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VIV fatigue reliability analysis of marine risers with uncertainties in the wake oscillator model

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Abstract
Uncertainties are rife in the fatigue life prediction of marine risers subjected to vortex-induced vibration (VIV). Industry deals with this issue by imposing large factors of safety that may not be properly justified, resulting in over-conservative riser designs in general. One important source of uncertainty arises from the VIV prediction models. This paper focusses on identifying the uncertainties of a wake oscillator model which approximates the fluctuating hydrodynamic force coupled with the riser equation of motion for nonlinear fluid-structure interaction analysis. This van der Pol-type oscillator relies on two wake coefficients which are described deterministically by empirical equations obtained via curve-fitting. However, the underlying data exhibit wide scatter; thus, it is proposed to model the two key coefficients as random variables. Based on experimental data, the joint probability density function of the variables is approximated. A new fast reliability approach is proposed for the VIV fatigue reliability analysis, while Monte Carlo simulations are performed for comparisons. Case studies of a vertical riser in a uniform flow show that the proposed method compares favorably with Monte Carlo in terms of predicting the failure probability as well as safety factors conforming to prescribed reliability levels. Moreover, this study reveals that the randomness of wake coefficients leads to large variability in the riser fatigue damage. The correlation between the coefficients should be properly incorporated as it affects the fatigue reliability of risers experiencing VIV.

Keywords
VIV, marine riser, fatigue damage, uncertainty, reliability analysis, wake oscillator
1. **Introduction**

Recently, there has been heightened attention paid to the modeling and response prediction of marine risers undergoing vortex-induced vibration (VIV). This is because as oil and gas developments move farther into deeper waters, risers become more vulnerable to VIV owing to the increased riser lengths, lower natural frequencies, and stronger ocean currents. The sustained oscillations from VIV can amplify the fluid oscillatory and mean loads, which in turn, severely limit drilling, production and exploration activities, resulting in excessive operational costs.

Modeling and prediction of VIV is a notoriously complicated task due to the multitude of system parameters involved in the fluid-structure interaction mechanism. Some hydrodynamic properties associated with the vortex shedding can change their characteristics with time, depending on response amplitudes and oscillation frequencies. For VIV of rigid cylinders with one or two degrees of freedom in uniform flows, the basic controlling parameters include the mass ratio, damping ratio, Reynolds number, aspect ratio, cylinder inclination and roughness [1–3], amongst others. For VIV of long flexible cylinders, additional meaningful parameters related to the elasto-geometric properties, natural frequencies and spatial modal shapes play a role in the multi-degree-of-freedom systems [4–6].

From a design perspective, one of the foremost concerns for VIV is fatigue assessment. It is widely acknowledged that VIV fatigue prediction is fraught with high levels of uncertainties. For this reason, very large safety factors are typically imposed to the VIV fatigue damage during design, for example a factor of 20 is routinely used in industry for critical components that cannot be inspected [7]. Such safety factors are selected from past experience and opinion of experts, and as such have no rational basis. The underlying level of reliability is also implicit, and may not conform to target annual failure probabilities such as $10^{-5}$. Although such practices are relatively simple and seem to have served the industry well in the past, in the face of new challenges involving novel riser designs and configurations, generic safety factors may no longer be justified. A more rational approach would be to invoke exploit modern reliability analysis techniques to account for individual sources of uncertainty.
There are at least three dominant sources of uncertainties inherent in VIV fatigue life assessment: (1) the environment, specifically the current profile, direction and velocity; (2) the fatigue analysis methodology; (3) the VIV model [8]. Knowledge of the environment can be elevated with more on-site data, but there is always a natural variability that is unpredictable. Fatigue analysis has its traditional sets of uncertainties associated with the wide scatter of the $S–N$ curve and impreciseness of the damage accumulation rule, and there is plenty of research on this subject. This paper focusses on the randomness within the VIV model itself. Irrespective of the type of VIV model, uncertainty is inevitable because the complicated fluid-structure phenomena depend on many factors such that each case is essentially unique, and calibrated empirical parameters will not always be relevant to the problem at hand.

Of late, there have been multiple studies concerning probabilistic VIV fatigue assessment, an indication of its growing importance. Khan and Ahmad [9] conducted a VIV fatigue reliability analysis of a deep water riser; nevertheless, the authors only considered random variables related to the fracture mechanics model. Leira et al. [10] also performed reliability analyses, which incorporated various random parameters such as added mass, damping parameters, Strouhal number and lift coefficient. The impreciseness of the VIV model is captured by a single random variable termed the “model uncertainty factor”, which is normally distributed with assumed mean and variance. Fontaine et al. [11] compared three commercial VIV design tools (SHEAR7, VIVA and VIVANA) against data from high quality straked riser experiments. The authors found that the safety factors for the software ranged from 1 to 15, well below the industry standard. Tognarelli et al. [8] analyzed the field measurements of several full-scale drilling risers to ascertain the safety factors pertaining to design software (SHEAR7 and VIVANA). Although the study yields valuable insight on the level of conservatism involved, the authors cautioned that the safety factors should not be generalized to other riser designs.

This paper endeavors to characterize the uncertainties inherent in the wake oscillator model, which has been highly attractive for VIV prediction in the time domain due to computational robustness. The van der Pol-type oscillator, coupled with the structural equation of motion, captures many VIV traits, such as the fluctuating hydrodynamic force, as well as the self-excited and self-limiting response observed in
many experiments. A key capability of wake oscillator models is the estimation of maximum response amplitudes, depending on empirical coefficients which have been tuned a priori. Among the well-known wake-body models [12], those of Skop and Balasubramanian [13] and Facchinetti et al. [14] are often used, requiring only two empirical coefficients, thus simplifying the calibration with different experimental tests. Facchinetti et al. [14] proposed constant empirical coefficients; unfortunately, these values cannot be applied to VIV predictions in different ranges of system parameters. In contrast, Skop and Balasubramanian [13] presented analytical expressions for empirical coefficients that capture variable physical and experimental data. These are practically useful since the prediction model could be applied to different cylinders in a variety of flow conditions.

Gabbai and Hiebert [15] performed sensitivity analyses on the Facchinetti model in order to investigate the relative importance of five different parameters. The authors assigned uniform distributions for all parameters and made assumptions on the variance, arguing that the exact distributions are unknown. In this paper, the model of Skop and Balasubramanian [13] is evaluated by allowing for randomness in the two empirical coefficients, which exhibit wide scatter in the calibrated testing data. To the authors’ knowledge, this is an original endeavor. The objective is to better understand the effect of uncertainty of the wake oscillator coefficients on VIV fatigue reliability. To this end, the probability distributions of the coefficients will be characterized based on the actual data, rather than being assumed. Reliability analysis will be performed, and the probability of failure is estimated by Monte Carlo simulations. As Monte Carlo is too costly for practical design, a new fast reliability methodology suitable for routine usage will be developed.

2. Modeling of riser VIV

2.1 Preliminaries

In what follows, a vertical top-tensioned riser (TTR) undergoing cross-flow VIV due to a uniform current is considered. The TTR has a fully-submerged length $L_R$, external diameter $D$ and typical pinned-pinned supports. The incoming flow $V$ is arbitrary such that the riser oscillates in the transverse direction.
The TTR is treated as a flexural tensioned beam that accounts for both bending and axial rigidities. With zero displacements and curvatures at end supports and by non-dimensionalizing all displacement-related variables with respect to $D$, the partial-differential equation governing the cross-flow motion $v(x, t)$ of the TTR reads [6]

$$
\ddot{v} + \frac{c}{M + M_a} \dot{v} + \delta v'' - \alpha (T_r(x) v')' = \frac{H_y(x, t)}{(M + M_a) D},
$$

(1)

where primes denote differentiation with respect to the dimensionless axial coordinate $x$ and overdots denote time derivatives. The constant mechanical parameters are the viscous damping coefficient $c$, 

$$\delta = EI(M+M_a)D^4, \quad \alpha = T_R(M+M_a)D^2,$$

in which $M$ is the riser mass including contents, $M_a$ the potential fluid added mass ($M_a = C_A \rho A_f$), $\rho$ the fluid density, $A$ the cross-sectional area of the displaced volume, $C_A$ the added mass coefficient ($C_A \approx 1$ for a circular cylinder [1]), $T_R$ the top tension and $EI$ the bending stiffness. The riser tension is spatially non-uniform due to gravity. The variable-tension function, normalized by $T_R$, is $T_r(x) = 1 - x D W_e / T_R$, with $W_e$ being the riser submerged weight.

2.2  \textit{Wake oscillator model}

The unsteady hydrodynamic lift force $H_y$ leading to cross-flow VIV may be expressed as

$$H_y(x, t) = \frac{1}{2} \rho D V^2 C_L(x, t) = \frac{1}{2} \rho D V^2 \left( Q(x, t) - 2 \gamma \dot{v}(x, t) / \omega_s \right),
$$

(2)

where $C_L(x, t)$ is the space-time varying lift coefficient associated with vortex shedding, $\gamma$ the fluid-added damping coefficient, $\omega_s$ the vortex-shedding frequency (rad/s) with $\omega_s = 2 \pi St V/D$ and $St$ the Strouhal number. In Eq. (2), the last expression was introduced by Skop and Balasubramanian [13] to account for the fluctuation of lift coefficient through the fluid wake variable $Q$ and for the stall term which captures the limited response at zero structural damping. The variable $Q$ may be governed by a van der Pol wake oscillator whose equation reads [13]

$$\ddot{Q} - \omega_s^2 G \left( C_{L0}^2 - 4 Q^2 \right) \dot{Q} + \omega_s^2 Q = \omega_s F \dot{v},
$$

(3)
where \( C_{L0} \) is a lift coefficient of the stationary circular cylinder. \( F \) and \( G \) are empirical wake coefficients which may be derived as functions of system parameters defining both the flow and cylinder properties in the experiments. Following Skop and Balasubramanian [13] who collected some experimental cross-flow VIV data and derived the steady-state solutions of coupled oscillators, variable \( F \) and \( G \) depend on two parameters: the cylinder maximum amplitude per diameter \( A_{\text{max}} \) and the frequency ratio \( \omega_{s,A}/\omega_n \) with \( \omega_{s,A} \) being the vortex frequency at \( A_{\text{max}} \). To summarize, expressions for \( F \) and \( G \) are rewritten as [13]

\[
F = \frac{\hat{\mu}(S_G + \gamma)^2}{2} \left( \delta_A^2 + 4 \right) \left( \Psi_A - \delta_A \right) \tag{4}
\]

\[
G = \frac{F}{2C_{L0}^2(S_G + \gamma) \delta_A \left( \delta_A^2 + 4 \right)}, \tag{5}
\]

\[
\delta_A = \left\{ -\left( 8X_A - 1 \right) + \sqrt{\left( 8X_A - 1 \right)^2 + 48X_A \left( 4X_A - 1 \right)} \right\} \frac{1}{6X_A}, \tag{6}
\]

\[
X_A = \left\{ \frac{(S_G + \gamma) A_{\text{max}}}{C_{L0}} \right\}^2, \quad \Psi_A = \frac{2}{\hat{\mu}(S_G + \gamma)} \left[ \omega_{s,A}/\omega_n - 1 \right]. \tag{7}
\]

In above equations, \( S_G \) is the Skop-Griffin parameter [13] with \( S_G = \xi / \hat{\mu} \) depending on the mass ratio \( \hat{\mu} \) where \( \hat{\mu} = \rho D^2 / 8\pi^2 St^2 \left( M + M_a \right) \) and the damping ratio \( \xi \) (structural damping/critical damping). Accordingly, both \( F \) and \( G \) depend on the given mass-damping parameter \( (m^* \xi) \) which is well known to characterize both the maximum amplitude and the lock-in range of the cylinder [1–2].

### 2.3 Riser dynamics

A modal analysis is now applied. By rearranging Eqs. (1) and (3) in their first-order differential forms and assuming that the wake and the cylinder concurrently oscillate according to a lock-in condition with \( \omega_n \approx \omega_s \), both \( v \) and \( Q \) are postulated in terms of a full eigenbasis by letting

\[
\dot{v} = A \rightarrow v(x,t) = \sum_{n=1}^{\infty} \phi_n(x)f_n(t), \quad A(x,t) = \sum_{n=1}^{\infty} \phi_n(x)p_n(t), \tag{8}
\]

\[
\dot{Q} = B \rightarrow Q(x,t) = \sum_{n=1}^{\infty} \phi_n(x)d_n(t), \quad B(x,t) = \sum_{n=1}^{\infty} \phi_n(x)e_n(t). \tag{9}
\]
where \( \varphi_n \) are the transverse modal shape functions and \( \omega_n \) the associated natural frequencies in still water. Accurate information on \( \omega_n \) and \( \varphi_n \) is essential: these can be obtained analytically in closed-form for constant riser tension. However, for variable tension and pinned-pinned supports, \( \omega_n \) and \( \varphi_n \) are semi-analytically and numerically determined based on a Fourier sine-based series by postulating [4]

\[
\varphi_n(x) = \sum_{n=1}^{N_s} \gamma_n \sin\left(\frac{n\pi x D}{L_R}\right).
\]  

(10)

The eigenfunction coefficients (\( \gamma_n \)) are determined via a Galerkin approach, depending on the number of sine functions (\( N_s \)) retained to yield a convergence solution of frequencies and mode shapes. In Eq. (8), \( f_n \) and \( p_n \) are generalized displacement and velocity coordinates of the riser whereas \( d_n \) and \( e_n \) in Eq. (9) are generalized displacement and velocity coordinates of the wake, respectively. By substituting Eqs. (8)–(9) into Eqs. (1)–(3), performing the standard Galerkin approach with zero displacements and curvatures at end boundaries, and applying the orthonormalization of modes [4], a reduced-order multi-mode model governing the nonlinear riser-wake interaction reads

\[
\dot{f}_n = p_n, \quad \dot{p}_n = -2\frac{\xi_n}{s} \omega_n p_n - \omega_n^2 f_n + \frac{\xi_n}{s_G} \omega_n^2 (d_n - 2\gamma p_n / \omega_n),
\]  

(11)

\[
\dot{d}_n = e_n, \quad \dot{e}_n = \omega_G \sum_{i,j,k}^{\infty} \sum_{i,j,k}^{\infty} \left[ \int_0^{L_p/D} \phi_{ij}, \phi_{jk} \phi_{ik} dx \right] d_i e_k.
\]  

(12)

The multi-mode interaction effect, which cannot be neglected [5], is accounted for in Eq. (12) through the wake dynamics governed by cubic nonlinearities.

3. Modeling of random variables

3.1 Selection of random variables

It is necessary to select the random variables to be included in the fatigue reliability analysis. Attention is placed on the uncertainties of the parameters governing the wake oscillator model. The empirical wake coefficients \( F \) and \( G \) are undoubtedly subject to experimental variability. Both \( F \) and \( G \) are functionally related to \( A_{\text{max}} \) and \( \omega_s, \omega_n \) (Eq. (7)). Since \( A_{\text{max}} \) and \( \omega_s, \omega_n \) are the parameters that were
actually measured, it is expedient to select them (instead of \( F \) and \( G \)) as random variables as their uncertainties can be more easily characterized. Other random variables will be discussed in Section 3.5.

3.2 Data analysis for \( A_{\text{max}} \) and \( \omega_{s,A}/\omega_n \)

This section focusses on data analysis for \( A_{\text{max}} \), \( \omega_{s,A}/\omega_n \), based on the experimental datasets collated in [13] from 13 other references published between 1964–1982. These references reported experiments conducted under different setups in air or water with uniform flows in a sub-critical flow range. Different scaled models were considered including a spring-mounted, pivoted and cantilevered rigid cylinder, as well as a taut cable or flexible cylinder, with different mass-damping ratios (0.02 < \( S_G < 3.70 \)). For the taut cable, the reported excited string-based modes were between the second and fourth. Note that there are in total 64 datasets in 13 references, each reporting \( S_G \) and \( A_{\text{max}} \). However, \( \omega_{s,A}/\omega_n \) values were given in only 43 out of the 64 sets.

From the data points, a plot of \( A_{\text{max}} \) against \( S_G \) is reproduced in Fig. 1, while Fig. 2 depicts the trend for \( \omega_{s,A}/\omega_n \) versus \( S_G \). Skop and Balasubramanian [13] fitted data to empirical formulae based on the least squares method, and arrived at the formulae:

\[
A_{\text{max}} = \frac{0.385}{\sqrt{0.12 + S_G^2}}, \quad \omega_{s,A} = \frac{1.216 + 0.084}{1 + 2.66S_G^2} \cdot \omega_n \cdot \omega_n.
\] (13)

According to Eq. (13), once \( S_G \) has been ascertained, \( A_{\text{max}} \) and \( \omega_{s,A}/\omega_n \) are assumed to be deterministic. Nonetheless, it is apparent from Figs. 1 and 2 that there is considerable scatter, and Eq. (13) merely represents the average trends (in Ref. [13] Fig. 1 is plotted on a log-log graph; hence the scatter is less striking).

To account for the scatter in the experimental observations, Eq. (13) is recast as

\[
A_{\text{max}} = A_{\text{max}}^* + \Delta A^*, \quad \omega_{s,A} = \omega^* + \Delta \omega^* \cdot \omega_n
\] (14)

where \( A_{\text{max}}^* \) and \( \omega^* \) are the design values, while \( \Delta A^* \) and \( \Delta \omega^* \) are the residuals. Note that \( A_{\text{max}}^* \) and \( \omega^* \) are not the mean values, because the means of \( \Delta A^* \) and \( \Delta \omega^* \) are non-zero, i.e. there is mean-bias, as will be
elaborated upon later. In principle, the statistical properties of the residuals may depend on $S_G$, but this dependency is difficult to incorporate on account of the limited sample size. Hence, it is assumed that $\Delta A^*$ and $\Delta \omega^*$ do not depend on $S_G$. This assumption is consistent with the least-squares approach used to derive Eq. (13). Fig. 3(a)-(b) show the histograms of $\Delta A^*$ and $\Delta \omega^*$, normalized such that they conform to probability densities. In Section 3.3, continuous probability density functions will be assigned to each of $\Delta A^*$ and $\Delta \omega^*$, while their correlation will be examined in Section 3.4.

3.3 Fitting data to probability density functions

Fitting a probability density function (pdf) to data is a common problem in statistics. It is appropriate to first introduce some terminologies. For a random variable $X$ with pdf $f_X(\cdot)$, the $j$th statistical moment about a value $\eta$ is defined as

$$m_j^{(\eta)} = E[(X - \eta)^j] = \int_{-\infty}^{\infty} (x - \eta)^j f_X(x)dx$$

(15)

where $E[\cdot]$ denotes the expectation. The first two moments are the mean and variance, given respectively by $\mu_X = E[X]$ and $\sigma_X^2 = E[(X - \mu_X)^2]$. If data are fitted to a normal distribution, the mean and variance are necessary and sufficient information.

However, the data appear to be asymmetrical, implying that the normal distribution may not be optimal. In such a situation, the higher moments provide useful additional information. In particular, the third and fourth standardized moments are the skewness $\gamma_X$ and kurtosis $\kappa_X$, i.e.

$$\gamma_X = E[(X - \mu_X)^3]/\sigma_X^3, \quad \kappa_X = E[(X - \mu_X)^4]/\sigma_X^4$$

(16)

The skewness quantifies the asymmetry of the distribution, while the kurtosis measures the “peakedness”. For a normal distribution, $\gamma_X = 0$ and $\kappa_X = 3$.

The first four moments for $\Delta A^*$ and $\Delta \omega^*$ are calculated, and listed in Table 1. The means of the residuals are non-zero, which signify that Eq. (13) has mean-bias; fortunately the bias is relatively minor. It is perceived that the skewness and kurtosis deviate from those of a normal distribution, suggesting that
a non-normal distribution may be more appropriate for the two random variables. Owing to the relatively small sample size, the choice of distribution is far from obvious. This is especially problematic for $\Delta \omega'$, as the data is ostensibly tri-modal, congregating at the mean, and the two extremes. In the absence of more samples, it is hard to conclude whether sampling variability is the underlying cause.

When the actual distribution type is unclear, a common technique in statistics is to fit data to a flexible distribution type. Such an approach dates back to over a century, when Pearson [16] devised a system of distributions that satisfies a set of given four moments. Although systems of distributions are generally very flexible, they are difficult to implement and have other known drawbacks [17]. This study adopts a recently proposed distribution, known as the Shifted Generalized Lognormal Distribution (SGLD) [17]. The pdf for SGLD reads

$$f_x(x) = \frac{1}{2^{1/2r} \sigma \Gamma(1 + 1/r)(x - b)} \exp \left( - \frac{1}{r \sigma} \left| \ln \left( \frac{x - b}{\theta} \right) \right|^r \right), \quad b < x < \infty$$

where $\Gamma(\cdot)$ denotes the gamma function. The distribution has four parameters; while $b$ and $\theta$ are for location and scale, respectively, while $\sigma$ and $r$ are shape parameters (here, $\sigma$ is not the standard deviation). The cumulative distribution function (cdf) is expressed as

$$F_x(x) = \frac{1}{2} + \frac{1}{2} \sgn \left( \frac{x - b}{\theta} - 1 \right) g \left( \frac{1}{r} \left| \frac{\ln \left( (x - b) / \theta \right)}{\sigma} \right| \right), \quad b < x < \infty$$

where $g(v, x) = \left( \int_0^x t^{v-1} e^{-t} dt \right) / \Gamma(v)$ is the incomplete gamma function ratio. The inverse cdf $F^{-1}_x(\cdot)$ also possesses an analytical form, which can be derived from Eq. (18) [17].

The SGLD has several advantages. It is highly flexible and able to approximate many theoretical distributions. Moreover, it includes several important distributions as special or limiting cases, such as the normal ($\sigma \to 0, r = 2$), lognormal ($b = 0, r = 2$), Laplace, exponential and uniform distributions. The parameters of SGLD can be readily computed for a given set of four moments [17]. Eq. (17) is only valid for positive skewness, characterized by a pdf with a long tail on the right-hand side. For negative skewness, one should consider $|x|$, and subsequently mirror the pdf about $x = \mu_x$. This operation is
equivalent to replacing $x$ by $2\mu_X - x$ on the right-hand-side of Eq. (17). The same procedure applies to Eq. (18), except that the resulting expression should be subtracted from unity. In addition, as $\sigma \to 0$ as the skewness vanishes, and Eq. (17) approaches the symmetrical exponential power distribution, which has a symmetrical pdf given by [18]

$$f_X(x) = \frac{1}{2r^{1/r} \sigma (1 + 1/r)} \exp \left( -\frac{1}{r \sigma^r} |x-b|^r \right), \quad -\infty < x < \infty$$  \hspace{1cm} (19)

In practice, Eq. (19) should be used when $|\gamma_X|$ is zero or very small, say $|\gamma_X| < 0.05$, to avoid numerical difficulties when computing the SGLD parameters.

The fitted pdfs for $\Delta A^*$ and $\Delta \omega^*$ using SGLD are shown in Fig. 3(a)-(b). The normal distribution, which satisfies only the first two moments, is included for comparison. In both cases, the fit by SGLD appears to be better than the normal distribution. In addition to a visual comparison, a chi-squared goodness-of-fit test is also performed as reported in Appendix A.

3.4 Correlation between wake oscillator variables

Since $A_{\max}$ and $\omega_{\omega_A}/\omega_n$ have been extracted from the same set of experiments, and both parameters pertain to the same wake oscillator model, it is reasonable to expect that $A_{\max}$ and $\omega_{\omega_A}/\omega_n$, and therefore, $\Delta A^*$ and $\Delta \omega^*$ may be correlated. Fig. 4 shows a scatter plot of $\Delta \omega^*$ against $\Delta A^*$ based on the 43 sets of experiments in which both $A_{\max}$ and $\omega_{\omega_A}/\omega_n$ are available. The correlation coefficient $\rho_{\Delta A^*\Delta \omega^*}$ is calculated, and found to be 0.361, suggesting a fair degree of correlation. It is thus worthwhile to study the impact of this correlation on the fatigue reliability.

The Nataf transformation [19] is a common technique for constructing the joint probability density function (jpdf) of correlated non-Gaussian random variables, entailing only the marginal pdfs and the correlation structure [19]. The two dependent variables $X_1$ and $X_2$ are each transformed to dependent Gaussian variables $Y_1$ and $Y_2$, according to

$$y_1 = \Phi^{-1}[F_{X_1}(x_1)], \quad y_2 = \Phi^{-1}[F_{X_2}(x_2)]$$  \hspace{1cm} (20)
where \( \Phi() \) is the standard normal cumulative distribution function (cdf), while \( F_{x_1}() \) and \( F_{x_2}() \) are the marginal cdfs. The Nataf transformation approximates the jpdf of \( X_1 \) and \( X_2 \) by

\[
f_{x_1,x_2}(x_1,x_2) = f_{x_1}(x_1)f_{x_2}(x_2) \frac{\phi_2(y_1,y_2,\rho_{y_1y_2})}{\phi(y_1)\phi(y_2)}
\]

where \( \phi() \) and \( \phi_2() \) are, respectively, the univariate and bivariate standard normal pdf, while \( \rho_{y_1y_2} \) is the correlation coefficient for \( Y_1 \) and \( Y_2 \). The relationship between \( \rho_{y_1y_2} \) and \( \rho_{x_1x_2} \) is given as [19]

\[
\rho_{x_1x_2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{x_1 - \bar{X}_1}{\sigma_{x_1}} \right) \left( \frac{x_2 - \bar{X}_2}{\sigma_{x_2}} \right) \phi_2(y_1,y_2,\rho_{y_1y_2}) \, dy_1 \, dy_2
\]

where \( x_i = F_{x_i}^{-1}[\Phi(y_i)] \). Since \( \rho_{x_1x_2} \) is available, whereas the unknown \( \rho_{y_1y_2} \) is implicit in Eq. (22), an iterative procedure is required. Let \( X_1 = \Delta A^* \) and \( X_2 = \Delta \omega^* \). Noting that \( \rho_{x_1x_2} = 0.3607 \), solving Eq. (22) by iteration yields \( \rho_{y_1y_2} = 0.3692 \).

In reliability analysis, it is often convenient, or even imperative, to transform the original random variables \( X \) in physical space to uncorrelated standard normal variables \( U \) (standard normal space). In general, this task can be achieved through the Rosenblatt transformation [19]. Here, since \( Y_1 \) and \( Y_2 \) are correlated standard normal variables, it is easier to apply the Cholesky factorization technique to decompose the correlation matrix [19]. Without going into details, the transformation for the bivariate case may be written as

\[
Y_1 = \sqrt{1 - \rho_{y_1y_2}^2} U_1 + \rho_{y_1y_2} U_2 \quad Y_2 = U_2
\]

By combining Eqs. (20) and (23), the transformation between the physical and standard normal space is obtained as

\[
X_1 = F_{x_1}^{-1}[\Phi(\sqrt{1 - \rho_{y_1y_2}^2} U_1 + \rho_{y_1y_2} U_2)] \quad X_2 = F_{x_2}^{-1}[\Phi(U_2)]
\]
3.5 Other random variables

Let $\mathbf{X} = [X_1 \ X_2 \ldots X_7]^T$ represent the vector of random variables considered in the fatigue reliability analysis. In addition to $X_1 = \Delta A^*$ and $X_2 = \Delta \omega^*$, five other random variables (assumed to be independent) are included. Table 2 summarizes the variable description, distribution type, median and coefficient of variation (CoV) (standard deviation divided by the mean). The reason for adopting the median is for convenience; the median is invariant when the variable is transformed, e.g. to normal space.

The damping ratio $X_3 = \xi$ and lift coefficient $X_4 = C_{L0}$ are treated as random variables since they are seldom known precisely. Unfortunately, data for $\xi$ and $C_{L0}$ are not available to characterize their distributions, thus assumptions are made; $\xi$ and $C_{L0}$ are assigned the lognormal distribution, a common choice for positive quantities. For instance, Kim et al. [20] modeled $\xi$ for a steel structure using the lognormal distribution; with median $\xi_{0.5} = 0.01$, and $\text{CoV} \;\xi = \sigma_\xi / \mu_\xi = 0.3$; these are also the values adopted herein. The location $\mu$ and scale $\sigma$ parameters of the lognormal distribution are obtained as

$$\mu = \ln(\xi_{0.5}) \quad \text{and} \quad \sigma = \sqrt{\ln(1 + \epsilon_\xi^2)}.$$ 

DNV [21] suggests that environmental variables are lognormal with CoVs in the range 0.05 – 0.10. In addition, riser mass is also uncertain owing to fluctuations in internal fluid contents. DNV [21] recommends the normal distribution for riser mass, with CoV ranging from 0.05 – 0.10. Hence, the current velocity $X_5 = V$ and riser mass $X_6 = M$ are modeled by the lognormal and normal distributions respectively, both with CoVs of 0.08. It is worth clarifying that $\xi$ and $M$ will influence both $F$ and $G$ through $S_G$. Although the residuals $\Delta A^*$ and $\Delta \omega^*$ are independent of $\xi$ and $M$, the design values $A^\text{\max}_\text{\max}$ and $\omega^\text{\max}$ are affected by $\xi$ and $M$, being functions of $S_G$ (cf. Eq. (13)).

Fatigue strength of materials is subject to high variability, and cyclic load tests of nominally identical specimens can yield diverse results. The basis for practical fatigue design is the $S$–$N$ curve, which will be elaborated in Section 4.1. Scatter in the $S$–$N$ curve is commonly accounted for through the $S$–$N$ parameter $X_7 = \bar{a}$ (see Section 4.1). Wirsching and Chen [22] recommended that $\bar{a}$ should follow a
lognormal distribution, and suggested CoVs for $S$–$N$ curves of different classes; for class F2 that applies to steel risers, the suggested CoV is 0.56.

For lognormal variables, the transformation to normal space is given by

$$X_i = \exp(U \sigma_i + \mu_i), \quad i = 3, 4, 5, 7$$

(25)

where $\mu_i$ and $\sigma_i$ are parameters of the lognormal distribution.

4. Fatigue reliability analysis

4.1 Fundamentals of fatigue analysis

The $S$–$N$ curve specifies the number $N$ of cycles to failure under a cyclic load of constant stress range $S$. Typically, the $S$–$N$ curve is fairly linear on a log-log graph. Correspondingly, the relationship can be conveniently represented as

$$N(S) = \bar{a} S^{-m}$$

(26)

where $\bar{a}$ and $m$ are parameters determined from laboratory tests. These parameters are readily available for steel risers pertaining to various conditions [23]; typically $m$ ranges from 3 to 5 for steel.

The single-mode VIV response is typically periodic; however responses involving multiple modes may not be the case. In order to make use of the $S$–$N$ curve for stresses with irregular time histories, the standard practice is to employ the rainflow counting algorithm [24] to extract the stress cycles, and subsequently invoke the Miner-Palmgren rule [25], which states that the fatigue damage $D$ is accumulated according to

$$D = \sum_{i=1}^{\infty} \frac{1}{N(S_i)} = \frac{1}{\bar{a}} \sum_{i=1}^{\infty} S_i^m$$

(27)

where the index $i$ refers to the $i$th stress cycle. Failure is assumed to occur when $D$ reaches unity, in the absence of safety factors.

4.2 Reliability analysis and classical solution techniques

The classical reliability problem involves evaluating the following multi-dimensional integral
\[ P_f = \Pr \left[ G(\mathbf{X}) \leq 0 \right] = \int_{G(\mathbf{X}) < 0} \ldots \int f_\mathbf{X}(\mathbf{x}) d\mathbf{x} \]  \hspace{1cm} (28)

where \( P_f \) is the probability of failure, \( \mathbf{X} = [X_1, X_2, \ldots X_n]^T \) denotes the vector of dependent random variables with jpdf \( f_\mathbf{X}(\mathbf{x}) \), and \( G(\mathbf{X}) \) is a limit state function. If one wishes to follow classical convention such that \( G(\mathbf{X}) \leq 0 \) represents a limit state violation, the limit state function can be defined as \( G(\mathbf{X}) = D_{\text{fail}} - D(\mathbf{X}) \), where \( D_{\text{fail}} \) is the failure threshold. In practice, however, it is more intuitive to directly interpret \( D(\mathbf{X}) \) as the performance function, with \( D(\mathbf{X}) \geq D_{\text{fail}} \) construing failure.

Reliability methods can in general be grouped into three categories: (1) direct numerical integration, (2) Monte Carlo simulation, and (3) approximate techniques. For the case of direct numerical integration, the multi-dimensional integration is computed numerically via, for example, the trapezium rule. The drawback is that the computational effort increases exponentially with the problem dimension. Thus, Monte Carlo simulation (MCS) is usually preferred when the problem involves many random variables, since the sampling variability is not governed by the dimension. MCS relies on repeated random sampling, and it can be computationally costly, especially if low failure probabilities are being assessed. This has led to the development of efficient but approximate methods, the most popular of which is perhaps the First Order Reliability Method (FORM) [19]. In FORM, the random variables and the limit state function are transformed to standard normal space. The next step is to locate the design point \( \mathbf{u}^* \), the point on the failure surface closest to the origin. The failure probability is then approximated as \( P_f = \Phi(-\beta) \), where the reliability index \( \beta \) is the distance of \( \mathbf{u}^* \) from the origin.

However, FORM has several shortcomings, one of which is the difficulty of searching for the design point using algorithms that involve numerical differentiation [27]. This limitation is particularly detrimental if the limit state function is not smooth, as is often the case for responses from time domain simulation of dynamic systems. Another drawback seldom mentioned in the literature is that the design point for each FORM analysis is specific to a particular response quantity of interest. Hence, FORM is suitable for component reliability analysis, but is inappropriate for system reliability analysis. In the present context, a particular node in the riser system needs to be identified prior to the FORM analysis,
and the results are not applicable to other nodes. Lastly, the limit state function must be predefined; as such it is ineffective to use FORM to study failure probabilities under different thresholds.

The high computational cost of conventional MCS can be alleviated by a class of methods known as Enhanced Monte Carlo (EMC) [26], which relies on the extrapolation of failure probabilities from moderate levels to high thresholds (low probabilities). Unlike most other variance reduction methods such as importance sampling, EMC is applicable for system reliability analysis, and although EMC is not considered herein, it will be interesting to explore its use in a future work.

4.3 Point estimate method

This study adopts the point estimate method (PEM) [27], a technique that has gained traction in the past decade. It has the advantage of obviating iteration, and being “derivative-free”, i.e. there is no need to compute the derivatives of the performance function. An added benefit is that the evaluation points are pre-determined; accordingly the results can be subsequently post-processed for any node or response quantity of interest. The moments of the performance function (here it is $D(X)$) are first estimated. Thereafter the performance function’s cdf $f_{D_i}()$ is approximated from the moments, and $P_f$ can be obtained accordingly. It is usual to consider the first four moments [27], and here SGLD will be used to construct the distribution of $D$.

The random variables are first transformed to standard normal space, so that the performance function is now evaluated in terms of $U$, i.e. $D(U)$. Let $D_i(U_i)$ be a single-variable function representing the response when all entries of the vector $U$ are zero, except the $i$th entry, which is equal to $u_i$. The $R$th raw moments of $D_i(U_i)$ are point estimated by evaluating $D_i(U_i)$ at $K$ different points, i.e.

$$
\int [D_i(u_i) - \mu_i] f_{U_i}(u_i) du_i \approx \sum_{k=1}^{K} P_k [D_i(u_{ik}) - \mu_i]^k
$$

(29)

where $u_{ik}$ are the estimating points and $P_k$ are the corresponding weights. Because $U_i$ is a standard normal variable, the estimating points and weights can be derived by Gauss-Hermite quadrature with weighting function $\exp(-U_i^2 / 2)$. Specifically, for $K = 7$, Ref. [27] gives $u_{i1} = -u_{i7} = 1.154$, $u_{i2} = -u_{i6} = 2.367$, $u_{i3} =$
\[-u_{i5} = 3.750, u_{i4} = 0; P_1 = P_7 = 0.2401, P_2 = P_6 = 3.076 \times 10^{-2}, P_3 = P_5 = 5.483 \times 10^{-4}, P_4 = 16/35.\] Hence, the first four moments of \(D_i\) are

\[
\mu_{D_i} = \sum_{k=1}^{K} P_k D_i(u_{ik}), \quad \sigma_{D_i}^2 = \sum_{k=1}^{K} P_k \left[ D_i(u_{ik}) - \mu_{D_i} \right]^2,
\]

\[
\gamma_{D_i} = \sigma_{D_i}^3 \sum_{k=1}^{K} P_k \left[ D_i(u_{ik}) - \mu_{D_i} \right]^3, \quad \kappa_{D_i} = \sigma_{D_i}^4 \sum_{k=1}^{K} P_k \left[ D_i(u_{ik}) - \mu_{D_i} \right]^4.
\]

(30)

The foregoing procedure is repeated for each variable, i.e. \(D_1, D_2, \text{ etc.}\) are evaluated in turn. The total number of estimating points for \(n\)-dimensional problem amounts to \(n(K - 1) + 1\), as the estimation point at the origin is shared between all variables. Subsequently, \(D(U)\) is approximated as

\[
D(U) = \sum_{i=1}^{n} \left[ D_i - D(\mu_U) \right] + D(\mu_U)
\]

where it is noted that \(\mu_U = 0\). Because \(D_1, D_2, \text{ etc.}\) are independent, the four moments of \(D(U)\) can be formulated as

\[
\mu_D = D(\mu_U) + \sum_{i=1}^{n} \left[ \mu_{D_i} - D(\mu_U) \right], \quad \sigma_D^2 = \sum_{i=1}^{n} \sigma_{D_i}^2
\]

\[
\gamma_D = \sigma_D^3 \sum_{i=1}^{n} \gamma_{D_i} \sigma_{D_i}^3, \quad \kappa_D = \sigma_D^4 \left( \sum_{i=1}^{n} \kappa_{D_i} \sigma_{D_i}^4 + 6 \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{D_i}^2 \sigma_{D_j}^2 \right).
\]

(32)

The PEM has certain drawbacks, as mentioned in the discussion papers [28, 29]. First, it should be noted that \(D(U)\) would be normally distributed if it were a linear function of \(U\). However, if the nonlinearity is strong, the skewness and kurtosis of \(D(U)\) will deviate substantially from a normal curve, and the fitted distribution becomes less reliable, particularly at the tails. In the present problem, one anticipates \(D(U)\) to be highly nonlinear because of VIV, as well as the nonlinear relation between the damage and stress (cf. Eq. (26)). Accordingly, PEM in its native form may not be so ideal.

A simple but effective refinement is herein proposed. By recognizing that \(D(U)\) is strictly positive, it is logical to establish the performance function as the log of \(D(U)\), i.e. \(L = \ln[D(U)]\). It is expected that
L will be closer to a normal distribution, and the density fit will be better, while \( D \) will also be ensured to be non-negative. The main steps for the refined PEM are summarized as follows.

i. Take the log of the fatigue damage at the evaluation points.

ii. Apply the classical PEM to calculate the four moments of \( L \).

iii. Fit a distribution to \( L \) using SGLD (or the exponential power distribution in the case of zero or small skewness).

iv. Recover the cdf and pdf of \( D \) according to

\[
F_D(d) = f_L(\ln d) \quad \text{and} \quad f_D(d) = (1/d) f_L(\ln d)
\]

respectively; the latter follows from the change-of-variable theorem [19].

Practical design is usually semi-probabilistic, relying on a series of “partial safety factors” to account for the uncertainty due to a load and resistance mechanism. In fatigue design, the safety factor is traditionally imposed on the fatigue damage. DNV [21] recommends a safety factor of 10 for high safety class, i.e. when structural failure has severe consequences. For VIV fatigue, an even higher safety factor of 20 is the industry standard because of the associated uncertainties. However, such generic safety factors do not accommodate the exact nature of uncertainties for a particular application. The more rational approach is the use of partial safety factors for each type of uncertainty.

Let \( \lambda_R \) denote the safety factor corresponding to reliability \( R \). Then, the reliability (i.e. complement of failure probability) can be expressed as

\[
R = 1 - P_f = 1 - \Pr[D > \lambda_R D(X_{\text{design}})] = F_D(\lambda_R D(X_{\text{design}}))
\]

(33)

where \( D(X_{\text{design}}) \) is the damage evaluated at the design values \( (X_1 = X_2 = 0, \text{median values for } X_3 \text{ to } X_6) \). This is in fact an inverse reliability problem in which \( \lambda_R \) is sought for a prescribed \( P_f \). Using the refined PEM, \( F_D(\cdot) \) is available in analytical form; thus, it is straightforward to solve for \( \lambda_R \).

4.4 Summary of key assumptions

It is worth summarizing the key assumptions made in this study.

(1) As the wake oscillator model was originally developed and experimentally calibrated for cross-flow-only VIV, the effect of mean drag and in-line VIV that could be important is disregarded.
(2) By focusing on the hydrodynamic nonlinearities, other geometrical nonlinear effects associated with large displacement and top tension fluctuation are neglected.

(3) In common with much of the literature, the modal damping ratio is assumed to be equal for all eigenmodes, resulting in a unique \( S_G \) that can be related to experiment data for rigid cylinder tests.

(4) For simplicity, a single slope \( S-N \) model is used; however recent studies [30] have reported that a bi-linear \( S-N \) model is more appropriate for fatigue reliability assessment of risers.

(5) For a realistic fatigue assessment, all current velocities and their associated probability of occurrence over the design life should be considered. However, analysis of multiple current velocities is exceedingly time consuming for MCS. For expediency, current velocity is modeled as a random variable to reflect its fluctuation over the design life; however it is assumed to be constant for each realization.

5. Case studies

5.1 Baseline case (two random variables)

Consider a baseline case with a vertical steel TTR, whose key properties are summarized in Table 2. The riser model has \( m^* = 1.52 \) \((m^* = M / (\rho \pi D^2 / 4))\) and aspect ratio \((L_r/D)\) of 1866. As uniform flows entail greater fatigue damage than sheared flows [31], the uniform current profile with velocity of 0.4 m/s is considered as a baseline case. Since the associated Reynolds number is about \(1 \times 10^5\), the use of wake oscillator in the sub-critical flow regime is justified. A similar uniform current profile (0.5 m/s) for a 500-m-riser has recently been considered by Wang et al. [32]. Five modes are found to be sufficient for solution convergence with the third (second symmetric) mode being the dominant mode excited by VIV. The stress is the sum of the axial and bending contributions, and it is evaluated at the mid-length, although the methodologies are equally applicable to other locations. The dynamic simulations are performed until steady state is reached; the initial transients are omitted from the fatigue analysis. For convenience, the fatigue damage presented hereafter will be normalized as
\[
\hat{D}(X) = \frac{D(X)}{D(X_{\text{design}})}
\] (34)

In the baseline case, only two random variables \(X_1\) and \(X_2\) are considered; the uncertainties of other random variables are assumed to be incorporated in their respective safety factors. The \(S-N\) exponent \(m\) is designated as 3 for the baseline case, a larger value is investigated in Section 5.2. The other \(S-N\) parameter \(\bar{a}\) vanishes upon normalization by Eq. (34). Both MCS and direct numerical integration are performed to benchmark the approximate reliability approaches. For MCS, a large sample size of \(4 \times 10^4\) is considered. Numerical integration can be carried out in the physical space \(X\), correlated normal space \(Y\), or uncorrelated normal space \(U\). In principle, working in physical space is less ideal as the appropriate domain of integration is harder to ascertain. Here, the integration will be performed in \(Y\)-space, and the normalized damage response \(\hat{D}(Y_1, Y_2)\) is sampled in a regular grid with an interval of 0.1, and the domain of integration for \(Y_1\) and \(Y_2\) ranges from \(-4.2\) to \(4.2\). The integrand in Eq. (28) is simply replaced by \(f_Y(\cdot)\), and likewise the performance function is transformed accordingly. The advantage of \(Y\)-space is that the response surface is easier to interpret since \(Y_1\) maps directly to \(X_1\), while \(Y_2\) represents \(X_2\) (cf. Eq. ((20))). Besides, once the VIV dynamic simulations have been completed, it is straightforward to put 
\[
\rho_{Y_1Y_2} = 0
\]
to study the consequences if the random variables are independent. Fig. 5 depicts the surface plot of \(\hat{D}(Y_1, Y_2)\).

For the PEM approach, it is essential to work in \(U\)-space. Fig. 6(a)-(b) shows the point estimates of \(\hat{D}_1(U_1)\) and \(\hat{D}_2(U_2)\). Unlike the numerical integration approach, simulations have to be performed separately for different correlations. For the case of uncorrelated random variables (represented by dotted lines), \(Y = U\), thus the trends are equivalent to those of \(Y_1\) and \(Y_2\), which also characterize the behavior of \(\Delta A^*\) and \(\Delta \omega^*\) (albeit by a nonlinear scaling of the horizontal-axis). With this in mind, it is observed from Fig. 6(a) that the damage increases with \(\Delta A^*\), which is expected as a larger \(A_{\text{max}}\) corresponds to more pronounced vibrations and higher stresses. Referring to Fig. 6(b), the damage decreases with \(\Delta \omega^*\); however, this behavior is incidental and depends on the system characteristics, in particular the natural
frequencies and vortex shedding frequencies. It is found that if, for example, the current is raised to 0.5 m/s, the trend is no longer decreasing, but follows an inverted U shape with the peak occurring close to $U_2 = 0$.

Tables 4 and 5 compare the four moments of $\hat{D}$ and $\ln \hat{D}$ obtained by various methods, for the cases with and without correlation, respectively. For the conventional PEM, the moments of $\hat{D}$ are directly estimated. Instead, the refined PEM first evaluates the moments of $\ln \hat{D}$, and $f_D(\cdot)$ is derived according to the procedure outlined in Section 4.3. The moments of $\hat{D}$ can then be ascertained from $f_D(\cdot)$. The safety factors (specific to the case of two random variables) corresponding to reliability levels of 0.9, 0.99, 0.999 and 0.9999 are determined using the different methods. Fig. 7 plots the pdf of $\hat{D}$ for the baseline case with correlated variables; the MCS curve is omitted for better clarity. Fig. 8(a)-(b) plots the failure probability for different failure thresholds, corresponding to the correlated and uncorrelated cases, respectively. The MCS curves are truncated for $P_f < 10^{-3}$ due to large sampling variability associated with small probabilities. It is helpful to put these results in perspective by including the curves corresponding to the assumption that $\hat{D}$ obeys a normal distribution; the mean and variance are obtained from the numerical integration method.

The discussion focusses initially on the significance of correlation. It is found that the presence of correlation diminishes the uncertainty in the damage; the CoV is reduced by a non-negligible 20%. Correspondingly, the failure probability for the correlated case is also lower. The reason is that the damage increases with $\Delta A^*$, but decreases with $\Delta \omega^*$. As such, the positive correlation between $\Delta A^*$ and $\Delta \omega^*$ has an opposing effect that diminishes the uncertainty. It should be noted, however, that the decreasing trend for $\Delta \omega^*$ is circumstantial, and the correlation may enhance the uncertainty for other systems. The salient point is that the correlation between $\Delta A^*$ and $\Delta \omega^*$ should not be neglected.

Next, the performances of the various methods are compared. In theory, MCS should produce the same results as direct numerical integration. The results presented in the tables and graphs confirm this, except for some discrepancy in the kurtosis $\hat{D}$. Nevertheless, this disparity is not excessive, considering
the sensitivity of the kurtosis to the domain of integration (for numerical integration) and the sampling variability (for MCS). The skewness and kurtosis of \( \hat{D} \) deviate considerably from a normal distribution. According to Zhao and Ono [29], such scenarios are the bane of the conventional PEM, owing to multiple reasons. For one, \( \gamma_\beta \) and \( \kappa_\beta \) are likely to be poorly estimated, as borne out by Tables 4 and 5. More estimation points can alleviate the problem, but only to a certain extent as Eq. (31) can become inaccurate when the performance function is strongly nonlinear (and thus non-Gaussian). Besides, even if the moments have been properly estimated, the pdf fit is likely to be less favorable when the skewness and kurtosis are large.

On the other hand, refined PEM seeks to evaluate the moments of \( \ln \hat{D} \), whose skewness and kurtosis are close to a normal distribution. Hence, they can be estimated quite precisely, and the moments of \( \hat{D} \) that are indirectly calculated by refined PEM also agree well with numerical integration. Referring to Fig. 8, it is evident that the normal distribution is inadequate, grossly under predicting \( P_f \). The conventional PEM is better than the normal distribution, but is still unsatisfactory, especially for the tail region. Conversely, it is encouraging to find that refined PEM estimates \( P_f \) and the case-specific safety factors accurately even for low failure probabilities, surpassing the expectations of a fast approximate approach.

Tables 4 and 5 also report the values of \( \hat{D}(U = 0) \), which is the damage assessed at the origin in standard normal space. Since \( \hat{D}(U = 0) \) is close to unity, the discrepancy between \( D(U = 0) \) and \( D(X_{\text{design}}) \) is minor. However, \( \mu_\hat{D} \) exceeds unity by a substantial margin, occurring because the uncertainties, in conjunction with the nonlinearity of the performance function, shifts the mean of \( \hat{D} \) upwards compared to its deterministic value. Certainly, this effect is detrimental, and should be accounted for in practice. Interestingly, the conventional PEM predicts the first moment \( \mu_\hat{D} \) very well, but its performance progressively deteriorates with higher moments. Finally, it is clear from the pdf curves that there is considerable variability of \( \hat{D} \) about the design value of unity. This variability can also be inferred from
the coefficient of variation (CoV) of \( \hat{D} \). The implication is that the uncertainties from the wake oscillator model have significant impact on the fatigue reliability.

5.2 Varying S–N parameter

The S–N exponent \( m \) is raised to 5, keeping all other parameters unchanged. In this section, only two variables \( X_1 \) and \( X_2 \) are considered here, and they are correlated. Figs. 9 and 10 compare the pdf and \( P_f \) curves generated by the various methods, while key results are summarized in Table 6. One may infer from Eq. (27) that \( m \) governs nonlinearity between the stress and damage. Accordingly, an increase in \( m \) amplifies the nonlinearity of the performance function. The stronger nonlinearity manifests itself in several ways. First, \( \mu_{\hat{D}} \) is now larger, which implies that the shift in the mean of \( \hat{D} \) from the deterministic value is more pronounced. Second, \( \hat{D} \) is more skewed, and the third and fourth moments are now further from the normal distribution. As a result, conventional PEM is much less accurate than before. Fortunately, the performance of the refined PEM is still respectable, by virtue of the fact that despite the heightened nonlinearity, the skewness and kurtosis of \( \ln \hat{D} \) remain close to a normal distribution.

One drawback of PEM (both the basic and refined versions) is the difficulty of obtaining error bounds or confidence intervals for the probability curves. The error does not originate from sampling variability (such as MCS), but is of a systematic nature and can be traced to three separate sources, namely (1) estimating the moments by point estimates (Eq. (30)), (2) the approximation associated with Eq. (31), and (3) four moments do not uniquely define a pdf.

5.3 Reliability analysis with all random variables

A more comprehensive reliability analysis is performed with seven random variables (cf. Table 2 and Section 3.5), with \( m = 3 \). Because numerical integration is prohibitive in high dimensions, MCS will serve as the benchmark. Fig. 11 shows the point estimates for \( \hat{D}_i(U_i) \), where \( i = 3 \) to 7. As \( \xi \) is magnified, one would expect the response, and therefore the stress and associated damage to diminish,
and the behavior of $\hat{D}_3(U_3)$ is consistent with this principle. This is similar to the trend for $\hat{D}_4(U_4)$ where the damage decreases with increasing $C_{L0}$, because the wake damping term is governed by $C_{L0}$ (cf. Eqs. (5) and (12)); the variation in $G$ is such that the response diminishes as $C_{L0}$ increases. In contrast, $\hat{D}_5(U_5)$ increases exponentially with the flow velocity $V$ (and Reynolds number). This behavior is consistent with recent VIV experimental results of Swithenbank et al. [33] based on flexible cylinders whose response amplitudes increase with Reynolds number. The trend of increasing $\hat{D}_6(U_6)$ with mass may be explained by the change of system wake/riser natural frequencies such that a near-resonance becomes properly tuned, leading to a more perfect lock-in condition. Lastly, because damage is inversely proportional to $\bar{a}$, $\hat{D}_7(U_7)$ decreases with $U_7$.

It is of interest to understand the relative importance of the different variables on the damage uncertainty. To this end, the conventional PEM is used to calculate $\sigma_{D_i}^2$ ($i = 1$ to $7$), which is the variance contributed by each variable (cf. Eq. (30)). The conventional PEM may not necessarily estimate the higher moments well, but its treatment of the second moment is still respectable. Further, for the purpose of this exercise, it is assumed that $X_1$ and $X_2$ are independent, so that each $U_i$ can be mapped to the corresponding $X_i$. The calculations yield $\sigma_{D_i}^2 = 0.5273, 0.3625, 0.0486, 0.0584, 0.8760, 0.3201, 0.6533$ for $i = 1$ to $7$ respectively. From the results of $\sigma_{D_i}$, and also by reference to Fig. 11, it may be inferred that both wake oscillator parameters are significant for VIV fatigue reliability, with $A_{\text{max}}$ being the more influential of the two. In contrast, $\xi$ and $C_{L0}$ appear to be inconsequential, and for practical purposes can be treated as deterministic parameters. Mass and current velocity are also critical uncertainties as they directly influence the lock-in condition associated with vortex shedding and riser natural frequencies. As expected, the damage is highly sensitive to the variability of the $S$–$N$ parameter.

Fig. 12 shows the pdf obtained by the refined PEM and MCS. Fig. 13 compares the $P_f$ curves for two cases; the first case comprises all seven random variables, whereas in the second case, $A_{\text{max}}$ and $\omega_{s,A}/\omega_n$ are constants with five random variables. The purpose is to understand the significance of $A_{\text{max}}$. 

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and $\omega_{s, A}/\omega_n$ in the context of other uncertainties. It is apparent from Fig. 13 that the $P_f$ curves for the two cases are vastly dissimilar, indicating that $A_{\text{max}}$ and $\omega_{s, A}/\omega_n$ are important uncertainties that should not be ignored. The refined PEM is in excellent agreement with MCS for the case with five random variables. However, with seven random variables, refined PEM is less accurate. One possible reason may be that the wake oscillator coefficients have complex interactions with current velocity and mass that are not entirely captured by the refined PEM, bearing in mind that point estimates are evaluated by varying a single parameter at a time.

Table 7 presents the detailed results of the reliability analyses, as well as the safety factors. The safety factors are notably higher when the uncertainties of $A_{\text{max}}$ and $\omega_{s, A}/\omega_n$ are considered. At first glance, the safety factor of 31.16 for one particular case may appear excessive. In fact, this is not so, when one considers that a factor of 20 is routinely applied to the VIV fatigue damage, in addition to partial safety factors inherent in other load and resistance mechanisms. For instance, the $S–N$ curve used in practice is two standard deviations below the mean value. If, instead the median $\bar{\sigma}$ is adopted as in the present reliability analysis, the safety factor has to be much higher than 20.

6. Conclusions

In the Skop and Balasubramanian wake oscillator model [13], the calibrated functions of $A_{\text{max}}$ and $\omega_{s, A}/\omega_n$ given by Eq. (13) were considered to be deterministic. However, due to appreciable scatter in the experimental database, it is proposed to model $A_{\text{max}}$ and $\omega_{s, A}/\omega_n$ as random variables for riser VIV fatigue analysis. The pdfs of the residuals ($A^*_{\text{max}}$ and $\omega^*$) are fitted to data using a four parameter distribution [17], while the dependency between $A^*_{\text{max}}$ and $\omega^*$ is approximated by the Nataf transformation.

The aim of reliability analysis is to predict the failure probability $P_f$. Here, the concern is also to determine safety factors for use in a semi-probabilistic design approach. As direct numerical integration and Monte Carlo simulation (MCS) are computationally demanding, a very fast but approximate approach is proposed, based on refining the conventional point estimate method (PEM), which is known to perform poorly when the response is markedly non-Gaussian.
Simulations are performed on a top tensioned riser subjected to a uniform current profile. The VIV fatigue damage is calculated at mid-length of the riser. The baseline case considers only $A_{\text{max}}^*$ and $\omega^*$ as random variables. Using direct integration method and MCS as benchmark, the refined PEM is found to be quite accurate in estimating $P_f$, the pdf of the damage, and the safety factors, notwithstanding the highly non-Gaussian nature of the fatigue damage. The fatigue damage has large variability, arising from the randomness of both $A_{\text{max}}^*$ and $\omega^*$. Further, it is shown that the correlation between $A_{\text{max}}^*$ and $\omega^*$ should be properly represented as it affects the fatigue reliability.

A more comprehensive reliability analysis is performed with five additional random variables, specifically the damping ratio, lift coefficient, current velocity, riser mass and $S$–$N$ parameter $\tilde{a}$. Comparison is made between seven random variables and five random variables ($A_{\text{max}}^*$ and $\omega^*$ are omitted). It is found that the uncertainties of $A_{\text{max}}$ and $\omega_{S,N}/\omega_n$ have significant impact on $P_f$ and the safety factors. Thus, it is crucial to consider the uncertainties of the wake oscillator coefficients to ensure a safe design. Although a vertical riser and uniform flow is considered, the methodologies presented herein are generic and should be applicable to other riser systems [4–5] and non-uniform flow cases [6]. Fatigue contributions from in-line VIV should also be recognized. These can be verified through further studies.

Acknowledgements

The authors are grateful to the Early Career Researcher International Exchange Awards supported by the Scottish Funding Council through (GRPe-SFC) through the research project “Advanced Analysis & Design Tools for Vortex-Induced Vibrations of Offshore Structures”. The first author also acknowledges financial support from the NUS Start-up Grant, No. R-302-000-099-133.

Appendix A

The chi-squared test is performed to determine how well the SGLD and normal distributions are able to fit the data for $\Delta A^*$ and $\Delta \omega^*$. For each distribution type and dataset, bins with intervals of equal probability are first demarcated. A common rule is that the expected (i.e. theoretical) counts $E_i$ in each bin
should be at least 5 \( (i \) represents the bin number). Since the sample size is small, \( E_i \) is chosen to be close to 5 to maximize the number of bins \( N \). The chi-squared statistic is defined as

\[
\chi^2 = \sum_i^N \frac{(O_i - E_i)^2}{E_i}
\]

where \( O_i \) is the observed counts. The hypothesis that the data have arisen from a specified distribution is rejected if \( \chi^2 > \chi^2_{1-\alpha, N-p-1} \), where \( \chi^2_{1-\alpha, N-p-1} \) is the chi-square critical value with significance level \( \alpha \) and degrees-of-freedom \( N - p - 1 \), and \( p \) is the number of estimating parameters (\( p = 4 \) and 2 for SGLD and normal distribution respectively).

The results are summarized in Table 8. Taking \( \alpha = 5\% \), it is found that for \( \Delta A^* \), the normal distribution can be rejected, whereas SGLD is a possible candidate. For \( \Delta \omega^* \), both distributions are acceptable. The SGLD has a smaller (better) \( \chi^2 \) value compared to the normal distribution, suggesting a better fit; however the critical value is also lower due to more estimating parameters. It should be cautioned that the chi-squared test does not accord higher weight to the tails, which are particularly critical for reliability analysis. The SGLD is expected to model the tail more accurately since it accounts for the higher moments.

References


Figure Captions

Fig. 1  Plot of $A_{\text{max}}$ against $S_G$

Fig. 2  Plot of $\omega_s/\omega_n$ against $S_G$

Fig. 3  Comparison of pdfs obtained from data and SGLD fit: (a) $\Delta A^*$; (b) $\Delta \omega^*$

Fig. 4  Scatter plot of $\Delta \omega^*$ against $\Delta A^*$

Fig. 5  Surface plot of the normalized damage $\hat{D}(Y_1, Y_2)$ for the baseline case

Fig. 6  Point estimates for the baseline case: (a) $\hat{D}_1(U_1)$; (b) $\hat{D}_2(U_2)$

Fig. 7  Probability density of the normalized damage with $m = 3$ and correlated wake oscillator variables

Fig. 8  Probability of failure for different thresholds with $m = 3$: (a) correlated case; (b) uncorrelated case

Fig. 9  Probability density of the normalized damage with $m = 5$

Fig. 10  Probability of failure for different thresholds with $m = 5$

Fig. 11  Point estimates for the other random variables, $\hat{D}_i(U_i)$

Fig. 12  Probability density of the normalized damage for comprehensive reliability analysis with seven random variables

Fig. 13  Probability of failure for comprehensive reliability analysis

Table Captions

Table 1  Statistical properties of $\Delta A^*$ and $\Delta \omega^*$

Table 2  Characteristics of other random variables

Table 3  Properties of vertical top-tensioned riser

Table 4  Results for the baseline case, correlated random variables

Table 5  Results for the baseline case, uncorrelated random variables

Table 6  Results for case that the $S$–$N$ exponent is increased to $m = 5$

Table 7  Results for comprehensive reliability analysis

Table 8  Summary of the chi-squared test
### Table 1: Statistical properties of $\Delta A^*$ and $\Delta \omega^*$

<table>
<thead>
<tr>
<th>Variable</th>
<th>$X_1 = \Delta A^*$</th>
<th>$X_2 = \Delta \omega^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample size</td>
<td>64</td>
<td>43</td>
</tr>
<tr>
<td>Mean</td>
<td>$-0.0071$</td>
<td>$-0.0017$</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.1351</td>
<td>0.1127</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.8427</td>
<td>0.4668</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.963</td>
<td>2.506</td>
</tr>
</tbody>
</table>

*Parameters of the SGLD*

- $b$: $-0.6376$  
- $\theta$: 0.6167  
- $\sigma$: 0.1922  
- $r$: 1.598

### Table 2: Characteristics of other random variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Distribution type</th>
<th>Median</th>
<th>CoV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_3$</td>
<td>Damping ratio $\xi$</td>
<td>Lognormal</td>
<td>0.01</td>
<td>0.3</td>
</tr>
<tr>
<td>$X_4$</td>
<td>Lift coefficient $C_{l0}$</td>
<td>Lognormal</td>
<td>0.28</td>
<td>0.2</td>
</tr>
<tr>
<td>$X_5$</td>
<td>Current velocity $V$</td>
<td>Lognormal</td>
<td>0.4 m/s</td>
<td>0.08</td>
</tr>
<tr>
<td>$X_6$</td>
<td>Riser mass $M$</td>
<td>Normal</td>
<td>87.73 kg/m</td>
<td>0.08</td>
</tr>
<tr>
<td>$X_7$</td>
<td>$S$–$N$ parameter $\bar{a}$</td>
<td>Lognormal</td>
<td>$\bar{a}_0$</td>
<td>0.56</td>
</tr>
</tbody>
</table>

### Table 3: Properties of vertical top-tensioned riser

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>$2.05 \times 10^{11}$ N/m$^2$</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>0.268 m</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>14 mm</td>
</tr>
<tr>
<td>Water depth</td>
<td>500 m</td>
</tr>
<tr>
<td>Density of water</td>
<td>1025 kg/m$^3$</td>
</tr>
<tr>
<td>Top tension/submerged weight</td>
<td>8.3</td>
</tr>
</tbody>
</table>
Table 4 Results for the baseline case, correlated random variables

<table>
<thead>
<tr>
<th></th>
<th>Numerical integration</th>
<th>Monte Carlo simulation</th>
<th>Refined PEM</th>
<th>Conventional PEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{D}(U = 0) )</td>
<td>1.018</td>
<td>1.018</td>
<td>1.018</td>
<td>1.018</td>
</tr>
<tr>
<td>Moments of ( \hat{D} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu_{\hat{D}} )</td>
<td>1.144</td>
<td>1.145</td>
<td>1.146</td>
<td>1.144</td>
</tr>
<tr>
<td>( \sigma_{\hat{D}} )</td>
<td>0.534</td>
<td>0.534</td>
<td>0.546</td>
<td>0.512</td>
</tr>
<tr>
<td>CoV of ( \hat{D} )</td>
<td>0.467</td>
<td>0.467</td>
<td>0.476</td>
<td>0.448</td>
</tr>
<tr>
<td>( \gamma_{\hat{D}} )</td>
<td>1.228</td>
<td>1.265</td>
<td>1.319</td>
<td>0.673</td>
</tr>
<tr>
<td>( \kappa_{\hat{D}} )</td>
<td>5.433</td>
<td>5.803</td>
<td>5.827</td>
<td>3.702</td>
</tr>
<tr>
<td>Moments of ln ( \hat{D} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu_{\ln\hat{D}} )</td>
<td>0.033</td>
<td>0.034</td>
<td>0.032</td>
<td>N.A.</td>
</tr>
<tr>
<td>( \sigma_{\ln\hat{D}} )</td>
<td>0.452</td>
<td>0.450</td>
<td>0.459</td>
<td>N.A.</td>
</tr>
<tr>
<td>( \gamma_{\ln\hat{D}} )</td>
<td>0.013</td>
<td>0.024</td>
<td>0.073</td>
<td>N.A.</td>
</tr>
<tr>
<td>( \kappa_{\ln\hat{D}} )</td>
<td>2.609</td>
<td>2.613</td>
<td>2.746</td>
<td>N.A.</td>
</tr>
<tr>
<td>Safety factors</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_{0.9} )</td>
<td>1.866</td>
<td>1.866</td>
<td>1.872</td>
<td>1.825</td>
</tr>
<tr>
<td>( \lambda_{0.99} )</td>
<td>2.794</td>
<td>2.780</td>
<td>2.898</td>
<td>2.578</td>
</tr>
<tr>
<td>( \lambda_{0.999} )</td>
<td>3.763</td>
<td>3.836</td>
<td>3.918</td>
<td>3.235</td>
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<tr>
<td>( \lambda_{0.9999} )</td>
<td>4.902</td>
<td>N.A.</td>
<td>4.981</td>
<td>3.856</td>
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Table 5 Results for the baseline case, uncorrelated random variables

<table>
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<th>Monte Carlo simulation</th>
<th>Refined PEM</th>
<th>Conventional PEM</th>
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<tbody>
<tr>
<td>$\hat{D}(U = 0)$</td>
<td>1.018</td>
<td>1.018</td>
<td>1.018</td>
<td>1.018</td>
</tr>
<tr>
<td>Moments of $\hat{D}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{\hat{D}}$</td>
<td>1.202</td>
<td>1.210</td>
<td>1.202</td>
<td>1.197</td>
</tr>
<tr>
<td>$\sigma_{\hat{D}}$</td>
<td>0.702</td>
<td>0.710</td>
<td>0.702</td>
<td>0.640</td>
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<tr>
<td>CoV of $\hat{D}$</td>
<td>0.584</td>
<td>0.587</td>
<td>0.584</td>
<td>0.535</td>
</tr>
<tr>
<td>$\gamma_{\hat{D}}$</td>
<td>1.616</td>
<td>1.649</td>
<td>1.603</td>
<td>0.788</td>
</tr>
<tr>
<td>$\kappa_{\hat{D}}$</td>
<td>7.557</td>
<td>7.779</td>
<td>7.270</td>
<td>3.764</td>
</tr>
<tr>
<td>Moments of $\ln\hat{D}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{\ln\hat{D}}$</td>
<td>0.031</td>
<td>0.037</td>
<td>0.031</td>
<td>N.A.</td>
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<tr>
<td>$\sigma_{\ln\hat{D}}$</td>
<td>0.554</td>
<td>0.554</td>
<td>0.555</td>
<td>N.A.</td>
</tr>
<tr>
<td>$\gamma_{\ln\hat{D}}$</td>
<td>0.031</td>
<td>0.040</td>
<td>0.052</td>
<td>N.A.</td>
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<tr>
<td>$\kappa_{\ln\hat{D}}$</td>
<td>2.615</td>
<td>2.618</td>
<td>2.694</td>
<td>N.A.</td>
</tr>
<tr>
<td>Safety factors</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_{0.9}$</td>
<td>2.128</td>
<td>2.152</td>
<td>2.124</td>
<td>2.063</td>
</tr>
<tr>
<td>$\lambda_{0.99}$</td>
<td>3.536</td>
<td>3.555</td>
<td>3.562</td>
<td>3.020</td>
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<tr>
<td>$\lambda_{0.999}$</td>
<td>5.123</td>
<td>5.322</td>
<td>5.066</td>
<td>3.845</td>
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<tr>
<td>$\lambda_{0.9999}$</td>
<td>7.004</td>
<td>N.A.</td>
<td>6.691</td>
<td>4.617</td>
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</table>
Table 6 Results for case that the S–N exponent is increased to \( m = 5 \)

<table>
<thead>
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<th>Numerical integration</th>
<th>Monte Carlo simulation</th>
<th>Refined PEM</th>
<th>Conventional PEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{D}(U = 0) )</td>
<td>1.023</td>
<td>1.023</td>
<td>1.023</td>
<td>1.023</td>
</tr>
<tr>
<td>Moments of ( \hat{D} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu_{\hat{D}} )</td>
<td>1.370</td>
<td>1.371</td>
<td>1.381</td>
<td>1.361</td>
</tr>
<tr>
<td>( \sigma_{\hat{D}} )</td>
<td>1.100</td>
<td>1.106</td>
<td>1.149</td>
<td>0.958</td>
</tr>
<tr>
<td>CoV of ( \hat{D} )</td>
<td>0.803</td>
<td>0.807</td>
<td>0.832</td>
<td>0.704</td>
</tr>
<tr>
<td>( \gamma_{\hat{D}} )</td>
<td>2.434</td>
<td>2.610</td>
<td>2.584</td>
<td>1.297</td>
</tr>
<tr>
<td>( \kappa_{\hat{D}} )</td>
<td>15.63</td>
<td>18.48</td>
<td>15.75</td>
<td>7.578</td>
</tr>
<tr>
<td>Moments of ( \ln \hat{D} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu_{\ln \hat{D}} )</td>
<td>0.047</td>
<td>0.049</td>
<td>0.045</td>
<td>N.A.</td>
</tr>
<tr>
<td>( \sigma_{\ln \hat{D}} )</td>
<td>0.737</td>
<td>0.735</td>
<td>0.749</td>
<td>N.A.</td>
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<tr>
<td>( \gamma_{\ln \hat{D}} )</td>
<td>0.005</td>
<td>0.015</td>
<td>0.062</td>
<td>N.A.</td>
</tr>
<tr>
<td>( \kappa_{\ln \hat{D}} )</td>
<td>2.615</td>
<td>2.619</td>
<td>2.764</td>
<td>N.A.</td>
</tr>
<tr>
<td>Safety factors</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_{0.9} )</td>
<td>2.738</td>
<td>2.738</td>
<td>2.763</td>
<td>2.533</td>
</tr>
<tr>
<td>( \lambda_{0.99} )</td>
<td>5.304</td>
<td>5.270</td>
<td>5.667</td>
<td>4.456</td>
</tr>
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<td>( \lambda_{0.999} )</td>
<td>8.690</td>
<td>8.646</td>
<td>9.324</td>
<td>6.689</td>
</tr>
<tr>
<td>( \lambda_{0.9999} )</td>
<td>13.47</td>
<td>N.A.</td>
<td>13.88</td>
<td>9.335</td>
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Table 7 Results for comprehensive reliability analysis

<table>
<thead>
<tr>
<th></th>
<th>Seven random variables</th>
<th>Five random variables</th>
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<tbody>
<tr>
<td></td>
<td>Monte Carlo simulation</td>
<td>Refined PEM</td>
</tr>
<tr>
<td>Moments of $\tilde{D}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{\tilde{D}}$</td>
<td>1.568</td>
<td>1.561</td>
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<tr>
<td>$\sigma_{\tilde{D}}$</td>
<td>1.912</td>
<td>2.005</td>
</tr>
<tr>
<td>CoV of $\tilde{D}$</td>
<td>1.220</td>
<td>1.285</td>
</tr>
<tr>
<td>$\gamma_{\tilde{D}}$</td>
<td>3.822</td>
<td>4.141</td>
</tr>
<tr>
<td>$\kappa_{\tilde{D}}$</td>
<td>35.02</td>
<td>35.00</td>
</tr>
<tr>
<td>Moments of $\ln\tilde{D}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{\ln\tilde{D}}$</td>
<td>-0.148</td>
<td>-0.156</td>
</tr>
<tr>
<td>$\sigma_{\ln\tilde{D}}$</td>
<td>1.187</td>
<td>1.173</td>
</tr>
<tr>
<td>$\gamma_{\ln\tilde{D}}$</td>
<td>-0.468</td>
<td>-0.427</td>
</tr>
<tr>
<td>$\kappa_{\ln\tilde{D}}$</td>
<td>3.291</td>
<td>3.481</td>
</tr>
<tr>
<td>Safety factors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_{0.9}$</td>
<td>3.643</td>
<td>3.587</td>
</tr>
<tr>
<td>$\lambda_{0.99}$</td>
<td>9.103</td>
<td>9.730</td>
</tr>
<tr>
<td>$\lambda_{0.999}$</td>
<td>17.51</td>
<td>18.86</td>
</tr>
<tr>
<td>$\lambda_{0.9999}$</td>
<td>N.A.</td>
<td>31.16</td>
</tr>
</tbody>
</table>
Table 8 Summary of the chi-squared test

<table>
<thead>
<tr>
<th></th>
<th>$\Delta A^*$</th>
<th>$\Delta \omega^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of bins $N$</td>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>Expected counts $E_i$</td>
<td>4.923 (constant)</td>
<td>5.375 (constant)</td>
</tr>
<tr>
<td><strong>SGLD</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observed counts $O_i$</td>
<td>6, 4, 1, 6, 9, 5, 2, 5, 6, 7, 4, 4, 5</td>
<td>5, 5, 2, 9, 7, 5, 4, 6</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>10.34</td>
<td>5.558</td>
</tr>
<tr>
<td>Critical value</td>
<td>15.51</td>
<td>7.815</td>
</tr>
<tr>
<td><strong>Normal distribution</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observed counts $O_i$</td>
<td>4, 5, 2, 6, 13, 2, 4, 5, 6, 6, 2, 4, 5</td>
<td>5, 6, 6, 5, 9, 2, 3, 7</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>19.69</td>
<td>6.302</td>
</tr>
<tr>
<td>Critical value</td>
<td>18.31</td>
<td>11.07</td>
</tr>
</tbody>
</table>
Fig. 1  Plot of $A_{\text{max}}$ against $S_G$

Fig. 2  Plot of $\omega_{s,A}/\omega_n$ against $S_G$
Fig. 3  Comparison of pdfs obtained from data and SGLD fit: (a) $\Delta A^*$; (b) $\Delta \omega^*$
Fig. 4  Scatter plot of $\Delta \omega^*$ against $\Delta A^*$

Fig. 5  Surface plot of the normalized damage $\hat{D}(Y_1, Y_2)$ for the baseline case
Fig. 6  Point estimates for the baseline case: (a) $\hat{D}_1(U_1)$; (b) $\hat{D}_2(U_2)$
Fig. 7  Probability density of the normalized damage with \( m = 3 \) and correlated wake oscillator variables.
Fig. 8 Probability of failure for different thresholds with $m = 3$: (a) correlated case; (b) uncorrelated case

(a)

(b)
Fig. 9  Probability density of the normalized damage with $m = 5$

Fig. 10  Probability of failure for different thresholds with $m = 5$
Fig. 11 Point estimates for the other random variables, $\hat{D}_i(U_i)$
Fig. 12  Probability density of the normalized damage for comprehensive reliability analysis with seven random variables

Fig. 13  Probability of failure for comprehensive reliability analysis