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NON-G-COMPLETELY REDUCIBLE SUBGROUPS OF THE EXCEPTIONAL ALGEBRAIC GROUPS

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Abstract. Let G be an exceptional algebraic group defined over an algebraically closed field k of characteristic $p > 0$ and let H be a subgroup of G. Then following Serre we say H is G-completely reducible or G-cr if, whenever H is contained in a parabolic subgroup P of G, then H is in a Levi subgroup of that parabolic. Building on work of Liebeck and Seitz, we find all triples (X, G, p) such that there exists a closed, connected, simple non-G-cr subgroup $H \leq G$ with root system X.

1. Introduction

Let G be an algebraic group defined over an algebraically closed field k of characteristic $p > 0$ and let H be a subgroup of G. Then following Serre [Ser05] we say H is G-completely reducible or G-cr if, whenever H is contained in a parabolic subgroup P of G, then H is in a Levi subgroup of that parabolic. This is a natural generalisation of the notion of a group acting completely reducibly on a module V: if we set $G = GL(V)$ then saying H is G-completely reducible is precisely the same as saying that H acts semisimply on V.

This notion is important in unifying some other pre-existing notions and results. For instance, in [BMR05], it was shown that a subgroup H is G-cr if and only if it satisfied Richardson's notion of being strongly reductive in G. It also allows one to state some previous results due to Liebeck–Seitz and Liebeck–Saxl–Testerman on the subgroup structure of the exceptional algebraic groups in a particularly satisfying form.

Assume G is connected and simple[1] of one of the five exceptional types and let X be a irreducible root system. The result [LS96, Theorem 1] asserts a number $N(X, G)$ such that if a subgroup H of G is closed, connected and simple, with root system X, then H is G-cr whenever the characteristic p of k is bigger than $N(X, G)$. In particular if p is bigger than 7 then the authors show that all closed, connected, reductive subgroups of G are G-cr. There is some overlap in that paper with the contemporaneous work of [LST96]. If H is a simple subgroup of rank greater than half the rank of G, then [Theorem 1, *ibid.*] finds all conjugacy classes of simple subgroups of G; the proofs indicate where these conjugacy classes are G-completely reducible. With essentially one class of exceptions, all such subgroups, including the non-G-cr subgroups, can be located in so-called subsystem subgroups of G. We shall mention these in greater detail later.

More recently, [Ste10] and [Ste13] find all conjugacy classes of simple subgroups of exceptional groups of types G_2 and F_4. One consequence of this is to show that the numbers $N(X, G)$ found above can be made strict. (One needs only to change $N(A_1, G_2)$ from 3 to 2.) The main purpose of this article is to make *all* the $N(X, G)$ strict. That is, for each of the five types of exceptional algebraic group G, for each prime $p = \text{char } k$ and for each irreducible root system X, we give in Table 1 of Theorem 1 an example $H = E(X, G, p)$ of a connected, closed, simple non-G-cr

[1]This means, as usual, that the centre $Z(G)(k)$ is finite, and $G(k)/Z(G(k))$ is abstractly simple.

subgroup H with root system X, precisely when this is possible. In other words we classify the triples (X, G, p) where there exists a connected, closed, simple non-G-cr subgroup H with root system X. Moreover, in all but two cases (where $(X, G, p) = (G_2, E_7, 7)$ or $(G_2, E_8, 3)$), we can give a construction of $E(X, G, p)$ in a subsystem subgroup.

Our main theorem can thus be viewed as the best possible improvement of the result [LS96, Theorem 1], in the spirit of that result. Before we state our main theorem in full, we need a couple of definitions: Let Φ be the root system of G corresponding to a choice of Borel subgroup B containing a maximal torus T and for $\alpha \in \Phi$, let U_α denote the T-root subgroup corresponding to α. A subset R of the root system Φ is called a *closed subsystem* (or just a *subsystem*) if whenever $\alpha, \beta \in R$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in R$; and whenever $\alpha \in R$, $-\alpha \in R$. The subsystems of Φ are easily determined by the Borel–de Siebenthal algorithm[2]. A *subsystem subgroup* of G is a semisimple, closed, connected subgroup Y which is normalised by a maximal torus T of G. It follows that a subsystem subgroup Y is of the form $\langle U_\alpha | \alpha \in R\rangle$ where R is a closed subsystem of Φ or (Φ, p) is $(B_n, 2)$, $(C_n, 2)$, $(F_4, 2)$ or $(G_2, 3)$ and R lies in the dual of a closed subsystem.

Most of our examples $H = E(X, G, p)$ are described in terms of an embedding of H into a subsystem subgroup M. Here we describe M just by giving its root system. For further notation in the table referenced by the theorem, see Section 2 below.

Theorem 1. *Let G be an exceptional algebraic group defined over an algebraically closed field k of characteristic $p > 0$. Suppose there exists a non-G-cr closed, connected, simple subgroup H of G with root system X. Then (X, G, p) has an entry in Table 1.*

Conversely, for each (X, G, p) given in Table 1, the last column guarantees an example of a closed, connected, simple, non-G-cr subgroup $E(X, G, p)$ with root system X.

In particular we can improve on [LS96, Theorem 1]. In the table in Corollary 2 we have struck out the primes which were used in the hypotheses in [*loc. cit.*]. This is done partly to show where we have made improvements but mainly to facilitate reading the proof of the first part of Theorem 1.

Corollary 2. *Let G be an exceptional algebraic group over a field k of characteristic p. Let X be an irreducible root system and let $N(X, G)$ be a set of primes defined by the table below. (For instance, $N(B_2, E_8) = \{2, 5\}$.) Suppose H is a closed, connected, reductive subgroup of G with root system having simple components $X_1, \ldots, X_n$. Then if $p \notin \bigcup_i N(X_i, G)$, H is G-cr.*

	$G = E_8$	E_7	E_6	F_4	G_2
$X = A_1$	≤ 7	≤ 7	≤ 5	≤ 3	~~3~~ 2
A_2	~~5~~ 3 2	~~5~~ 3 2	3 2	3 ~~2~~	
B_2	5 ~~3~~ 2	~~3~~ 2	~~3~~ 2	2	
G_2	7 ~~5~~ 3 2	7 ~~5~~ ~~3~~ 2	~~3~~ 2	2	
A_3	2	2	~~2~~		
B_3	2	2	2	2	
C_3	3 ~~2~~	~~2~~	~~2~~	~~2~~	
B_4	2	~~2~~	~~2~~		
C_4, D_4	2	2	~~2~~		

The hypothesis $p > N(X, G)$ is to be found throughout the literature. Certainly [LS96, Theorems 2–8] depend on it, as do [LS98, Theorems 9–10] and [TZ13, Prop. 4.1, Proof]. It should be possible

[2]cf. [BDS49] or [Bou82, IV. Ex. §4.4]; this is now given a thorough treatment in [MT11, §13.2].

G	X	p	Example $E = E(X, G, p)$
G_2	A_1	2	$E \hookrightarrow A_1\tilde{A}_1$; $x \mapsto (x, x)$
F_4	A_1	2	$E \hookrightarrow A_2$; $V_3\|E = V(2) \cong L(2)/k$
		3	$E \hookrightarrow A_2^2$; $(V_3, V_3)\|E = (L(2), L(2))$
	A_2	3	$E \hookrightarrow A_2^2$; $x \mapsto (x, x)$
	B_2	2	$E \leq D_3$
	G_2	2	$E \leq D_4$
	B_3	2	$E \leq D_4$
E_6	A_1	2	$E(A_1, F_4, 2) \leq F_4$, as above
		3	$E(A_1, F_4, 3) \leq F_4$, as above
		5	$E \hookrightarrow D_5$; $V_{10}\|E = T(8) \cong k/L(8)/k$
	A_2	2	$E \hookrightarrow A_5$; $V_6\|E = V(20) \cong L(10)^{[1]}/L(01)$
		3	$E \hookrightarrow A_2^3$; $x \mapsto (x, x, x)$
	B_2	2	$E \hookrightarrow A_4$; $V_5\|E = V(10) \cong L(10)/k$
	G_2	2	$E \hookrightarrow C_4$; $V_8\|E = T(10) \cong k/L(10)/k$
	B_3	2	$E \hookrightarrow C_4$; $V_8\|E = T(100) \cong k/L(100)/k$
E_7	$E \leq E_6$	$2, 3, 5$	each of the subgroups of E_6 above
	A_1	7	$E \leq A_7$; $V_8\|E = V(7) \cong L(1)^{[1]}/L(5)$
	A_3	2	$E \leq A_3A_3 \leq A_7$; $V_8\|E = L(100) + L(001)$
	G_2	7	E in an E_6-parabolic of G.*
	C_4	2	$E \hookrightarrow A_7$; $V_8\|E = L(1000)$
	D_4	2	$E \leq C_4$ above
E_8	$E \leq E_7$	$2, 3, 5, 7$	each of the subgroups of E_7 above
	B_2	5	$E \leq D_8$; $V_{16}\|E = T(20) \cong k/L(20)/k$
	G_2	3	E in a D_7-parabolic of G.*
	C_3	3	$E \leq D_8$; $V_{16}\|E = T(010) + k \cong k/L(010)/k + k$
	B_4	2	$E \leq A_8$; $V_9\|E = V(1000) \cong L(1000)/k$

* These subgroups have no proper reductive overgroup in G; on the construction of these, see Remark 5.2.

TABLE 1. Simple non-G-cr subgroups of type X in the exceptional groups

without too much extra work to use our theorem to improve results or shorten proofs in these instances, by replacing the hypothesis '$p > N(X, G)$' by '$p \notin N(X, G)$'.

2. NOTATION

When discussing roots or weights, we use the Bourbaki conventions [Bou82, VI. Planches I-IX]. We use some of the representation theory for algebraic groups whose notation we have taken largely consistent with [Jan03]. For an algebraic group G, recall that a G-module is a comodule for the Hopf algebra $k[G]$; in particular every G-module is a $kG(k)$-module, where $G(k)$ is the group of k-points of G. Let B be a Borel subgroup of a reductive algebraic group G, containing a maximal torus T of G and containing all the negative root groups of G. Let $X(T)$ denote the the character group of T and let X^+ be the subset of dominant weights. Recall that for each dominant weight $\lambda \in X^+$ for G, the space $\mathrm{H}^0(\lambda) := \mathrm{H}^0(G/B, \lambda) = \mathrm{Ind}_B^G(\lambda)$ is a G-module with highest weight λ and with socle $\mathrm{Soc}_G\mathrm{H}^0(\lambda) = L(\lambda)$, the irreducible G-module of highest weight λ. The Weyl module of highest weight λ is $V(\lambda) \cong \mathrm{H}^0(-w_0\lambda)^*$ where w_0 is the longest element in the Weyl group. We identify $X(T)$ with $\mathbb{Z}^r$ for r the rank of G and for $\lambda \in X^+ \cong \mathbb{Z}^r_{\geq 0} \leq X(T)$, write $\lambda = (a_1, a_2, \dots, a_r) = a_1\omega_1 + \cdots + a_r\omega_r$ where ω_i are the fundamental dominant weights; a $\mathbb{Z}_{\geq 0}$-basis of X^+. Put also $L(\lambda) = L(a_1, a_2, \dots, a_r)$, where we will often omit commas when the a_i are all in single digits. When $0 \leq a_i < p$ for all i, we say that λ is a (p-)restricted weight and we write $\lambda \in X_1$. Recall that any module V has a Frobenius twist $V^{[n]}$ induced by precomposing the action with the nth power of the Frobenius map $F : G \to G$. Steinberg's tensor product theorem states that $L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_n)^{[n]}$ where $\lambda_i \in X_1$ and $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^n\lambda_n$ is the p-adic expansion of $\lambda \in X^+$. We refer to λ_0 as the restricted part of λ.

The right derived functors of $\mathrm{Hom}(V, *)$ are denoted by $\mathrm{Ext}^i_G(V, *)$ and when $V = k$, the trivial G-module, we have the identity $\mathrm{Ext}^i_G(k, *) = H^i(G, *)$ giving the Hochschild cohomology groups.

We make no attempt to establish the isogeny type of the groups we discuss. Thus we will simply refer to groups by their root systems. Let us recall some standard G-modules: when G is classical, there is a 'natural module' which we refer to by V_{nat}; or, where there is no confusion, by V_m where m is the dimension of V_{nat}. It is always the Weyl module $V(\omega_1)$, which is irreducible unless $p = 2$ and G is of type B_n; in the latter case it has a 1-dimensional radical. Certain properties of these modules are described in [Jan03, II.8.21]. Of importance to us is the fact that when $G = \mathrm{SL}_n$, $\bigwedge^r(L(\omega_1)) = L(\omega_r)$ for $r \leq n-1$. We use this fact without further reference.

Recall that F_4 has a 26-dimensional Weyl module which we denote 'V_{26}'. When $p \neq 3$, V_{26} is the irreducible representation of high weight $0001 = \omega_4$. When $p = 3$, V_{26} has a one-dimensional radical, with a 25-dimensional irreducible quotient of high weight 0001. The group E_6 (resp. E_7, E_8) has a module of dimension 27 (resp. 56, 248) of high weight ω_1 (resp. ω_7, ω_8) which is irreducible in all characteristics. We refer to this module as V_{27} (resp. V_{56}, $\mathrm{Lie}(E_8)$).

We will often want to consider restrictions of simple G-modules to reductive subgroups H of G. Where we write $V_1|V_2|\dots|V_n$ we list the composition factors V_i of an H-module. For a direct sum of H-modules, we simply write $V_1 + V_2$. Where a module is uniserial, we will write $V_1/\dots/V_n$ to indicate the socle and radical series: here the head is V_1 and the socle V_n. On rare occasions we also use V/W to indicate a quotient. It will be clear from the context which is being discussed.

Recall also the notion of a tilting module as a module V such that V and V^* both have good filtrations, i.e. filtrations with sections being modules of the form $\mathrm{H}^0(\lambda)$. Let us record in a lemma some key properties of tilting modules which we use:

Lemma 2.1. (i) *For each $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$ of high weight λ.*
(ii) *A direct summand of a tilting module is a tilting module.*
(iii) *The tensor product of two tilting modules is a tilting module.*
(iv) $\mathrm{Ext}^1_G(T(\lambda), T(\mu)) = 0$*; in particular* $\mathrm{H}^1(G, T(\lambda)) = 0$.

Proof. For (i), see [Don93, 1.1(i)]; (ii) follows from [Jan03, Proposition II.4.16]; (iii) is an important result of Mathieu, which was first proved (in a case-by-case fashion) in most cases by Donkin. [Don93, 1.2]; (iv) follows from [Jan03, II.4.13 (2)]. □

A common example of a tilting module arises from considering the case where V_1 is a an irreducible Weyl module, V_2 is a two-step uniserial Weyl module with submodule V_1 and quotient W. Then the module T which is uniserial with successive factors $V_1/W/V_1$ is an indecomposable tilting module.

Recall that a parabolic subgroup P of G has a Levi decomposition, $P = LQ$, so that

$$1 \to Q \to P \xrightarrow{\pi} L \to 1$$

is exact, where Q is the unipotent radical of the P. Recall also $L = L'Z(L)$ with the derived subgroup L' being semisimple. We have, in addition to the notion on G-complete reducibility from the introduction, another useful notion. Let H be a subgroup of a reductive group G. Then H is said to be G-irreducible or G-ir if H is in no proper parabolic subgroup of G.

3. Outline

Theorem 1 has two facets. The first proves that if $p \notin N(X, G)$ for $N(X, G)$ as defined in Corollary 2, then X is G-cr. The second proves the existence of the examples given in Table 1 and proves that they are non-G-cr.

The proof of the first part runs along the same lines as that of [LS96, Theorem 1]: Assume H is a closed, connected, simple non-G-cr subgroup of G. Then H is a subgroup of $P = LQ$. Let $\bar{H}$ be its image in L'. Almost all the time, $H \cap Q = \{1\}$ as group-schemes and so we have $HQ = \bar{H}Q$ and H is a complement to Q in $\bar{H}Q$. Then the possibilities for H are parameterised by $\mathrm{H}^1(\bar{H}, Q)$; in fact, in any case, the possibilities for H are parameterised by $\mathrm{H}^1(\bar{H}, Q^{[1]})$ (cf. Lemma 4.4).

From [ABS90], Q has a filtration $Q = Q_1 \geq Q_2 \geq Q_3 \dots$ with successive quotients being known (usually semisimple) L-modules. So if we have $\mathrm{H}^1(\bar{H}, (Q_i/Q_{i+1})^{[1]}) = 0$ for each i, then (by Lemma 4.4(ii)) $\mathrm{H}^1(\bar{H}, Q^{[1]}) = 0$ and H is conjugate to $\bar{H}$.

Now, for an exceptional algebraic group G over k of characteristic p and a irreducible root system X we consider possible embeddings $\bar{H} \leq L'$ where $\bar{H}$ is an L'-irreducible subgroup (which can be determined using Lemma 4.8 for classical groups and by working down through the maximal subgroups of Lemma 4.9 for exceptional groups). The composition factors V of the restrictions of the L-modules Q_i/Q_{i+1} are investigated, and then conditions for the vanishing of $\mathrm{H}^1(\bar{H}, V)$ found, for all relevant V. (Usually the dimensions of the composition factors are too small to admit non-vanishing of $\mathrm{H}^1(\bar{H}, V)$.)

With essentially one class of exceptions (again coming from Lemma 4.4), one can reduce to the case where V is of the form $L(\lambda) \otimes L(\mu)^{[1]}$ with $L(\lambda)$ non-trivial and restricted. There are any number of computer programs one can use to calculate the values of $\mathrm{H}^1(\bar{H}, V)$ where μ is 0.[3] Assuming $\mu = 0$, since the possible dimension of V is limited to the order of a subset of roots of G, finding all possible values of λ and p such that with $\mathrm{H}^1(\bar{H}, V) \neq 0$ is then a finite process.

For the proof of the second part of Theorem 1, for each of the remaining cases (where some composition factor V of $Q^{[1]}$ has $\mathrm{H}^1(\bar{H}, V) \neq 0$), we exhibit a non-$G$-cr subgroup H with the required root system over the required characteristic. In almost all cases we can give an example in a classical subgroup of G. Here is it easy to see when it is in a parabolic subgroup using Lemma 4.8. Where this is not possible, we can assert the existence of such a group using a cohomological argument.

[3]We use the data on Frank Lübeck's website which accompanies [Lüb01]. One can establish the structure of a Weyl module using the data there, thence use the dimension-shift [Jan03, II.4.14] to calculate $\mathrm{H}^1(\bar{H}, V)$

4. Preliminaries

One needs to be careful about the notion of complements in semidirect products of algebraic groups. These are treated systematically in [McN10]. We recall some of the main facts.

Definition 4.1 (cf. [McN10, 4.3.1])**.** Let $G = H \ltimes Q$ be a semidirect product of algebraic groups as in [Jan03, I.2.6]. A closed subgroup H' of G is a *complement* to Q if it satisfies the following equivalent conditions:

(i) Multiplication is an isomorphism $H' \ltimes Q \to G$.
(ii) $\pi_{H'} : H' \to H$ is an isomorphism of algebraic groups
(iii) As group-schemes, $H'Q = G$ and $H' \cap Q = \{1\}$.
(iv) For the (abstract) groups of k-points, one has $H'(k)Q(k) = G(k)$, $H'(k) \cap Q(k) = \{1\}$ and $\mathrm{Lie}(H') \cap \mathrm{Lie}(Q) = 0$.

Remark 4.2. See [Ste, §3.2] for a discussion. Note that [LS96] uses item (iv) above as its definition of a complement, omitting the last condition on Lie algebras.

For the following definition, suppose H is an algebraic group which acts morphically on Q. Then we denote the result of the action of $m \in H(k)$ on $q \in Q(k)$ by q^m; if H acts by conjugation, we have $q^m = m^{-1}qm$.

Definition 4.3. A regular map $\gamma : H \to Q$ of varieties is a 1-*cocycle* if $\gamma(nm) = \gamma(n)^m\gamma(m)$ for each $n, m \in H(k)$. We write $Z^1(H, Q)$ for the set of 1-cocycles.

For $\gamma, \delta \in Z^1(H, Q)$, we say $\gamma \sim \delta$ if there is an element $q \in Q(k)$ with $q^{-h}\gamma(h)q = \delta(h)$ for each $h \in H(k)$. We write $\mathrm{H}^1(H, Q)$ for the set of equivalence classes of 1-cocycles $Z^1(H, Q)/\sim$.

We recall some results from [Ste].

Lemma 4.4. (i) *The set of* 1*-cocycles* $Z^1(H, Q)$ *is in bijection with the set of complements to* Q *in* HQ*. Two cocycles are equivalent if the corresponding complements are conjugate by an element of* $H(k)$*.*

(ii) *Suppose* H *is a closed, connected, reductive subgroup of a parabolic subgroup* $P = LQ$ *of* G *and denote by* $\bar{H}$ *the subgroup of* L *given by the image of* H *under the quotient map* $\pi : P \to L$*.*

Then as abstract groups $H(k)$ *is a complement to* $Q(k)$ *in* $\bar{H}(k)Q(k)$*; and either (1)* H *is a complement to* Q *in* $\bar{H}Q$*; or (2)*

(a) $p = 2$*;*
(b) *There exists a component* SO_{2n+1} *of the semisimple group* $H/Z(H)^\circ$*;*
(c) *the image of this component in* $\bar{H}/Z(\bar{H})^\circ$ *is isomorphic to* Sp_{2n}*; and*
(d) *the natural module for* Sp_{2n} *appears in a filtration of* Q *by* $\bar{H}$*-modules.*

In case (2), H *corresponds to a cocycle* $\gamma \in Z^1(\bar{H}, Q^{[1]})$ *such that* $[\gamma]$ *has no preimage in* $\mathrm{H}^1(\bar{H}, Q)$ *under the inclusion* $\mathrm{H}^1(\bar{H}, Q) \to \mathrm{H}^1(\bar{H}, Q^{[1]})$*. Moreover there is a bijection between the set of conjugacy classes of closed, connected, reductive subgroups* H *of* $\bar{H}Q$ *with* $HQ/Q \cong \bar{H}$ *and the set* $\mathrm{H}^1(\bar{H}, Q^{[1]})$*.*

(iii) *In a filtration of a unipotent algebraic* H*-group* Q *by* H*-modules (such as that given by Lemma 4.10) if each composition factor* V *satisfies* $\mathrm{H}^1(H, V) = 0$ *then* $\mathrm{H}^1(H, Q) = 0$*.*

Proof. (i) is [Ste, Lemma 3.2.2]; (ii) is [Ste, Lemma 3.6.1]. For (iii), such a filtration is 'sectioned' in the sense of [Ste, Definition 3.2.7] using [Ste, Lemma 3.2.8]. Now one uses the exact sequence of non-abelian cohomology in [Ste, 2.1(i)] inductively. (See the discussion in [Ste, §3.2] on the validity of this sequence for regular (Hochschild) cohomology.) □

It follows from the next lemma that in almost all cases the cohomology group $\mathrm{H}^1(G,V)$ for a semisimple algebraic group G satisfies $\mathrm{H}^1(G,V) \cong \mathrm{H}^1(G,V^{[1]})$. This fact allows us to reduce our considerations to simple modules with non-trivial restricted parts.

Lemma 4.5. *Let G be a simple algebraic group and V a simple G-module. Then $\mathrm{H}^1(G,V) \cong \mathrm{H}^1(G,V^{[1]})$ unless G is Sp_{2n} and V is its $2n$-dimensional natural module.*

Moreover $\mathrm{H}^1(G,V^{[1]})$ is isomorphic to its generic cohomology $\mathrm{H}^1_{gen}(G,V)$.[4]

Proof. See [Jan03, II.12.2, Remark], [CPSvdK77, 7.1] and [LS96, 1.3]. □

There are many papers finding the dimensions of $\mathrm{Ext}^n_H(L,M)$ with H of low rank and L,M simple. Taking $L=k$, one gets the following result. For completeness, we have included more data than necessary for our purposes.

Lemma 4.6. *Let L be a simple module for a simple algebraic group H where H is one of SL_2, SL_3, Sp_4 over an algebraically closed field of any characteristic p; G_2 for $p=2,3$ or $p \geq 13$; or SL_4, Sp_6 or Sp_8 when $p=2$. Then $\mathrm{H}^1(H,L)$ is at most one-dimensional, and is non-zero if and only if L is a Frobenius twist of one of the modules in the following table.*

In the table we also give some useful dimension data in certain cases.

H	*p*	*L*	*dim L*
SL_2	*any*	$L(p-2)\otimes L(1)^{[1]}$	$2p-2$
SL_3	$p\geq 3$	$L(p-2,p-2)$	$(p-1)^3-1$
		$L(1,p-2)\otimes L(1,0)^{[1]}$	54 *for* $p=5$
		$L(p-2,1)\otimes L(0,1)^{[1]}$	54 *for* $p=5$
	$p=2$	$L(1,0)\otimes L(1,0)^{[1]}$	9
		$L(0,1)\otimes L(0,1)^{[1]}$	9
Sp_4	$p\geq 5$	$L(0,p-3)$	
	$p\geq 3$	$L(2,p-2)\otimes L(0,1)^{[1]}$	125 *for* $p=3$
		$L(p-2,1)\otimes L(1,0)^{[1]}$	≥ 64
	$p=2$	$L(1,0)^{[1]}$	4
		$L(0,1)$	4
G_2	$p\geq 13$	$L(p-5,0)$	
		$L(p-2,1)\otimes L(1,0)^{[1]}$	
		$L(4,p-4)\otimes L(1,0)^{[1]}$	
		$L(3,p-2)$	
		$L(3,p-2)\otimes L(0,1)^{[1]}$	
	$p=3$	$L(1,1)$	49
		$L(0,1)\otimes L(1,0)^{[1]}$	49
	$p=2$	$L(1,0)$	6
		$L(0,1)\otimes L(1,0)^{[1]}$	84

H	*p*	*L*	*dim L*
SL_4	$p=2$	$L(1,0,1)$	14
		$L(0,1,0)\otimes L(1,0,0)^{[1]}$	24
		$L(0,1,0)\otimes L(0,0,1)^{[1]}$	24
		$L(1,0,1)\otimes L(0,1,0)^{[1]}$	84
Sp_6	$p=2$	$L(1,0,0)^{[1]}$	6
		$L(1,0,1)$	48
		$L(0,1,0)\otimes L(1,0,0)^{[1]}$	84
Sp_8	$p=2$	$L(1,0,0,0)^{[1]}$	8
		$L(0,1,0,0)$	26
		$L(1,0,1,0)$	246
		$L(1,0,1,0)\otimes L(0,1,0,0)$	6396
		$L(1,0,1,0)\otimes L(0,1,0,0)$	6396
		$L(0,1,0,1)$	416

[4]The generic cohomology is the group $\lim_{r\to\infty}\mathrm{H}^1(G(p^r),V)$, where $G(q)$ is the fixed points $G(k)^{F^r}$ of G under the rth power of the Frobenius morphism $F: G\to G$. It is achieved by all $r\geq r_0$ for some $r_0=r_0(\Phi,V)$. It is also (see [Wan85]) the value of $\mathrm{H}^1(G(k),V(k))$, where $G(k)$ is the algebraic group viewed as an abstract group.

Proof. These are special cases from [Cli79] for SL_2, [Yeh82, 4.2.2] for SL_3, [Ye90] for Sp_4, $p \geq 3$, [LY93] for G_2 ($p \geq 13$), [Sin94, Proposition 2.2] for Sp_4 ($p = 2$), [Sin94, Proposition 3.4] for G_2 ($p = 3$), [DS96, II.§2.1.6, II.§2.2.4, II.§3.3.6, III.§2.2.4] for SL_4, Sp_6, G_2 and Sp_8, respectively, when $p = 2$. □

Lemma 4.7. *Let $G = G_2$ over a field of characteristic 5 and let L be a simple module for G with $\mathrm{H}^1(G, L) \neq 0$. Then $\dim L > 97$.*

Proof. One reduces to the case where the restricted part of L is non-trivial using Lemma 4.5. Start with the case that L is restricted. One can use the data from [Lüb01] to establish that all Weyl modules of dimension less than 97 are irreducible. But [Jan03, II.4.13] shows that $\mathrm{H}^1(G, L(\lambda)) \cong \mathrm{H}^0(G, \mathrm{H}^0(\lambda)/\mathrm{Soc}_G(\mathrm{H}^0(\lambda))) = 0$.

If M is not restricted, then it is isomorphic to $M_1 \otimes M_2^{[1]}$ for M_1 restricted and M_2 non-trivial. The lowest dimensions M_1 and M_2 can have is 7 each, the next is 14, but $14 \times 7 = 98 > 97$, so we conclude $M_1 = L(1, 0)$ and $M_2 = L(1, 0)^{[r]}$. Now by [LS96, 1.15] (or the linkage principle), one gets $\mathrm{H}^1(G, M) = 0$. □

The next lemma establishes L'-irreducible embeddings $\bar{H} \leq L'$ when L' is of classical type: it helps determine when a subgroup H is in a parabolic of a classical subgroup M of G.

Lemma 4.8 ([LS96, p32-33])**.** *Let G be a simple algebraic group of classical type, with natural module $V = V_G(\omega_1)$, and let H be a G-irreducible subgroup of G.*

(i) *If $G = A_n$, then H acts irreducibly on V*
(ii) *If $G = B_n$, C_n, or D_n with $p \neq 2$, then $V|H = V_1 \perp \cdots \perp V_k$ with the V_i all non-degenerate, irreducible, and inequivalent as X-modules.*
(iii) *If $G = D_n$ and $p = 2$, then $V|H = V_1 \perp \cdots \perp V_k$ with the V_i all non-degenerate, $V_2|H, \ldots, V_k|H$, irreducible and inequivalent, and if $V_1 \neq 0$, H acting on V_1 as a B_{m-1}-irreducible subgroup where $\dim V_1 = 2m$.*

On a couple of occasions we need to know the reductive maximal subgroups of exceptional algebraic groups. A subgroup listed as $\tilde{X}$ indicates it is a subsystem subgroup corresponding to short roots.

Lemma 4.9 (c.f. [LS04, Theorem 1])**.** *Let G be an exceptional group not of type E_8 and let M be a closed, connected, reductive maximal subgroup of G without factors of type A_1. Then M is in the following list*

G	*Subsystem M*	*Non-subsystem M*
G_2	A_2, $\tilde{A}_2$ $(p = 3)$	
F_4	B_4, $C_4(p = 2)$, $A_2\tilde{A}_2$	G_2 $(p = 7)$
E_6	A_2^3	A_2 $(p \neq 2, 3)$, G_2 $(p \neq 7)$, C_4 $(p \neq 2)$, F_4, A_2G_2.
E_7	A_7, A_2A_5	A_2 $(p \geq 5)$, G_2C_3

A filtration for unipotent radicals of parabolics by explicitly calculable L-modules is given in [ABS90, Theorem 2 (& Remark 1)]; to find the isomorphism types of the composition factors is a simple calculation using the root system of G. We use this theorem on occasion in the sequel; see [Ste13, 4.3.1, Proof] for examples. Summarising the results for our situation, we get:

Lemma 4.10 ([LS96, 3.1]). *Let $G = E_6$, E_7 or E_8 and let $P = LQ$ be a parabolic subgroup of G. There is a filtration of $Q = Q_1 \geq Q_2 \geq \dots Q_n = 1$ for which each Q_i is normal in P, each Q_i/Q_{i+1} is a vector group, and there is an L'-isomorphism of Q_i/Q_{i+1} to a simple L'-module. If L_0 is a simple factor of L', the possible high weights λ of the L_0-composition factors, and their dimensions are as follows:*

(i) $L_0 = A_n$: $\lambda = \omega_j$ or ω_{n+1-j} $(j = 1, 2, 3)$, dimensions $\binom{n+1}{j}$;
(ii) $L_0 = D_n$: $\lambda = \omega_1$, ω_{n-1} or ω_n, dimensions $2n$, 2^{n-1} and 2^{n-1} resp.;
(iii) $L_0 = E_6$: $\lambda = \omega_1$ or ω_6, dimension 27 each;
(iv) $L_0 = E_7$: $\lambda = \omega_7$, dimension 56.

Corollary 4.11. *With the hypotheses of the lemma, let V be an L'-composition factor of Q and suppose L' does not contain a component of type A_1. Then (i) either $\dim V \leq 60$ or $G = E_8$, $L' = D_7$ and V is a spin module for L' of dimension 64; (ii) if $G = E_7$, $\dim V \leq 35$; if $G = E_6$, $\dim V \leq 20$.*

Proof. If L' is itself simple, the statement follows from the lemma. Also, if $G = E_6$ then there are only 36 positive roots, so part (i) is clear. If $G = E_7$ then there are only 63; Levi subgroups of rank no more than 2 are easily ruled out, and for any other, $\dim Q \leq 60$. So for (i) we may assume $G = E_8$ with L' non-simple. The possibilities for L' are A_2A_2, A_2A_3, A_2A_4, A_3A_3, A_3A_4, A_2D_4 and A_2D_5. Since V is simple, it must be a tensor product of simple modules for the two factors, with the simple modules occurring in the lemma. One checks that the highest dimension possible for this is when $L = A_3A_4$, $V = L(\omega_2) \otimes L(\omega_2)$ with $\dim V = 6 \times 10 = 60$.

For the second part, if $G = E_7$ and L' is simple this follows from Lemma 4.10, the largest case occurring when $L' = A_6$. If L' is not simple, then it is A_4A_2, A_3A_2 or A_2A_2. Then the largest possible dimension comes from the first option and is at most $10 \times 3 = 30 \leq 35$-dimensional. If $G = E_6$ it is easy to see the largest possible dimension of V occurs when $L' = A_5$ and $V = L(\omega_3)$, with $\dim V = 20$. □

5. Proof of Theorem 1

In [Ste10] and [Ste13] we found all semisimple non-G-cr subgroups of G where G is G_2 and F_4 respectively. So the result follows for these cases. It remains to deal with the cases $G = E_6$, E_7 and E_8. We hone [LS96, Theorem 1] to show that if H is a closed, connected, simple subgroup of G with root system X and p is not in our list $N(X, G)$ in Corollary 2, then H is G-cr. Then we check that the examples given in Table 1 are indeed non-G-cr.

$p \notin N(X, G)$ implies that H is G-cr. *Proof of the first statement of Theorem 1:*

Looking for a contradiction, we will assume H is a closed, connected, simple, non-G-cr subgroup of G; then we can make the following assumption, using Lemma 4.4:

> *We have $H \leq P = LQ$ with $\bar{H}$ being L-ir, and either (i) H is a complement to Q in $\bar{H}Q$ and there exists an $\bar{H}$-composition factor V of Q with $\mathrm{H}^1(\bar{H}, V) \neq 0$; or (ii) $p = 2$, $H = \mathrm{SO}_{2n}$, $\bar{H} = \mathrm{Sp}_{2n}$ and $V = L(\omega_1)$ appears as an $\bar{H}$-composition factor of Q.*

Since [LS96, Theorem 1] deals with the case that p is greater than any of the numbers in the table in Corollary 2, we need only deal with those we have struck out. Thus, the cases to consider are

$$(X,G,p) \in \{(B_2,\bullet,3),\ (G_2,\bullet,5),\ (G_2,E_6,3),\ (G_2,E_7,3),\ (A_2,\bullet,5),\ (A_3,E_6,2),\\ (B_4,E_6,2),\ (B_4,E_7,2),\ (D_4,E_6,2),(C_3,\bullet,2),\ (C_4,E_6,2)\},$$

where $\bullet$ can be replaced by E_6, E_7 or E_8.

By Corollary 4.11 the largest possibility for the dimension of V occurs when $G = E_8$, $L' = D_7$ and V has dimension 64. By Lemma 4.7, there is no such V when $H = G_2$ and $p = 5$. This rules out the case $\underline{(G_2,\bullet,5)}$.

Suppose H is of type B_2 and $p = 3$. By Lemma 4.6, V has dimension at least 64 and so by Corollary 4.11, we have $G = E_8$, $L' = D_7$ and V is a spin module. Since $\bar{H}$ is D_7-irreducible, it must act on the natural module V_{14} for L' as specified in Lemma 4.8. Checking [Lüb01], one finds the non-trivial simple restricted representations of dimension no more than 14 up to Frobenius twists are $L(0,1)$, $L(1,0)$, $L(0,2)$, $L(2,0)$ with dimensions 4, 5, 10 and 14, respectively. But $L(0,1)$ is the natural representation for Sp_4, thus carries a symplectic structure, which cannot be non-degenerate. Hence $V_{14}|\bar{H} = L(2,0)$; moreover, as $L(2,0)$ is an irreducible Weyl module when $p = 3$, the embedding $\bar{H} \hookrightarrow L'$ can be seen as the reduction mod p of an embedding $\bar{H}_{\mathbb{Z}} \hookrightarrow L'_{\mathbb{Z}}$. Now [LS96, Proposition 2.12] gives that $V_{\mathbb{Z}}|\bar{H}_{\mathbb{Z}}$ is the irreducible Weyl module $V(1,3)$. Using [Lüb01] one can calculate the composition factors of a reduction mod 3 of this module; one sees that $V|\bar{H}$ has composition factors $L(1,3)|L(2,1)|L(0,1)$. Since none of these modules appears in Lemma 4.6, this rules out $(X,G,p) = \underline{(B_2,\bullet,3)}$.

By Corollary 4.11 the largest possibility for the dimension of V when $G = E_7$ is 35; when $G = E_6$ it is 20. Then dimension considerations using Lemma 4.6 also rule out $(X,G,p) = \underline{(A_2,E_6,5),\ (A_2,E_7,5),}$ $\underline{(G_2,E_6,3) \text{ and } (G_2,E_7,3)}$.

For $\underline{(A_2,E_8,5)}$, since V has dimension at least 54 by Lemma 4.6, we must have $L' = E_7$, D_7 or A_7; but simple E_7-modules are self-dual, so if one of the possibilities for V coming from Lemma 4.6 appeared as a composition factor of $Q|\bar{H}$, so would its dual, but together these are bigger than $\dim Q$. Thus we may assume that $L' = A_7$ or D_7. If $L' = A_7$ then since V must have dimension at least 54, we must have V a composition factor of the L'-module $L(\omega_3) = \bigwedge^3(L(\omega_1))$. Since $\bar{H}$ is L'-ir, $\bar{H}$ must act irreducibly on the natural 8-dimensional module V_8 for L'. A check of [Lüb01] forces $V_8|\bar{H} = L(1,1)$. However, $\bigwedge^3 L(1,1)$ has highest weight $(2,2)$ in the dominance order (it also has high weights $(3,0),(0,3),(1,1) < (2,2)$), but the weights appearing in Lemma 4.6 are all higher than these in the dominance order.

We claim there are no D_7-irreducible embeddings of A_2. From [Lüb01] the simple modules of dimension no more than 14 up to duals and Frobenius twists are $L(0,3)$ (dim 10), $L(1,0) \otimes L(0,1)^{[r]}$ (dim 9), $L(1,0) \otimes L(1,0)^{[1]}$ (dim 9), $L(1,1)$ (dim 8), $L(2,0)$ (dim 6), $L(1,0)$ (dim 3) and the trivial module k, where $r > 0$. Any direct factor of $V_{14}|A_2$ must be non-degenerate; hence self-dual, by Lemma 4.8. This rules out all but $L(1,1)^{[r]}$ and k as factors. But then there must be a 6-dimensional trivial submodule, contradicting Lemma 4.8. This completes the case $(A_2,E_8,5)$.

Consider next the case $(X,G,p) = \underline{(A_3,E_6,2)}$. By Corollary 4.11 we have $\dim V \leq 20$ so Lemma 4.6 shows that V must be 14-dimensional; this forces $L' = D_5$ or A_5. Examining low dimensional representations for A_3, it is easy to see using Lemma 4.8 that there is no D_5-irreducible embedding $\bar{H} \hookrightarrow D_5$, so we must have $\bar{H} \hookrightarrow L' = A_5$ by $V_6|\bar{H} = L(0,1,0)$. Here, Q has factors $L(\omega_3) = \bigwedge^3(V_6)$ and a trivial module. Now $L(0,1,0)$ has weights $\pm(0,1,0), \pm(1,0,-1), \pm(1,-1,1)$, so $\bigwedge^3 L(0,1,0)$ has dominant weights $(0,0,2)$, $(2,0,0)$ and $(0,1,0)$ (the sums of distinct triples of those in V_6). These do not appear as the high weights of simple modules in Lemma 4.6. Thus $\mathrm{H}^1(\bar{H}, \bigwedge^3 L(0,1,0)) = 0$ and this case is ruled out.

In case $\underline{(B_4, E_6, 2)}$ we must have $\bar{H} \leq D_5$, with Q having the structure of a spin module for $L' = D_5$. But then $Q|\bar{H} := V \cong L(0001)$ using [LS96, 2.7] is a spin module for $\bar{H}$ with $V(0001) = L(0001)$. So $\mathrm{H}^1(B_4, V) = 0$ and this case is ruled out.

If $(X, G, p) = \underline{(D_4, E_6, 2)}$ then clearly $L' = D_4$. But then from Lemma 4.10 we see that any module appearing as an L'-composition factor of Q is $L(\omega_i)$ for $i \in 1, 3, 4$. All of these modules are irreducible Weyl modules, hence satisfy $\mathrm{H}^1(L', L(\omega_i)) = 0$. Thus this case is ruled out.

For $\underline{(B_4, E_7, 2)}$, we could have $L' = D_5$ or E_6. If $L' = D_5$ then by Lemma 4.10 its composition factors on Q are either spin modules or natural modules. The arguments in case $(B_4, E_6, 2)$ apply to show that $\mathrm{H}^1(\bar{H}, V) = 0$ if V is the restriction of a spin module, so we can assume V is the restriction of the natural module. The subgroup B_n is the stabiliser of a non-degenerate 1-space of the natural module of D_{n+1}; when $p = 2$, it acts as $T(\omega_1) \cong k/L(\omega_1)/k$. Thus by Lemma 2.1, $\mathrm{H}^1(\bar{H}, V) = 0$ in this case also.

If $L' = E_6$ then by Lemma 4.9 we must have $B_4 \leq F_4 \leq E_6$. We claim that this is conjugate in E_6 to the subgroup $B_4 \leq D_5 \leq E_6$ (hence is not L'-ir). For this, take the $M = D_5$ subsystem corresponding to roots $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_0\}$, where α_0 is the longest root. Let τ denote the standard graph automorphism transposing $x_{\pm\alpha_1}(t)$ with $x_{\pm\alpha_5}(t)$, $x_{\pm\alpha_3}(t)$ with $x_{\pm\alpha_4}(t)$ and fixing the other simple root groups. Since the subsystem subgroup M is stable under τ, the fixed points $M_\tau \cong B_4$ of M are contained in the $(E_6)_\tau \cong F_4$. Hence our $B_4 \leq D_5$ is also a subgroup of F_4, hence conjugate to any other; this proves the claim. This rules out the case $(B_4, E_7, 2)$.

Since there are no embeddings of a subgroup of type C_4 into any proper Levi of E_6, the case $\underline{(C_4, E_6, 2)}$ is ruled out too.

Lastly take case $\underline{(X, G, p) = (C_3, \bullet, 2)}$. We need an L'-ir embedding of $\bar{H}$ in L' and an H-composition factor V of Q with $\mathrm{H}^1(H, V) \neq 0$. We will see this is impossible. Note that under our standing assumption at the beginning of this section, V is not $L(100)$ since then H would be of type B_3. From Lemma 4.9, the possibilities for an L'-irreducible embedding of $\bar{H}$ in $L' = E_6$, or $L' = E_7$ require C_3 to be B_4-ir, C_4-ir, or A_7-cr. This cannot happen: by Lemma 4.8 and [Lüb01] the only possibility for the action of $\bar{H}$ on the natural module for C_4 or A_7 is $V_8|\bar{H} = L(001)$. This representation, ρ say, admits a quadratic form. Now the Frobenius twist of this, $\rho \circ F$ can be written as $\rho' \circ \tau$, where $\tau : C_3 \to B_3$ is the exceptional isogeny and $\rho' : B_3 \to D_4$ is the spin representation (one checks the high weights match). Thus the image of $\bar{H}$ is of type B_3. Now by Lemma 4.4 any non-G-cr subgroup with image $\bar{H} \cong B_3$ is also of type B_3, a contradiction.

Thus L' is of classical type and the possibilities are $L' = A_5$ and $L' = D_7$ (if $G = E_8$). If $G = E_6$, L' has to be type A_5, with Q having L'-composition factors k and $L(\omega_3) = \bigwedge^3 L(\omega_1)$. Hence Q has $\bar{H}$-composition factors which are k or in $\bigwedge^3 L(100)$ (which itself has composition factors $L(001)|L(100)^2$). Since these do not appear in Lemma 4.6 this case is ruled out. Similarly if $G = E_7$ or E_8 with $L' = A_5$, we must also consider the restrictions of $L(\omega_2)$ and its dual, $L(\omega_4)$ to $\bar{H}$. These are $\bigwedge^2 L(100) \cong \bigwedge^4 L(100)$ which also contain no composition factors with non-trivial H^1. If $G = E_8$ and $L' = D_7$, then the action of C_3 on $L(010)$ gives an irreducible embedding of C_3 into L'. The natural module $V_{D_7}(\lambda_1)|C_3 = L(010)$ and $\mathrm{H}^1(\bar{H}, L(010)) = 0$ so we must consider the restriction of the spin module $V_{\lambda_7}|\bar{H}$. The latter is $L(110)$ by [LS96, 2.12], but $\mathrm{H}^1(\bar{H}, L(110)) = 0$ by Lemma 4.6 a final contradiction.

This completes the proof of the first statement of Theorem 1.

$p \in N(X, G)$ **implies the existence of a non-G-cr subgroup H with root system X.** *Establishing the entries in Table 1:*

The examples given in Table 1 when $G = G_2$ and F_4 were shown already in [Ste10, Theorem 1] and [Ste13, Theorem 1(A)(B)] to be non-G-cr, so we need only deal with the cases $G = E_6$, E_7 and E_8.

The proof of many of these cases is similar. Let $H = E(X, G, p)$ for one of the examples in Table 1. We locate H within a parabolic subgroup of G and establish the embedding $\bar{H} \leq L$. Next we take a low-dimensional faithful G-module V and calculate the restriction to H and $\bar{H}$ of this G-module; in all cases under consideration these will be non-isomorphic. Thus we can conclude that since $V|H \ncong V|\bar{H}$, H is not even $GL(V)$-conjugate to $\bar{H}$, let alone G-conjugate to $\bar{H}$. Further, in all the cases we consider where we have given a subgroup H that is non-F_4-cr, we find it is also non-E_r-cr for $6 \leq r \leq 8$ using the embeddings $F_4 \leq E_6 \leq E_7 \leq E_8$; for this the following result of J.-P. Serre is very helpful.

Proposition 5.1 ([Ser05, Prop. 3.2])**.** *Let H be a subgroup of a Levi subgroup L of a parabolic subgroup of G. Then H is L-cr if and only if H is G-cr.*

Thus once we have established that a subgroup H is non-E_6-cr, it is also non-E_7-cr. Similarly any non-E_7-cr subgroup is also non-E_8-cr. Together with the results from [Ste10] and [Ste13], this reduces the list of cases $E(X, G, p)$ whose existence and non-G-cr-ness we must establish to the following:

$E(\bullet, E_6, \bullet)$ (all 8 cases from Table 1)	$E(A_1, E_7, 7)$
$E(G_2, E_7, 7)$	$E(C_4, E_7, 2)$
$E(D_4, E_7, 2)$	$E(B_2, E_8, 5)$
$E(G_2, E_8, 3)$	$E(A_3, E_7, 2)$
$E(C_3, E_8, 3)$	$E(B_4, E_8, 2)$

<u>$H = E(A_2, E_6, 2)$ or $E(B_2, E_6, 2)$ or $E(A_1, E_6, 2)$.</u> For these examples, we have given a non-completely reducible action of H on an $(n+1)$-dimensional space, putting H in a type A_n Levi subgroup L of E_6. Thus each is non-L-cr and thus by Proposition 5.1 is non-G-cr. (For the last case, observe that a long A_2 Levi subsystem subgroup of F_4 is still a long A_2 Levi subsystem subgroup considered as a subgroup of E_6.)

<u>$H = E(A_1, E_6, 3)$.</u> By [Ste13, Thm. 4.4.5(iv)], H is in a B_3-parabolic of F_4, hence a D_4-parabolic of E_6; in fact, $\bar{H} \leq A_1^2\tilde{A}_1 \leq B_3$ by $V_7|\bar{H} = L(1) \otimes L(1)^{[1]} + L(2)$. Thus $\bar{H} \leq A_1^4 \leq D_4$. On the other hand $H \leq A_2^3 \leq E_6$. From [LS04, Table 10.2] one gets $V_{27}|F_4 = T(\omega_4) \cong k/L(0001)/k \cong k/V(\omega_4)$ and thence one can use the restrictions $V(\omega_4)|H, \bar{H}$ from [Ste13, Table 5.1] to compare the restrictions $V_{27}|H, \bar{H}$ and conclude they are different.

<u>$H = E(A_1, E_6, 5)$.</u> The A_1-module $T(8) \cong k/L(8)/k$ is a direct summand of the 25-dimensional module $L(4) \otimes L(4) = T(4) \otimes T(4)$ by Lemma 2.1. The two tensor factors here admit orthogonal forms, so the tensor product does too. Hence we get a subgroup of type A_1 in GL_{25} which is

actually contained in SO_{25}. Indeed as the 10-dimensional direct factor $T(8)$ is the unique such, the duality must preserve this factor. Hence we get an $A_1 \leq \mathrm{SO}_{10} \times \mathrm{SO}_{15}$ and so projecting to the first orthogonal group, we get $H \leq \mathrm{SO}_{10}$ with $V_{10}|H = T(8)$.

Now, by Lemma 4.8 this subgroup is in a parabolic of SO_{10}. Considering dimensions of composition factors of Levi subgroups of D_5 acting on the natural module shows that H must in fact be in a D_4-parabolic of D_5 with $\bar{H}$ being D_4-irreducible and $V_8|\bar{H} = L(8)$. By [Ste13, 5.1] we have $V_{27}|D_4 = L(\omega_1)+L(\omega_3)+L(\omega_4)+k^3$. We wish to restrict this further to get $V_{27}|\bar{H}$. Note that since $L(8) \cong L(3)\otimes L(1)^{[1]}$, we have $\bar{H} \leq \mathrm{Sp}_4 \times \mathrm{Sp}_2 \leq D_4$. Let $\bar{H}'$ (resp. $\bar{H}''$) denote the projection of the $\bar{H}$ in the first (resp. second) factor. We have $\bar{H}'$ in a Levi factor of type A_3 where A_3 corresponds, say, to the nodes $1, 2, 3$ of the Dynkin diagram of D_4. Apply a graph automorphism τ to D_4 so that τA_3 corresponds to nodes 2, 3 and 4 of the Dynkin diagram. Note that if J is any subgroup of D_4 then $V_{27}|J = V_{27}|\tau J$, so it suffices to find the restrictions of $L_{D_4}(\omega_i)$ to $\tau\bar{H}$. The subgroup τA_3 is the obvious $D_3 \leq D_4$, so we have $L_{D_4}(\omega_1)|\tau A_3 = L(010)+k^2$, thus $L_{D_4}(\omega_1)|\tau\bar{H}' = \bigwedge^2(L(3))+k^2 = L(4)+k^3$, with $L_{D_4}(\omega_1)|\tau\bar{H} = L(4)+L(1)^{[1]}+k$ or $L_{D_4}(\omega_1)|\tau\bar{H} = L(4)+L(2)^{[1]}$. As $\bar{H}$ (hence $\tau\bar{H}$) is D_4-ir, it must be the latter, since $L(1)^{[1]}$ carries a symplectic form. Also from [LS96, 2.7] one sees that $L_{D_4}(\omega_3)|\tau A_3 \cong L_{D_4}(\omega_4)|\tau A_3 = L(100)+L(001)$ and so $L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\tau\bar{H}' = L(3)^2$. Thus $L_{D_4}(\omega_3) \cong L_{D_4}(\omega_4)|\tau\bar{H} = L(3)\otimes L(1)^{[1]} = L(8)$.

Finally we conclude that $V_{27}|\bar{H} = L(8)^2 + L(4) + L(2)^{[1]} + k^3$. In particular, $\bar{H}$ acts semisimply. On the other hand $V_{27}|D_5 = L(\omega_1) + L(\omega_5) + k$. But H does not act semisimply on $V_{10} = L(\omega_1)$. So $\bar{H}$ is not $GL(V_{27})$-conjugate to H, so neither is it E_6-conjugate to H. Thus it is non-E_6-cr.

<u>$H = E(G_2, E_6, 2)$ and $E(B_3, E_6, 2)$.</u> These cases are both similar as these subgroups are both non-F_4-cr subgroups of E_6 with the non-complete-reducibility detected on the F_4-module V_{26}. We give the latter as an example. We have $H \leq \tilde{D}_4 \leq C_4 \leq F_4 \leq G = E_6$. By considering the restriction of V_{nat} for C_4 to H, we see that H is in a C_3-parabolic of C_3 hence $\bar{H}$ is C_3. (Thus $\bar{H}$ is in fact the fixed point subgroup of $L' = A_5$ under the standard non-trivial graph automorphism τ of G.) Since $V_{27}|F_4 = L_{F_4}(\omega_1)+k$ we use [Ste13, 5.1] to get $V_{27}|\bar{H} = L(100)^2+L(010)+k$. In particular, $V_{27}|\bar{H}$ is semisimple. On the other hand, by [Ste13, 5.1], $V_{27}|H$ is $T(010)+k$, hence not semisimple. We conclude that H is non-G-cr.

<u>$H = E(A_2, E_6, 3)$.</u> As before, let τ denote the standard non-trivial graph automorphism of G. If G_τ denotes the fixed points of τ in G, we have $G_\tau \cong F_4$ such that the root groups corresponding to simple short roots are contained in the subsystem (of type A_2A_2) determined by the nodes in the Dynkin diagram of G on which τ acts non trivially. Thus H is contained in $A_2\tilde{A}_2 \leq F_4$ by $x \mapsto (x, x)$. It is shown in [Ste13, 4.4.1, 4.4.2] that this subgroup is in a B_3-parabolic of F_4 with $V_7|\bar{H} = L(11)$.

In [Ste13, 5.1] the restrictions of the F_4-module $V_{26} = V(0001) \cong L(0001)/k$ to H and $\bar{H}$ are calculated. Using this together with $V_{27}|F_4 = T(0001) \cong k/L(0001)/k$ we see that $V_{27}|\bar{H}$ cannot be the same as $V_{27}|H$: the former is an extension by the trivial module of $V_{26}|\bar{H} = L(11)^3 + k^5$ where the resulting module is self-dual, so must be $L(11)^3 + k^6$; the latter is a self-dual extension by the trivial module of $V_{26}|H = T(11) + T(11) + L(11)/k$, so must be $T(11)^3$.

This concludes all the 8 cases $(X, G, p) = (\bullet, E_6, \bullet)$.

<u>$H = E(A_1, E_7, 7)$.</u> Since $H \leq A_7$ with the natural module $V_8|H = L(1)^{[1]}/L(5)$, H is clearly in an A_5A_1-parabolic of A_7, with $V_8|\bar{H} = L(5)+L(1)^{[1]}$. Since $V_{56}|A_7 = L(\omega_2)+L(\omega_6) = \bigwedge^2 V_8 + \bigwedge^2 V_8^*$,

one sees that $V_{56}|\bar{H}$ will contain two direct factors isomorphic to $L(5) \otimes L(1)^{[1]} \cong L(12)$. Moreover these account for the only composition factors isomorphic to $L(12)$.

On the other hand, $\bigwedge^2 V_8|H \cong \bigwedge^2 V(7)$ is a submodule of $M := V(7) \otimes V(7)$. Since M is the tensor product of two Weyl modules, it also has a Weyl filtration; computing the character reveals that it has sections $V(14), V(12), V(10), V(8), V(6), V(4), V(2), V(0)$. (We are not interested in the order, but they do appear in the order given, with $V(14)$ a submodule, and $V(0)$ a quotient.)

The structure of these Weyl modules is well, known: if $p \leq n \leq 2p-2$ then $V(n) = L(n)/L(2p-2-n)$, while if $n < p$ then $V(n) = L(n)$. One sees that the composition multiplicity $[M : k] = 2$, with $[V(0) : k] = 1$ and $[V(12) : k] = 1$. One can also calculate the character of $\bigwedge^2 V_8$: it has composition factors $L(12)|L(8)|L(4)^2|k^2$.

We have $\mathrm{Hom}_H(k, V(7) \otimes V(7)) \cong \mathrm{Hom}_H(H^0(7), V(7)) = k$. So of the two trivial composition factors in M, only one is a submodule. The only composition factor in M extending k is $L(12)$; so it follows that $k/L(12)$ is a subquotient of M. But since $\bigwedge^2 V_8$ is a submodule of M containing all the trivial composition factors, it must contain this subquotient. Thus the composition factor $L(12)$ does not appear as a direct factor of $\bigwedge^2 V_8$. Therefore $V_{56}|H \not\cong V_{56}|\bar{H}$ and H is thus non-G-cr.

$\underline{H = E(A_3, E_7, 2)}$ We calculate the restriction to H of the A_7-module

$$\begin{aligned} L(\omega_2)|H \cong \bigwedge^2 L(\omega_1)|H &= \bigwedge^2 (L(100) + L(001)) \\ &= L(010) + L(010) + L(100) \otimes L(001) \\ &= L(010)^2 + T(101) \end{aligned}$$

where $T(101) \cong k/L(101)/k$ is the tilting module. By [LS96, 2.3], we have $V_{56}|A_7 = L(\omega_2) + L(\omega_6) = L(\omega_2) + L(\omega_2)^*$ so that $V_{56}|H = L(010)^4 + T(101)^2$.

Since all A_3A_3 subsystem subgroups of E_7 are conjugate (this is implied, for instance, by [LS96, Table 8.2]) and there is one such in D_6, we conclude $H \leq D_6$ with $V_{12}|H = L(010) + L(010)$. The dimensions of the composition factors imply that H is in an A_5-parabolic of D_6, with $\bar{H} \leq A_5$ embedded A_5-irreducibly via $V_6|\bar{H} = L(010)$.

Assume, looking for a contradiction, that H is conjugate to $\bar{H}$. By [LS96, 2.3] we have $V_{56}|D_6 = L(\omega_1)^2 + L(\omega_6)$. There are two possibilities for the A_5-subgroup containing $\bar{H}$, up to conjugacy; according to [LS96, 2.6] we have either $L_{D_6}(\omega_6)|A_5 = L(\omega_1) + L(\omega_1)^* + L(\omega_3)$ or $L_{D_6}(\omega_6)|A_5 = L(\omega_2) + L(\omega_2)^* + k^2$. The first of these possibilities would imply an A_3-submodule $\bigwedge^3(L(010)) \leq V_{56}|\bar{H}$, but this contains a high weight $(2, 0, 0)$; since $L(200)$ is not a composition factor of $V_{56}|H$ we cannot have $\bar{H}$ conjugate to H. Thus $\bar{H}$ must be in the second choice of A_5. We have $L(\omega_2)|\bar{H} = \bigwedge^2 L(010)$ is a submodule of $L(010) \otimes L(010)$. The latter is a tilting module; which one calculates is uniserial with $T(020) \cong k/L(101)/L(010)^{[1]}/L(101)/k$ and so $\bigwedge^2 L(010) \cong L(101)/k \cong V(101)$. Thus $V_{56}|\bar{H} = V(101) + V(101)^* + L(010)^4$. Therefore $V_{56}|H \not\cong V_{56}|\bar{H}$; this is a contradiction. Hence H is non-G-cr as required.

$\underline{H = E(C_4, E_7, 2) \text{ and } E(D_4, E_7, 2).}$ These cases are discussed in [LST96, 2.7, Proof]. There, H is shown to be in an E_6-parabolic and not conjugate to its image $\bar{H} \leq C_4 \leq F_4 \leq E_6 = L'$.

For $\underline{E(B_4, E_8, 2)}$ the embedding shown comes from the action of SO_9 on its (reducible) natural module. The resulting subgroup B_4 is non-A_8-cr, and in an $L' = A_7$-parabolic of E_8. The image $\bar{H}$ of the projection of H to L corresponds to the special isogeny $\mathrm{SO}_9 \to \mathrm{Sp}_8$ so that we are in the exceptional situation (ii)(2) of Lemma 4.4. Thus we have B_4 in an A_7-parabolic of E_8 and it is not even isomorphic to its image in the Levi subgroup, never mind conjugate. This case is concluded.

$H = E(C_3, E_8, 3)$, $E(B_2, E_8, 5)$. These cases contain similar arguments so we work through only the first in detail.

We do not wish to calculate the restriction $\mathrm{Lie}(E_8)|H$, so we take another approach. To see the existence of H as claimed, observe that since the natural module $L(100)$ for Sp_6 admits a symplectic form, the tensor square $M = L(100) \otimes L(100)$ admits an orthogonal form, with composition factors $L(200)|L(010)|k^2$. Since $L(100)$ is a tilting module, so is M; but from [Lüb01] we have $L(200) = V(200) = T(200)$ and $V(010) = L(010)/k$, so we must have $M \cong L(200) + T(010)$. Duality preserves these factors, so the 15-dimensional Sp_6-module $T(010)$ carries an orthogonal form. Thus we have a subgroup $H = \mathrm{Sp}_6 \leq \mathrm{SO}_{15} \leq \mathrm{SO}_{16}$ as claimed. By Lemma 4.8, this subgroup is in a D_7-parabolic of this D_8 with the natural module $V_{14}|\bar{H} \cong L(010) + k$. Clearly H is a complement to R in $\bar{H}R$ where R denotes the (abelian) unipotent radical of the D_7-parabolic in D_8, but H is not conjugate to $\bar{H}$. Hence by Lemma 4.4, H corresponds a non-trivial cocycle class in $\mathrm{H}^1(\bar{H}, R)$. In fact, using [ABS90, Theorem 2], one calculates R is abelian, isomorphic to $V_{14}|\bar{H} \cong L(010) + k$ as an $\bar{H}$-module, with $\mathrm{H}^1(\bar{H}, R) \cong \mathrm{H}^1(\bar{H}, L(010)) \cong k$.

Now, choosing subsystems carefully, one can arrange that the unipotent radical Q of the $L' = D_7$-parabolic of E_8 contains $Q \cap D_8 = R$ as a subgroup. Under these assumptions, one finds R coincides with $Z(Q)$, while $Q/Z(Q)|L' \cong L(\omega_7)$ (again, using [ABS90, *loc. cit.*]). From [LS96, 2.12] we see that the composition factors of $L(\omega_7)|\bar{H}$ are $L(110)|L(001)$. Thus $\mathrm{H}^0(\bar{H}, Q/Z(Q)) = 0$ and so by [Ste13, 3.2.14] the map $\mathrm{H}^1(\bar{H}, R) \cong \mathrm{H}^1(\bar{H}, Z(Q)) \to \mathrm{H}^1(\bar{H}, Q)$ is an injection. Hence the image of the cocycle class corresponding to H remains non-zero in $\mathrm{H}^1(\bar{H}, Q)$. So H is not Q-conjugate to $\bar{H}$. But by [BMRT13, 5.9(ii)] it follows that H is not G-conjugate to $\bar{H}$, hence H is not G-cr.

The case $E(B_2, E_8, 5)$ is entirely similar; one considers instead the action of B_2 on $L(10) \otimes L(10)$ which is again a tilting module, carrying an orthogonal form. The unique 15-dimensional direct factor isomorphic to $T(20)$ is preserved by this duality, giving $H \leq B_7 \leq D_8$. By Lemma 4.8, H is in a parabolic; by dimension considerations this must be a D_7-parabolic with $\bar{H} \leq D_7$, via $V_{14}|\bar{H} = L(20) + k$. We have $\mathrm{H}^1(\bar{H}, Z(Q)) = \mathrm{H}^1(\bar{H}, L(20) + k) = k$ and using [LS96, 2.12, 2.7] it can be shown that $Q/Z(Q)|\bar{H} = L(13) + L(11)$ giving $\mathrm{H}^0(\bar{H}, Q/Z(Q)) = 0$ too. The rest of the argument is the same.

Lastly, there are two cases where one cannot give a nice embedding in the manner we have done above. Let

$H = E(G_2, E_7, 7)$. We first indicate how to see the existence of this subgroup then show that it cannot have any proper reductive overgroup. By [LS04], when $p = 7$, F_4 has a maximal subgroup of type G_2 acting on V_{26} as $L(20)$. Set $\bar{H}$ to be this subgroup and regard $\bar{H}$ as subgroup of a Levi subgroup L of an E_6-parabolic $P = LQ$; note that $\bar{H}$ is E_6-irreducible. One has $V_{27}|F_4 = V_{26} + k$ so that $V_{27}|\bar{H} = L(20) + k$. Now, using [Lüb01], one has, when $p = 7$ that $V(20) \cong L(20)/k$. Thus $\mathrm{H}^1(\bar{H}, L(20)) = \mathrm{H}^0(\bar{H}, \mathrm{H}^0(20)/L(20)) = k$. Now $Q|L' \cong V_{27}$ or ${V_{27}}^*$ so one has $\mathrm{H}^1(\bar{H}, Q) = k$. But by [Ste13, 3.2.15] it follows that there is a non-G-cr subgroup H, which is a complement to Q in $\bar{H}Q$.

Suppose H had a proper reductive overgroup in G. By construction, it is not in an E_6 so by Lemma 4.9 it would have to lie in a subsystem subgroup of type A_7 or D_6. The only 8-dimensional representation up to Frobenius twists is $V_7 + k$. Thus we would have H contained L_1-irreducibly in a Levi subgroup L_1 of type A_6. Now by Proposition 5.1, it would follow that H is G-cr, a contradiction. If H were in D_6, an examination of 12-dimensional representations shows that it would have to act completely reducibly, hence H would be L_1-cr in some Levi subgroup L_1 of G, thus G-cr by Proposition 5.1, a contradiction. Thus H has no proper reductive overgroup in G as required.

$H = E(G_2, E_8, 3)$. This case is similar to $E(G_2, E_7, 7)$. Again we shall show the existence of this subgroup first. Set $\bar{H}$ to be the embedding of G_2 into $B_3 \times B_3 \leq D_7 = L'$ via $(V_7, V_7)|G_2 = (L(10), L(01))$. Thus the natural module for D_7, $V_{14}|\bar{H} \cong L(01) + L(10)$. Now the spin module $L_{D_7}(\omega_7)|B_3 \times B_3 \cong (L(001), L(001))$; hence one calculates $L_{D_7}(\omega_7)|\bar{H} = (L(10) + k) \otimes (L(01) + k) = L(11) + L(01) + L(10) + k$. We have that $Z(Q) = Q_2|\bar{H} = L(10) + L(01)$ and $Q/Q_2|\bar{H} = L(11) + L(01) + L(10) + k$. Now $\mathrm{H}^1(\bar{H}, Q/Q_2) \cong k$ and $\mathrm{H}^2(\bar{H}, Q_2) = 0$ (by a dimension shift). Thus by [Ste13, 3.2.11], we have that the map $\mathrm{H}^1(\bar{H}, Q) \to \mathrm{H}^1(\bar{H}, Q/Q_2)$ is surjective. Thus by [Ste13, 3.2.15] we have that there is exactly one non-G-cr subgroup H, a complement to Q in $\bar{H}Q$.

To see that H can have no proper reductive overgroup one checks [LS04, Cor. 2] to see that if $H \leq M$ for M a maximal connected reductive subgroup of E_8 then M must be D_8, A_8 or G_2F_4. Since $p = 3$, a subgroup of type G_2 in F_4 is in a D_4. The full connected normaliser of this is a $D_4D_4 \leq D_8$. Thus we would have $G_2 \leq A_8$ or D_8. Examining 9-dimensional representations for G_2, or 16-dimensional self-dual representations for D_8, we see that H stabilises a 2-space on the natural modules in each case. Thus $H \leq D_7$. Since there is only one class of such, H is conjugate to $\bar{H}$, a contradiction.

This completes the proof of Theorem 1.

Remark 5.2. In principle, it is possible to construct the subgroup $H(k)$ where $H = (G_2, E_8, 3)$ above explicitly by giving generators as products of root group elements $x_\alpha(t)$ of $E_8(k)$. Since the exact construction would be unedifying (potentially impossible in practice), we give a sketch of a recipe for the interested reader. A more formal description of this procedure can be found in [Ste13, §3.2.3]. Fix a maximal torus T of E_8 and let G_2 be a subgroup of a standard Levi subgroup L of the standard D_7-parabolic P of E_8.

The first task is to write generators of $G_2(k)$ (in our case its root groups) in terms of those of $D_7(k)$. Since G_2 is the fixed points of the triality automorphism of D_4, one can write the root groups of G_2 in terms of those of D_4. For instance, if α is the simple short root of G_2, one sets the image of $x_\alpha(t)$ in $D_4(k)$ as $x_{\alpha_1}(t)x_{\alpha_3}(t)x_{\alpha_4}(t) \in D_4(k)$, where $\{\alpha_1, \alpha_3, \alpha_4\}$ is an orbit of simple roots under triality. Then in fact, this subgroup is clearly in some B_3 subgroup of D_4, say that corresponding to the fixed points of the transposition $\alpha_3 \leftrightarrow \alpha_4$. So in fact, one may easily write the root group elements of G_2 in terms of those of B_3. Then the construction of $\bar{H}$ in a subgroup of type $B_3 \times B_3 \leq D_7$ allows one to write the root groups of $\bar{H}$ as products of those in D_7.

The next step is to construct an explicit cocycle $\gamma \in Z^1(G_2, L(11))$ which is not cohomologous to 0. For this, the induced module $\mathrm{H}^0(11)$ for G_2 can be constructed explicitly with a basis such that one knows the precise action of any root group element $x_\beta(t) \in G_2(k)$ on this basis.[5] The module $\mathrm{H}^0(11)$ contains as a two-step uniserial submodule W such that there is a short exact sequence

$$0 \to L(11) \to W \to k \to 0.$$

This short exact sequence corresponds to an element $[\gamma]$ of $\mathrm{Ext}^1_{G_2}(k, L(11)) \cong \mathrm{H}^1(G_2, L(11))$, where $\gamma \in Z^1(G_2, L(11))$. If necessary, replacing γ with a cohomologous cocycle, one can arrange that γ

[5] This can be done by taking an appropriate mod 3 reduction from the Weyl module $V_{\mathbb{Z}}(11)$ for $(G_2)_{\mathbb{Z}}$.

is trivial on a maximal torus T of G_2. Knowledge of the explicit action of the root group elements of $G_2(k)$ on (a basis of) $\mathrm{H}^0(11)$ allows one to deduce the image of these elements under γ.

The last step is to form the complement H to Q in $\bar{H}Q$. Firstly, since $Q/Q_2 \cong_{\bar{H}} L(11) + L(01) + L(10) + k$, one can set $G_2 \cong J := \{x\gamma(x) : x \in \bar{H}\} \leq \bar{H}Q/Q_2$, where $\gamma : \bar{H} \to Q/Q_2$ takes images in the direct factor $L(11)$. Here, the root group elements of J are expressed as root group elements of $\bar{H}Q/Q_2(k)$. Lastly one needs to lift J to H through Q_2 to a complement to Q in $\bar{H}Q$. Such a procedure is possible by the proof. It can be accomplished in practice by taking $j \in J$, noting that it lifts to an element $\hat{j} \in \bar{H}Q$ and that any other such lift is an element $\hat{j}q$ for q a product of root group elements of Q_2. There are Steinberg relations that the root group elements of H must satisfy to be isomorphic to G_2. Using the commutator relations in G we can deduce a suitable value of q for each element $j \in J$. (In practice, there is an 'obvious' lift $\hat{j}$ of j and the guess $q = 1$ usually suffices.) Now since the root groups of H generate it, this gives a construction of H.

For the case $H = E(G_2, E_7, 7)$, one may employ the above recipe again. One starts this time with the construction of the maximal subgroup $\bar{H} \cong G_2$ in E_6 from [Tes89]. To get the cocycle γ one can use the short exact sequence $0 \to L(20) \to \mathrm{H}^0(20) \to k \to 0$.

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