obvious at first sight. For instance, we have shown in section 2.3 that the phase shift between pressure and density is crucial in oscillatory convection, and that this information is lost if we consider only the leading-order terms in the equations.

We therefore suggest an alternative approach in deriving sound-proof equations, which is to take the energy principle idea of Ogura & Phillips (1962) and follow it to its logical conclusion. Specifically, we ask "what is the most general set of linear equations that is consistent with equations (3.1) and (3.3)?" In order for our equations to be physically meaningful, we will require that they also obey conservation of mass and entropy, i.e. our set of equations must include (2.2) and (2.3), and we require that ρ_1 and T_1 are linearly related to s_1 and p_1, with coefficients that depend only on the background state locally. (Note, however, that the background state must still satisfy the exact relations (2.6) and (2.7).) Finally, we require that the momentum equation includes a pressure gradient force of the form $-(1/\rho_0)\boldsymbol{\nabla}p_1$, and a buoyancy force that acts only in the vertical direction. The form of this buoyancy force we leave unspecified, though it must be a linear function of the thermodynamic perturbations. So we consider the equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = (A_p p_1 + A_\rho \rho_1 + A_s s_1 + A_T T_1)\boldsymbol{e}_z - \frac{1}{\rho_0}\boldsymbol{\nabla}p_1 + \boldsymbol{f}_1 \tag{3.5}$$

$$\rho_1 = B_s s_1 + B_p p_1 \tag{3.6}$$

$$T_1 = C_s s_1 + C_p p_1, \tag{3.7}$$

and we ask what conditions on the coefficients A_X, B_X, C_X are imposed by equations (2.2), (2.3), (3.1) and (3.3). Perhaps surprisingly, after some straightforward (but lengthy)

algebra, it can be shown that these restrictions remove almost all freedom in the choice of coefficients! The most general set of equations permitted is

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -s_1 \boldsymbol{\nabla} T_0 + T_1 \boldsymbol{\nabla} s_0 - \boldsymbol{\nabla}(p_1/\rho_0) + \boldsymbol{f}_1 \tag{3.8}$$

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{u}) = 0 \tag{3.9}$$

$$\frac{\partial s_1}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} s_0 = \frac{q_1}{T_0} \tag{3.10}$$

$$-\frac{\rho_1}{\rho_0^2} = \alpha H_{ps} s_1 \tag{3.11}$$

$$T_1 = H_{ss} s_1 + \alpha H_{ps} p_1, \tag{3.12}$$

where the only remaining free parameter, which we have chosen to call α, is an arbitrary function of the background state. Remarkably, the momentum equation (3.8) necessarily takes exactly the form of equation (2.1), so the only differences between the fully compressible equations and these sound-proof equations occur in the linearised equations of state, (3.11) and (3.12). To the best of our knowledge, equations (3.8)–(3.12) have never before been written down in exactly this form. They are similar to those of Cotter & Holm (2014), but more general because they include a general equation of state and non-adiabatic effects, and they also provide a unique and consistent prescription for the temperature perturbation T_1. We can determine the onset of oscillatory convection in these equations by following the same procedure that led to equation (2.30). Under the "Boussinesq" conditions (i)-(iv), and approximating the diffusive terms using equation (2.26), we find that marginal stability is achieved when

$$\tfrac{1}{2}\mathcal{N} \simeq \nu\kappa k_{\mathrm{h}}^4 + \frac{4\Omega^2 k_z^2}{k_{\mathrm{h}}^2}\left[\frac{\nu}{\kappa} + \alpha^2\left(\frac{c_p}{c_v} - 1\right)\left(\frac{2\Omega^2}{c^2 k_{\mathrm{h}}^2}\right)\right]. \tag{3.13}$$

For any non-zero value of α this implies that the onset of oscillatory convection in this system (at least under the standard Boussinesq conditions) occurs at a larger value of $\mathcal{N}$, and therefore at a higher Rayleigh number, than in the Boussinesq system.

4. Discussion

Although we have referred to equations (3.8)–(3.12) as "sound-proof", we have not explicitly shown that acoustic oscillations are absent from these equations. That this is the case can be demonstrated, for example, by deriving their exact dispersion relation for perturbations about some idealised background state. However, it can also be deduced simply from equation (3.11), which describes how the volume of a displaced parcel of fluid instantaneously adjusts to a value determined by the entropy inside the parcel and the local properties of the background state. By neglecting the effect of pressure perturbations on this adjustment, we remove the dynamical degree of freedom that permits the parcel to oscillate acoustically. A mathematical proof of this statement, which implicitly assumes the conservation of mass and entropy, has been given by Durran (2008).

The direct relation between ρ_1 and s_1 imposes a constraint on the velocity field, which can be deduced from equations (3.9)–(3.11):

$$\frac{1}{\alpha H_{ps} \rho_0^2} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{\nabla} s_0 = \frac{q_1}{T_0}. \tag{4.1}$$

Although α can be taken as any function of height, there are two significant special cases,

which correspond to the particular choices $\alpha = 0$ and $\alpha = 1$. When $\alpha = 0$, equation (4.1) reduces to the anelastic velocity constraint (3.4) and, in fact, our equations become almost identical to the particular version of the anelastic equations derived by Lantz (1992) and by Braginsky & Roberts (1995), which for brevity we will call the LBR equations. However, an important difference is that in our sound-proof system, with $\alpha = 0$, equation (3.11) states that $\rho_1 = 0$, whereas in the LBR equations ρ_1 is given as a function of the other thermodynamic perturbations. Formally this implies that mass is not conserved in the LBR equations, whereas mass conservation is built in to our linearised system. Nevertheless, because ρ_1 does not appear explicitly in equations (3.8) and (3.12), our equations are in fact mathematically equivalent to the LBR equations, and the only difference is conceptual. Lantz (1992) argued that these equations are the most natural generalisation of the Boussinesq equations to a density-stratified background and, indeed, we see that equation (3.13) becomes identical to its Boussinesq counterpart (2.19) in the case where $\alpha = 0$ (this means that the curves in figure 1 labelled as "Boussinesq" also correspond to the $\alpha = 0$ case of our sound-proof equations).

A significant practical advantage of the LBR equations over other sound-proof approximations is that they can be solved without ever having to explicitly calculate the pressure perturbation p_1, provided that perturbations to the heat flux are calculated from the gradient of s_1 rather than T_1 (Lantz 1992; Braginsky & Roberts 1995). In previous derivations of the LBR equations, this approximation has been justified by appealing to "turbulent diffusion" of entropy. However, setting $\alpha = 0$ implies a direct relation between T_1 and s_1 in equation (3.12). So in our version of the anelastic system there is no need to approximate the heat flux; instead we have an approximate equation of state (3.12) whose form is dictated by the energy principle (3.1).

Whilst the choice $\alpha = 0$ reduces equations (3.8)–(3.12) to the LBR anelastic equations, equation (3.13) suggests that the "best" sound-proof model is actually given by $\alpha = 1$, because in that case we exactly recover the corresponding compressible result (equation 2.30) from equation (3.13). (The choice $\alpha = -1$ can be discounted on physical grounds, because it would imply a positive correlation between ρ_1 and s_1.) If we set $\alpha = 1$ then equations (3.8)–(3.12) are equivalent to the linearisation of the "thermodynamically consistent" version of the pseudo-incompressible equations obtained by Klein & Pauluis (2012) (see also the "generalized pseudo-incompressible" equations of Vasil *et al.* (2013)). In fact, the results of Klein & Pauluis (2012) suggest how the method that we have proposed for standardising sound-proof models can be extended into the nonlinear regime. This is an essential step towards modelling systems that are well above the onset of convection, such as the interiors of stars, but is beyond the scope of the current paper.

A very similar set of pseudo-incompressible equations was recently studied numerically by Lecoanet *et al.* (2014). An important difference, however, is in the definition of the temperature perturbation T_1. We believe that many of the discrepancies Lecoanet *et al.* (2014) found between their solutions of the pseudo-incompressible equations and the fully compressible equations result from the incorrect definition of T_1 that they used, but further work will be required to confirm this. Having said that, not all of the issues with the pseudo-incompressible equations identified by Lecoanet *et al.* (2014) can be solved simply by redefining T_1. In particular, the velocity constraint (4.1) becomes ill-posed for horizontally-invariant perturbations in a fluid with impenetrable, thermally conducting boundaries. This difficulty does not arise in the stability analysis presented here, because convective motions necessarily have a finite horizontal lengthscale. In general, the pseudo-incompressible equations seem best suited to describing fluid motions on small horizontal scales, and are less accurate for motions on large horizontal scales (see e.g. Durran 2008).

One possible resolution is to allow the background state to be time dependent (e.g. O'Neill & Klein 2014), but this is beyond the scope of the analysis presented here.

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