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MAXIMAL SUBALGEBRAS OF CARTAN TYPE IN THE EXCEPTIONAL LIE ALGEBRAS

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Abstract. In this paper we initiate the study of the maximal subalgebras of exceptional simple classical Lie algebras $\mathfrak{g}$ over algebraically closed fields k of positive characteristic p, such that the prime characteristic is good for $\mathfrak{g}$. In this paper we deal with what is surely the most unnatural case; that is, where the maximal subalgebra in question is a simple subalgebra of non-classical type. We show that only the first Witt algebra can occur as a subalgebra of $\mathfrak{g}$ and give an explicit classification of when it is maximal in $\mathfrak{g}$.

1. Introduction

The maximal subalgebras of the simple finite-dimensional Lie algebras over the complex numbers were first classified by Dynkin [Dyn52a] and [Dyn52b]. The fact that in characteristic 0 there is such a good correpondence between connected closed subgroups of simple algebraic groups and subalgebras of their Lie algebras means that this classification lifts readily to a classification of maximal closed connected subgroups of the corresponding simple algebraic groups over algebraically closed fields of characteristic 0. Gary Seitz took up the case of achieving a classification of connected maximal closed subgroups of the simple algebraic groups over k, where k is an algebraically closed field of positive characteristic. This was achieved in [Sei87] for the classical algebraic groups and, under some fairly mild restrictions on the characteristic p of k in [Sei91] for the exceptional algebraic groups. Later, in work by Liebeck and Seitz [LS04], the latter classification (for exceptional algebraic groups) was completed to cover all characteristics and extended to all maximal, closed, positive dimensional subgroups (not necessarily connected). All these positive characteristic results rely on work of Donna Testerman [Tes88], [Tes89], [Tes92] which, particularly, classify and construct subgroups of type A_1. The extension of the original work on subgroups of the classical groups continues to evolve. See [BGMT15] for the latest developments.

In this paper we consider the analoguous question for modular Lie algebras, a more direct analogue of Dynkin's original work. Apart from the intrinsic motivation to generalise Dynkin's results to positive characteristic, it is worth mentioning that maximal subalgebras of modular Lie algebras play an important role in the classification due to Premet and Strade [PS06] of simple Lie algebras over algebraically closed fields of dimension $p > 3$ as they give rise to Weisfeiler filtrations [*op. cit.*, §2.4]. Whereas the classification of simple algebraic groups by their root data is by now very well-documented, the classification of simple Lie algebras is highly non-trivial and is thought likely to be out of reach for the primes $p = 2$ and $p = 3$. Let us recall the result for $p > 3$: a simple Lie algebra L is either classical (i.e. it is the Lie algebra of a simple algebraic group, or a central quotient thereof); of one of the four families of Cartan type simple Lie algebras W, K, S or H (either graded, or in case H or S, a filtered deformation); or $p = 5$ and it is one of the Melikyan algebras.

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Perhaps it is not surprising that the classification of maximal subalgebras of Lie algebras $\mathfrak{g}$ over algebraically closed fields of characteristic at least 5, even just those which come from algebraic groups G, is likely to be difficult. For instance, a fact one takes for granted when working with simple algebraic groups is a theorem of Borel and Tits which has as a corollary that all maximal non-reductive subgroups of G are parabolic. But the analagous statement for modular Lie algebras is not true in general. For instance when $p|n$, the maximal non-semsimple subalgebras of the simple Lie algebra $\mathfrak{psl}_n$ need not be parabolic. See O. K. Ten's work [Ten87a] for a classification of maximal non-semisimple subalgebras in the case that $\mathfrak{g}$ is a classical Lie algebra of type A–D. The paper [Ten87b] also gives a fairly coarse classification of the maximal semisimple subalgebra of these same Lie algebras. Lastly, it appears that the same author had at some point announced a result classifying the maximal subalgebras of Lie algebras of type G_2 when the characteristic is at least 5, but this remains unpublished.

We should mention one other significant piece of work on maximal subalgebras of simple Lie algebras, due to H. Melikyan [Mel05], who classifies in most cases, the maximal graded subalgebras of the Cartan type Lie algebras.

It is the point of this paper to initiate the study of maximal subalgebras of exceptional simple Lie algebras in good characteristic. Here one is fortunate that $p > 3$ and so the Premet–Strade classification of simple Lie algebras holds. In this paper we are concerned with what is surely the most unnatural case; that is, where the maximal subalgebra in question is a simple subalgebra of non-classical type.

The most straightforward non-classical algebra to describe is the first Witt algebra $W_1 = W(1; \underline{1})$ of dimension p, the Lie algebra of derivations of the truncated polynomial ring $k[X]/\langle X^p\rangle$, where p is the characteristic of k. When $p = 2$ it is not simple, when $p = 3$ it is isomorphic to $\mathfrak{sl}_2$, but further than that, it is simple and there are no more coincidences with other Lie algebras mentioned in the classification. It has a basis $\{\partial, X\partial, \dots, X^{p-1}\partial\}$, with structure constants given by $[X^i\partial, X^j\partial] = (j-i)X^{i+j-1}\partial$. The first Witt algebra does put in a number of guest appearances as subalgebras of $\mathfrak{g}$ and there are precisely four occasions when it is maximal up to conjugacy. The existence of p-subalgebras of type W_1 is essentially established by classifying the nilpotent element representing ∂.

Theorem 1.1. *Let $\mathfrak{g}$ be a simple classical Lie algebra of exceptional type over an algebraically closed field k of good characteristic $p > 0$. Suppose $W \cong W_1$ is a p-subalgebra of $\mathfrak{g}$. Let $\partial \in W$ be represented by the nilpotent element $e \in \mathfrak{g}$. Then the following hold:*

- (i) *e is a regular element in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and the root system associated to $\mathfrak{l}$ is irreducible.*
- (ii) *For h the Coxeter number of $\mathfrak{l}$, we have either $p = h + 1$ or $\mathfrak{l}$ is of type A_n and $p = h$.*
- (iii) *If e is regular in $\mathfrak{g}$ then W is unique up to conjugacy.*
- (iv) *If e is regular and $\mathfrak{g}$ is not of type E_6 then W is maximal.*
- (v) *If e is not regular in $\mathfrak{g}$ then W normalises a non-trivial abelian subalgebra of $\mathfrak{g}$, hence is not maximal.*

Conversely, suppose that $e \in \mathfrak{g}$ is nilpotent and (e, p) satisfies the conditions (i) and (ii) above. Then there exists a p-subalgebra isomorphic to W_1 with ∂ represented by e.

Remarks 1.2. (i). In the statement of the theorem, recall that since $\mathfrak{g} = \mathrm{Lie}(G)$ we have that $\mathfrak{g}$ inherits a restricted structure, leading to a p-map $\mathfrak{g} \to \mathfrak{g}; x \mapsto x^{[p]}$ which satisfies $\mathrm{ad}\, x^{[p]} = (\mathrm{ad}\, x)^p$; we say a subalgebra is a p-subalgebra of $\mathfrak{g}$ if it is closed under this map. Now, since we have

$\operatorname{ad} x^{[p]} = (\operatorname{ad} x)^p$ for any $x \in \mathfrak{g}$, it follows that any subalgebra is an ideal in its p-closure. As $\mathfrak{g}$ is simple, we have then that all maximal subalgebras really are p-subalgebras.

(ii). On the other hand, there do exist non-p-subalgebras of $\mathfrak{g}$ isomorphic to W_1 whenever there is a Levi subalgebra $\mathfrak{l}$ of type A_{p-1}, since one may then embed a copy of W_1 into this subalgebra via one of its p-dimensional non-restricted representations. The p-closure then contains the centre of the derived subalgebra of $\mathfrak{l}$.

(iii). If the nilpotent element e is regular in a proper Levi subalgebra of $\mathfrak{g}$ there can be many conjugacy classes of subalgebras of type W_1 containing e, in particular, those which are non-G-cr in the sense of [BMRT13].

(iv). The reader is invited to notice the pleasant fact that a subalgebra isomorphic to W_1 is maximal only if $p | \dim \mathfrak{g}$.

In some sense it is an artefact of the large dimensions of non-classical simple Lie algebras in good characteristic that they cannot fit inside the exceptional Lie algebras. For example, the Melikyan algebras only exist when the characteristic of k is 5 and the smallest one is 125-dimensional. Thus it cannot fit inside G_2, F_4 or E_6 and $p = 5$ is not a good prime for E_8. So it remains to rule out the existence of a 125-dimensional simple Lie algebra in E_7 of dimension just 133, which is not too hard: see Lemma 4.1. There could be more scope for finding, say, the first Hamiltonian algebra of dimension $p^2 - 2$, but in fact we show that this never appears as a subalgebra of $\mathfrak{g}$.

Theorem 1.3. *Let $\mathfrak{g}$ be a simple classical Lie algebra of exceptional type. Suppose p is a good prime for $\mathfrak{g}$ and let $\mathfrak{h}$ be a simple subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is either isomorphic to W_1 or it is of classical type.*

Remark 1.4. The conclusion of Theorem 1.3 does not extend to bad characteristic. Alex Kubiesa, an undergraduate student of the second author, has discovered a maximal simple subalgebra of F_4 over $\mathbb{F}_3$ of dimension 26. (And again, we have $26 | \dim \mathfrak{g}$.) There is strong evidence that this subalgebra is not isomorphic to the first contact algebra $K(3, [1, 1, 1])$. Of those known in characteristic 3, it also matches the dimension of an Ermolaev algebra, see [Str04, 4.4].

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2. Preliminaries

2.1. **Notation.** In the following G will be a simple algebraic group of exceptional type over an algebraically closed field k of characteristic p, and $\mathfrak{g} = \operatorname{Lie}(G)$ will be its Lie algebra. We assume that p is a good prime for the root system of G.

Fix a maximal torus T and a Borel subgroup B containing it and let Φ be the root system of G corresponding to T, with positive roots Φ^+ corresponding to B. If $S = \{\alpha_i\}$ represents the simple roots, one can express any other root β simply by giving the coefficients of an expression of β as a sum of the simple roots. We will use the Bourbaki ordering for this; hence the highest root of F_4, $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ is written as 2342 and the root $\alpha_2 + \alpha_4$ in E_6 is written as $\substack{00100\\1}$. We choose root vectors for T in $\mathfrak{g}$ and a basis for $\mathfrak{t} = \operatorname{Lie}(T)$ coming from a basis of subalgebras isomorphic to

$\mathfrak{sl}_2$ corresponding to each of the simple roots such that the collection of these is a Chevalley basis for $\mathfrak{g}$. We write these elements as $\{e_\alpha : \alpha \in \Phi\}$ and $\{h_\alpha : \alpha \in S\}$ respectively.

At various points, particularly in Appendix A, we will need to do various explicit computations with elements expressed in terms of the Chevalley basis. Such calculations can in principle be attempted by hand, but we use GAP to reduce time and error. We will wish to work with certain general elements of $\mathfrak{g}$ so we set up a Lie algebra in GAP over $\mathbb{Q}[x_1, \dots, x_{\dim \mathfrak{g}}]$. Since GAP works with a Chevalley basis, all structure constants are integral and reducing these modulo p give us analagous calculations over fields k of characteristic $p > 0$. For more details of the form of these calculations, see the appendix.

2.2. **Nilpotent orbits.** We work extensively with nilpotent orbits in good characteristic. Our main source for the theory is [Jan04]. Let us recall the following facts from this reference, which we will usually use without comment. Associated to each nilpotent element $e \in \mathfrak{g}$ is an orbit $\mathcal{O} = G.e$ of e under the adjoint action of G on $\mathfrak{g}$. We have $\dim G = \dim \mathcal{O} + \dim G_e$ where G_e is the centraliser of e in $\mathfrak{g}$. Since p is a (very) good prime for Φ, centralisers are smooth, and so $\dim G_e = \dim \mathfrak{g}_e$; and $\mathrm{Lie}(G_e) = \mathfrak{g}_e$. The nilpotent element e is said to be *distinguished* in some Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $\mathfrak{g}$ if each torus in L centralising e is contained in $Z(L)$. Every nilpotent element is distinguished in at least one Levi subalgebra. It is a result of Premet [Pre95] that there is at least one cocharacter $\tau : \mathbb{G}_m \to G$ *associated* to e. The cocharacter τ has the following properties: firstly, e is in the 2-weight space for τ, so that $\tau(t).e = t^2 e$; secondly τ evaluates in the derived subgroup of L, where L is a Levi subgroup with the property that e is distinguished in $\mathfrak{l} = \mathrm{Lie}(L)$. Any two associated cocharacters (with these properties) are conjugate by an element of G_e. Any cocharacter gives a grading of $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ where $\mathfrak{g}(i)$ is the ith weight space of τ on $\mathfrak{g}$. One has $[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j)$. If τ is associated to e, one has $e \in \mathfrak{g}(2)$ and $\mathfrak{g}(\geq 0) := \bigoplus_{i\geq 0} \mathfrak{g}(i)$ is a parabolic subalgebra $\mathfrak{p} = \mathrm{Lie}(P)$ of $\mathfrak{g}$ with $\mathfrak{g}(0)$ being a Levi subalgebra, and $\mathfrak{g}(>0) := \bigoplus_{i>0} \mathfrak{g}(i)$ is its nilradical, being $\mathfrak{g}(>0) = \mathrm{Lie}(R_u(P))$. The τ-grading on $\mathfrak{g}$ induces a grading on the centraliser $\mathfrak{g}_e$. One may write G_e as a semidirect product $C_e R_e$ with C_e reductive and R_e its unipotent radical. In this case, one has $\mathfrak{g}_e(0) = \mathrm{Lie}(C_e)$ and $\mathfrak{g}_e(>0) = \mathrm{Lie}(R_e)$.

The classification of nilpotent orbits is now well-established. For detailed data in the case that $\mathfrak{g}$ is exceptional and p is good for $\mathfrak{g}$, we are very grateful for the existence of [LT11], which gives complete tables of orbit representatives, associated cocharacters, and the explicit structure of C_e; i.e. its root system, in terms of the roots of G and $Z(C_e)^\circ$ in terms of the maximal torus T of G. Furthermore the authors give the component group C_e/C_e° and the structure of R_e in terms of modules for C_e. Since the calculation is used at one point, let us, by way of example of its usefulness, point out here that one can from such data read off the maximal value of i for which $\mathfrak{g}(i) \neq 0$. Let i be this value: then since $[e, \mathfrak{g}(i)] \subseteq \mathfrak{g}(i+2) = 0$, $\mathfrak{g}(i) \subseteq \mathfrak{g}_e(>0)$; thus $\mathfrak{g}(i)$ is a C_e-module explicitly listed in [LT11].

2.3. **Representations of W_1.** We assume that $p \geq 5$ in this section. Let us recall some of the representation theory of W_1. Recall that the Witt algebra W_1 can be given by a basis $\{\partial, X\partial, \dots, X^{p-1}\partial\}$ with commutator formula $[X^i\partial, X^j\partial] = (j-i)X^{i+j-1}\partial$. The simple modules for W_1 were determined in [Cha41]. In this paper we are almost exclusively interested in those irreducible representations which are restricted (also known as p-representations), i.e. those associated to the trivial central character in the universal enveloping algebra. These are all quotients of Verma modules parametrised by the integers λ from 0 to $p-1$. To describe them, let $\mathfrak{n}^+ = \langle X^2\partial, \dots, X^{p-1}\partial\rangle$ and $\mathfrak{b}^+ = \langle X\partial, \dots, X^{p-1}\partial\rangle$. Then as $\mathfrak{n}^+$ is an ideal of $\mathfrak{b}^+$ we may define a

1-dimensional $\mathfrak{b}^+$-module k_λ on which $\mathfrak{n}^+$ acts trivially and $X\partial$ acts by multiplication by λ. Then one defines the corresponding Verma module $Z^+(\lambda) = \mathfrak{u}(W_1) \otimes_{\mathfrak{u}(\mathfrak{b}^+)} k_\lambda$, where $\mathfrak{u}$ denotes the restricted enveloping algebra. It is easy to see that $Z^+(\lambda)$ is p-dimensional with basis $\{m_0, \dots, m_{p-1}\}$ and the action of $e_k = X^{k+1}\partial$ $(-1 \le k \le p-2)$ is given by

$$(*) \qquad e_k.m_j = (j + k + 1 + (k+1)\lambda)m_{j+k},$$

where we put $m_j = 0$ for j outside $\{0, \dots, p-1\}$. The $Z^+(\lambda)$ are all simple, except for $Z^+(0)$ and $Z^+(p-1)$. The former has a trivial simple quotient and the latter has a trivial submodule and $(p-1)$-dimensional simple quotient. We denote the corresponding simple quotient modules by $L(\lambda)$. Let us warn the reader to bear in mind that at later points, we will grade various modules for W_1 with $e_{-1} = \partial$ in grade 2, compatibly with the theory of nilpotent orbits. (So the adjoint action of ∂ will increase the grade, while decreasing the subscripts on the basis vectors in the modules just described.) Also slightly counterintuitively, $e_0 = X\partial$ will (in most cases at least) act with weight -1 on the graded pieces $\mathfrak{g}(2 + rp)$. Our choices are compatible with most of the literature.

One way one may recognise the high weight of a simple restricted module is the following: In each $L = L(\lambda)$ there is, up to scalars, a unique vector, m_0 killed by $\partial = e_{-1}$. By formula (*) $X\partial.m_0 = e_0.m_0 = (\lambda+1)m_0$ whenever $L(\lambda)$ is an irreducible Verma module. In the remaining two cases, one checks that for $L(p-1)$, $X\partial$ has weight 1 on a vector killed by ∂ and of course $X\partial$ has weight zero on the trivial module $L(0)$. Thus the action of ∂ and $X\partial$ on L determine L up to isomorphism. In particular, we may identify the adjoint module as $L(p-2)$: The element ∂ is killed by $\operatorname{ad}\partial$, and $[X\partial, \partial] = -\partial$. Thus $\lambda + 1 = -1$ modulo p and so $\lambda = p-2$.

If V is a finite-dimensional restricted W_1-module, with the weights of V known, and if sufficently many maximal or primitive vectors for ∂ are known, it is possible to list the composition factors of V. This works particularly in the case that V is compatibly graded with ∂ in grade 2 and $X\partial$ in grade 0 as we will show in the next lemma.

Lemma 2.1. *Suppose V is a finite-dimensional restricted $W = W_1$-module admitting a grading $V = \bigoplus_{i\in\mathbb{Z}} V(i)$ such that $\partial \cdot V(i) \subseteq V(i+2)$ and such that each $V(i)$ is stable under $X\partial$. Then there exists a unique semisimple W-module $V_s = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ with $V_s = \bigoplus_{i\in\mathbb{Z}} V_s(i)$ with $V_s(i) = V(i)$ as $X\partial$-modules and each $V_j = \bigoplus_{i\in\mathbb{Z}} V_j(i)$ a graded irreducible W-module.*

For this module V_s, the set of composition factors $[V|W]$ and $[V_s|W]$ coincide.

Proof. If V is irreducible then we simply take $V_s = V$ and $V_s(i) = V(i)$. Moreover there is only one choice for V_s as a W-module since both V and V_s are determined by the maximum r with $V(r) \neq 0$ and the weight of $X\partial$ on this necessarily 1-dimensional space.

Now take an irreducible submodule $V_1 \cong L(\lambda)$ of V. Again, V_1 is determined uniquely as a W-module by the weight of $X\partial$ on a vector v killed by ∂. Suppose this vector is $v = v_1 + v_2 + \cdots + v_r$ with $0 \neq v_j \in V(i_j)$, $i_1 > i_2 > \cdots > i_r$. We may write $V_1 = \langle w = v\rangle$ if $\lambda = 0$, $V_1 = \langle w, \partial w, \dots, \partial^{p-2}w = v\rangle$ if $\lambda = p-1$ or $V_1 = \langle w, \partial w, \dots, \partial^{p-1}w = v\rangle$ if $1 \le \lambda \le p-2$.

Now we regrade V so as to make V_1 a graded submodule. Let $V' := V$ as a W-module. Set $V'(i) = V(i)$ for $i > i_1$ and $i \neq i_1 \mod 2$. Further let $V'(i) = V(i)$ for $i < i_1$ if $\lambda = 0$, $i < i_1 - 2p + 4$ if $\lambda = p-1$ and $i < i_1 - 2p + 2$ if $\lambda \neq 0, p-1$. We now give special gradings to the remaining $1, p-2$ or $p-1$ parts. If $\{v_1, x_2, \dots, x_m\}$ is a basis for $V(i_1)$ then let $V'(i_1)$ be spanned by $\{v, x_2, \dots, x_m\}$. If $\lambda = 0$ then we have finished grading V'. Otherwise, v, hence also v_1 is in the image of ∂; say $\partial(v') = v$ and $\partial(v_1') = v_1$ with $v_1' \in V(i_1 - 2)$. If $\{v_1', x_2', \dots, x_{m'}'\}$ is a basis for $V(i_1-2)$ then let $V'(i_1 - 2)$ be spanned by $\{v', x_2', \dots, x_{m'}'\}$. We continue this process until we have finished grading

V'. The new W-module V' together with its grading satisfies the hypotheses of the lemma, that is $\partial \cdot V(i) \subseteq V(i+2)$ and $V(i)$ is stable under $X\partial$, however by construction we have ensured that V_1 is a graded submodule of V'.

Hence the quotient V'/V_1 is also graded, say $(V'/V_1) = \bigoplus (V'/V_1)(i)$. By induction, there exists a unique module $(V'/V_1)_s$ which is semisimple, with grading $(V'/V_1)_s = \bigoplus (V'/V_1)_s(i)$ satisfying $(V'/V_1)_s(i) = (V'/V_1)(i)$ as $X\partial$-modules and with a decomposition into graded irreducibles and such that the W-composition factors of V'/V_1 are the same as those of $(V'/V_1)_s$. We therefore set $V_s = (V'/V_1)_s \oplus V_1$, with the direct factor V_1 graded as it is in V'. Moreover since the highest weight space of V_1 is determined by the top i such that $V_1(i) \neq 0$ together with the weight of $X\partial$ on this space, this is the unique choice of V_s for which $V_s(i) = V(i)$ as $X\partial$-modules for all i. □

If we are in the situation of the lemma, we give an algorithm which produces the composition factors of V given just the restriction of $V(i)$ to $X\partial$, i.e. a list of $X\partial$-weights ℓ_i of $V(i)$ for each $i \in \mathbb{Z}$. By the lemma, we may assume that $V = V_s$ satisfying the conclusions of the lemma.

Proposition 2.2. *Let V be as in Lemma 2.1. For $i \in \mathbb{Z}$ with $V(i) \neq 0$, let ℓ_i be a list (with multiplicities) of the $X\partial$-weights on $V(i)$. Then the following algorithm determines the composition factors (with multiplicities) of V as a W-module:*

Algorithm. (i) Let $r \in \mathbb{Z}$ be maximal such that ℓ_r is nonempty. Pick $\mu \in \ell_r$.

(ii) Record a composition factor $U = L(\lambda)$ for $\lambda = \mu - 1$ if $\mu \neq 0, 1$ and $U = L(p-1)$, $L(0)$ if $\mu = 1, 0$ respectively. Form a new set of lists $\{\ell'_i\}$ by removing weights from $\{\ell_i\}$ in the following way: If $U = L(0)$ remove a 0-weight from ℓ_r, if $U = L(p-1)$ remove one weight $1, 2, \dots p-1$ from $\ell_r, \ell_{r-2}, \dots, \ell_{r-2p+4}$ respectively and otherwise remove one weight $\mu, \mu+1, \dots, \mu+p-1$ from $\ell_r, \ell_{r-2}, \dots, \ell_{r-2p+2}$.

(iii) If the new lists $\{\ell'_i\}$ are not all empty, repeat from Step (i).

Proof. By Lemma 2.1, we may assume that $V = \bigoplus_i V(i)$ is a semisimple W-module, and that it is a direct sum of simple graded W-modules $V = \bigoplus_j V_j$. Since we have $V(r) = \bigoplus_j V_j(r)$, there exists a simple submodule V_j such that $V_j(r) \neq 0$ and such that $X\partial$ has weight μ on $V_j(r)$. We have that ∂ kills $V_j(r)$, hence a vector of high weight μ on $V_j(r)$ uniquely determines the isomorphism type of the simple submodule $U = V_j$ as described in the algorithm. Moreover, V_j is a direct sum of one-dimensional graded pieces in the positions given in step (ii) of the algorithm. Proceeding with V/V_j in place of V, we may determine all composition factors (with multiplicities) of V. Moreover, replacing V by V/V_j corresponds to replacing the weights of V by the weights obtained after a single application of step (ii) in the algorithm. □

Remark 2.3. The lemma above and the subsequent algorithm can be applied in more generality than just the case that V is restricted. For example, all we use (together with the other hypotheses of the lemma) is the fact that V consists of composition factors whose isomorphism types are determined by the weight of $X\partial$ on a vector killed by ∂. Hence one can still use these results when V is not itself a restricted representation but all of whose composition factors are restricted. One can also use it in the case that V were known to consist of composition factors which are induced from k_λ where $\lambda \notin \mathbb{F}_p$. Finally, the same argument will also work for more general Lie algebras where there is a subalgebra $\mathfrak{n}$ which is strictly positively graded, with $\mathfrak{h}$ some subalgebra (such as a Cartan subalgebra) so that V consists of composition factors whose isomorphism type is determined by the weights of elements of $\mathfrak{h}$ on vectors killed by $\mathfrak{n}$.

Let us make a note for later that the extensions of the simple modules were determined in [BNW09] and independently, in [Ria11]. When $p = 2$, W_1 is no longer simple and when $p = 3$, $W_1 \cong \mathfrak{sl}_2$ for which the answer is well known.

We record the next easy lemma which will be of use in proving Theorem 1.1. In its proof we need the fact that the self-dual simple modules are $L((p-1)/2)$, $L(0)$ and $L(p-1)$. (This can be established by observing that for a self-dual module one needs the highest weight to be the negative of the lowest weight.) Otherwise one has $L(i)^* = L(p-1-i)$.

Lemma 2.4. *Suppose $p > 2$. In each of its non-trivial irreducible p-representations, the smallest classical simple Lie algebra $\mathfrak{h}$ of A–D type containing W_1 is $\mathfrak{psl}_p$ unless $V = L(p-1)$ and $\mathfrak{h} = \mathfrak{sp}_{p-1}$ or $V = L((p-1)/2)$ and $\mathfrak{h} = \mathfrak{so}_p$. Furthermore, the element $\partial \in W_1$ is represented by a nilpotent regular element of $\mathfrak{h}$.*

Proof. As explained above, a simple non-trivial p-representation V is self-dual if and only if $V = L(p-1)$ or $V = L((p-1)/2)$. One checks directly, c.f. [HS14, Lem. 11.7], that the action of W_1 on $L(p-1)$ preseves a symplectic form. Since the dimension of $L((p-1)/2)$ is odd, the action of W_1 must preserve an orthogonal form. Otherwise V is not self-dual and $\dim V = p$, so the actions here give $W_1 \subseteq \mathfrak{psl}_p$ of type A_{p-1}. For the last statement, examining the action of W_1 on V in each case, one sees that the element ∂ acts in each case with a single Jordan block on V, i.e. on the natural module for $\mathfrak{h}$. This shows that ∂ is regular in $\mathfrak{h}$. $\square$

2.4. **Representations of Hamiltonians.** Again we assume that $p \geq 5$. In Theorem 1.3 we claim that the only non-classical simple subalgebras of exceptional simple Lie algebras in good characteristic are isomorphic to W_1. Most possibilities can be ruled out on dimensional grounds (cf. Lemma 4.1), and we have a special argument to deal with the case of the Zassenhaus algebra $W(1;(2))$. It will remain to show that there are no subalgebras of $\mathfrak{g}$ isomorphic to the first restricted (graded) Hamiltonian algebra $H_2 = H(2;(1,1))^{(2)}$ of dimension p^2-2, the non-restricted (non-graded) Hamiltonian $H(2;(1,1);\Phi(\tau))^{(1)}$ of dimension p^2-1, the non-restricted (non-graded) Hamiltonian $H(2;(1,1);\Phi(1)) = H(2;(1,1);\Delta)$ of dimension p^2 or the second Witt algebra $W_2 := W(2;(1,1))$.[1] In the proof of the theorem we first show that there are no p-subalgebras of $\mathfrak{g}$ isomorphic to the minimal p-envelopes of these algebras. Since H_2 appears as a p-subalgebra of W_2 (indeed it is usually constructed in this way) it will suffice to show that there are no p-subalgebras $\mathfrak{h}$ isomorphic to H_2, $H(2;(1,1);\Phi(\tau))^{(1)}$ or $H(2;(1,1);\Phi(1))$. And for this, we will show that there is no restriction of the adjoint module $\mathfrak{g}|\mathfrak{h}$ compatible with a further restriction to a chosen p-subalgebra of $\mathfrak{h}$ isomorphic to W_1. (In Tables 3 and Table 6, we will have computed the composition factors of the restriction of $\mathfrak{g}$ to every possible p-subalgebra isomorphic to W_1.)

2.4.1. *Graded Hamiltonians.* Let us first recall a concrete description of $H := H_2$ by basis and structure constants, which can be found in [Kor78], for example, or (as we will opt for) generated from the general description given in [Str04, Ch. 2] in terms of elements of $W_2 = W(2;(1,1))$. We will use the divided power notation for the polynomial functions $\mathcal{O}_2 = k[X,Y]/(X^p,Y^p)$ (cf. *loc. cit.*). The algebra W_2 has basis $\{X^{(i)}Y^{(j)}\partial_X, X^{(i)}Y^{(j)}\partial_Y : 0 \leq i,j \leq p-1\}$ and is graded with $X^{(i)}Y^{(j)}\partial_X$ and

[1] For explicit descriptions of the Hamiltonian algebras and their minimal p-envelopes, see [Str04, §4.2] or [FSW14, §5], [Str09, §10.3] or [FSW14, §5], and [Str09, §10.4] respectively. The minimal p-envelopes of $H(2;(1,1);\Phi(\tau))^{(1)}$ and $H(2;(1,1);\Phi(1)) = H(2;(1,1);\Delta)$ can be computed by adding in the pth powers of elements of their bases as subalgebras of the restricted Lie algebra W_2; this is done explicitly in [FSW14, (5.25)-(5.27)] for $H(2;(1,1);\Phi(\tau))^{(1)}$ and we make the analogous comments below for the other case.

$X^{(i)}Y^{(j)}\partial_Y$ in degree $i+j-1$. We have that $H(2;(1,1)) := \{f\partial_X + g\partial_Y \in W_2 : \partial_Y(g) = -\partial_X(f)\}$. If we define $D(f) = \partial_X(f)\partial_Y - \partial_Y(f)\partial_X$ then when $p \geq 3$, the first graded Hamiltonian algebra is $H(2;(1,1))^{(2)} = \langle D(X^{(a)}Y^{(b)}) : 0 < a+b < 2(p-1)\rangle$, a simple ideal of $H(2;(1,1))$. Thus $H := H(2;(1,1))^{(2)}$ has basis

$$\{e_{i-1,j-1} := X^{(i-1)}Y^{(j)}\partial_Y - X^{(i)}Y^{(j-1)}\partial_X : 0 < i+j < 2p-2\},$$

where $X^{(-1)} = Y^{(-1)} = X^{-1} = Y^{-1}$ is understood to be zero. We see that H inherits from W_2 a grading $H = \bigoplus_{-1\leq i\leq 2p-4} H_i$, in which $e_{i-1,j-1}$ is in degree $i+j-2$. In addition H is restricted with toral element $e_{0,0} = (Y\partial_Y - X\partial_X)^{[p]} = Y\partial_Y - X\partial_X$ and with every other basis element satisfying $(X^{(i-1)}Y^{(j)}\partial_Y - X^{(i)}Y^{(j-1)}\partial_X)^{[p]} = 0$. In particular, $H_{(0)}$ is a subalgebra of H of codimension 2, $H_{(-1)} = \langle \partial_X, \partial_Y\rangle$ and the subalgebra $H_{(0)} = \bigoplus_{i\geq 0} H_i$ is the semidirect product of the subalgebra $H_0 = \langle Y\partial_X, Y\partial_Y - X\partial_X, X\partial_Y\rangle \cong \mathfrak{sl}_2$ and its p-ideal $H_{(1)} = \bigoplus_{i>0} H_i$.

The elements $x := \partial_X$ and $y := \partial_Y$ are important and span a vector space complement to $H_{(0)}$ in H. Note that $[x,y] = 0$. Since $H_{(1)}$ is an ideal in $H_{(0)}$, any representation of $H_0 \cong \mathfrak{sl}_2$ may be lifted to a representation of $H_{(0)}$ by insisting that $H_{(1)}$ act trivially. Because $H_0 \cong \mathfrak{sl}_2$, the irreducible p-representations $L(r)$ of H_0 are classified by the integers $0 \leq r \leq p-1$ with $L(r)$ of dimension $r+1$. Let us write $\hat{L}(r)$ for the corresponding module lifted to $H_{(0)}$.

In the adjoint representation of H on itself, the elements $e_{p-3,p-2}$ and $e_{p-2,p-3}$ span the highest weight space, i.e. the space killed by $H_{(1)}$. On these, the element $e_{0,0}$ has weights 1 and -1. By Frobenius reciprocity it follows that the adjoint module is a quotient of the Verma module $M(1) = \mathrm{Ind}_{u(H_{(0)},0)}^{u(H,0)} \hat{L}(1)$. (For more information on induced modules, see [SF88, §5.6].) The remaining p-representations of H_2 were first determined in [Kor78], though the most general reference, valid for higher rank Hamiltonian modular Lie algebras is [She88] together with certain corrections made in [Hol98]. For our purposes, the following statement is all we need:

Lemma 2.5. *A simple restricted representation of $H = H(2;(1,1))^{(2)}$ is isomorphic to one of $L_H(0) \cong k$, trivial; $L_H(1)$, the adjoint module; or the Verma module $L_H(r) = M(r)$ for $2 \leq r \leq p-1$ of dimension $(r+1)p^2$ obtained by inducing the module $\hat{L}(r)$ from $H_{(0)}$ to H.*

Now, let W be the p-subalgebra of H spanned by the elements $e_{0,j}$. It is easily seen that W is isomorphic to the Witt algebra W_1 with the element $X\partial$ represented by $h = e_{0,0}$ and the element ∂ represented by $y = e_{0,-1}$. Let $[V]$ denote the composition factors of a module V. We wish to calculate the composition factors $[L_H(r)|W]$ of the simple restricted representations of H to W according to the process described in the previous section.

Lemma 2.6. *The restrictions of simple restricted $H = H(2;(1,1))^{(2)}$-modules $L_H(r)$ to W are as follows. We have $[L_H(0)|W] = L(0)$, $[L_H(1)|W] = [\bigoplus_{j=1}^{p-2} L(j) \oplus L(p-1)^2]$, and*

$$[L_H(r)|W] = \left[\left(\bigoplus_{j=1}^{p-2} L(j) \oplus L(0)^2 \oplus L(p-1)^2\right)^{(r+1)}\right].$$

In particular, for a given p-representation of H, the restriction to W contains the same number of composition factors of type $L(j)$ such that $1 \leq j \leq p-2$.

Proof. The case $r = 0$ is clear. For $r = 1$ notice that $\operatorname{ad} y(e_{a,-1}) = [e_{0,-1}, e_{a,-1}] = 0$ for each $0 \leq a \leq p-2$ and that $\operatorname{ad} h(e_{a,-1}) = [e_{0,0}, e_{a,-1}] = (-a-1)e_{a,-1}$. Thus $[L_H(1)|W]$ contains at least one composition factor isomorphic to $L(r)$ with $1 \leq r \leq p-1$. The sum of the dimensions of these composition factors is p^2-p-1. Together these account for the full 0-weight space of h on $L_H(1)$; $X\partial$ acts non-trivially on the remaining weight spaces. It follows that there is a further composition factor isomorphic to $L(p-1)$, which exhausts the dimension of M.

For the remaining cases $2 \leq r \leq p-1$ we use the algorithm in Proposition 2.2; though this requires some set-up. We have $M(r) = \mathrm{Ind}_{u(H_{(0)},0)}^{u(H,0)} \hat{L}(r)$. We may take a basis $\{v_r, v_{r-2}, \dots, v_{-r}\}$ for $\hat{L}(r)$, where v_i is in the i-weight space for h. Then since $\langle x, y\rangle$ is a vector space complement for $H_{(0)}$ in H, we may take a basis

$$\{x^a y^b \otimes v_i : 0 \leq a \leq p-1,\ 0 \leq b \leq p-1,\ i = r-2c,\ 0 \leq c \leq r\}$$

of $M(r)$. The action of $z \in H$ on M is given by $z.(x^a y^b \otimes v_i) = (z x^a y^b) \otimes v_i$. Every vector in this basis is a weight vector for h. Since $[x, y] = 0$ we have $y.(x^a y^b \otimes v_i) = x^a y^{b+1} \otimes v_i$ and so (recalling $M(r)$ is a p-representation) each $x^a y^{p-1} \otimes v_i$ is killed by y.

One checks that the span of the vectors $\{x^a y^b \otimes v_r : 0 \leq a, b \leq p-1\}$ is a W-submodule, $M(r)_r$. To see this, the key calculation is that $e_{0,r}$ will commute with $x^a y^b$ in $u(H, 0)$ modulo vectors x, y; or $e_{a,b}$ with $a + b > 0$ together with $e_{-1,1}$, any of which kills v_r; or $e_{0,0}$, which stabilises v_r. Moreover we may grade $M(r)_r$ as $M(r)_r = \bigoplus M(r)_r(i)$ with $M(r)_r(2b)$ spanned by the vectors $\{x^a y^b \otimes v_r : 0 \leq a \leq p-1\}$. Then $M(r)_r$ satisfies the hypotheses of Proposition 2.2 and we may write down the composition factors according to the algorithm given there.

Thus let ℓ_i be the list with multiplicities of the h-weights on $M(r)_r(i)$. The highest graded piece is $M(r)_r(2p-2)$ which is spanned by vectors $x^a y^{p-1} \otimes v_r$ on which h has weight $a-(p-1)+r = a+1+r$ for each $0 \leq a \leq p-1$, thus there is a composition factor $L(a)$ for each $0 \leq 1 \leq p-1$. Remove the relevant h-weights according to part (ii) of the algorithm. This leaves ℓ_{2p-2} empty and we may continue with weights in ℓ_{2p-4}. We find additionally a copy of the W_1-module $L(p-1)$. Removing the weights of these, we are left just with the module $L(0)$. In the quotient of $M(r)$ by $M(r)_r$ we have a submodule $M(r)_{r-2} + M(r)_r \subseteq M(r)/M(r)_r$ spanned by $\{x^a y^b \otimes v_{r-2} : 0 \leq a \leq p-1, 0 \leq b \leq p-1\}$. We may grade this similarly via the powers of y and apply the algorithm again with the same result. Since there are $r+1$ values of i (corresponding to vectors $v_{-r}, \dots, v_r$) on which we perform this task, we are done. □

2.4.2. *Non-graded Hamiltonians.* The same task of finding a W_1 subalgebra and its composition factors on restricted representations can be performed for the minimal p-envelopes of the simple Lie algebras $H(2;(1,1);\Phi(\tau))^{(1)}$ of dimension p^2-1 and $H(2;(1,1);\Phi(1))$ of dimension p^2. By [Str09, §10.3, §10.4] the minimal p-envelope Z of each is (p^2+1)-dimensional. Each has a codimension 2 subalgebra $Z_{(0)}$ containing $Z_{(1)}$ of dimension p^2-4 as an ideal and with $Z_0 := Z_{(0)}/Z_{(1)} \cong \mathfrak{sl}_2$. As before, we lift all simple restricted representations from Z_0 to $Z_{(0)}$ by letting $Z_{(1)}$ act trivially and induce the Verma modules $M(r)$ from $Z_{(0)}$ to Z.

We will want to see that the only reducible Verma modules are $M(1)$ and $M(0)$. This is supplied by [FSW14, Theorem 5.3] for $H(2;(1,1);\Phi(\tau))^{(1)}$, but we will need to argue similarly in the other case. For this we need an explicit description of the elements of $H = H(2;(1,1);\Phi(1))$ in terms of elements of W_2. The latter has basis $\{X^i Y^j \partial_X, X^i Y^j \partial_Y : 0 \leq i, j \leq p-1\}$ and graded with $X^i Y^j \partial_X$ and $X^i Y^j \partial_Y$, in degree $i+j-1$. By [Str09, §10.4], H is spanned by the elements $\partial_X(f)\partial_Y - \partial_Y(f)\partial_X - X^{p-1} f \partial_Y$ for $f \in k[X,Y]/(X^p, Y^p)$. Applying this recipe to the monomial $f = X^i Y^j$, we see that H has basis

$$\{jY^{j-1}\partial_X + X^{p-1}Y^j\partial_Y, iX^{i-1}Y^j\partial_Y - jX^iY^{j-1}\partial_X : 1 \leq i \leq p-1, 0 \leq j \leq p-1\}$$

or in divided power notation,

$$\{Y^{(j-1)}\partial_X - X^{(p-1)}Y^{(j)}\partial_Y, X^{(i-1)}Y^{(j)}\partial_Y - X^{(i)}Y^{(j-1)}\partial_X : 1 \le i \le p-1, 0 \le j \le p-1\},$$

where $X^{(-1)} = Y^{(-1)} = X^{-1} = Y^{-1}$ is again understood to be zero.

Only the element $\partial_X - X^{(p-1)}Y\partial_Y$ has a pth power outside this set, *viz.* $-Y\partial_Y$, so that adding for example the element $X\partial_X + Y\partial_Y$ to this basis gives the basis of the minimal p-envelope Z of H.

Since Z is a p-subalgebra of $W_2 = W(2;(1,1))$, we induce a restricted descending filtration on Z from the natural grading $W_2 = \bigoplus_{d=-1}^{2p-3} W_d$, namely $Z_{(n)} := Z \cap W(2;(1,1))_{(n)}$, where $W(2;(1,1))_{(n)} = \bigoplus_{d\ge n} W(2;(1,1))_d$. One checks that this filtration has depth 1 and height $2p-4$: *a fortiori* we have $W(2;(1,1))_r = 0$ for $r \le -2$ or $r \ge 2p-2$, and $W(2;(1,1))_{(2p-3)} = W(2;(1,1))_{2p-3}$ is spanned by the two elements $X^{p-1}Y^{p-1}\partial_X$ and $X^{p-1}Y^{p-1}\partial_Y$ hence has no intersection with Z. We claim that the associated graded algebra $\operatorname{gr} Z$ is isomorphic to $H(2;(1,1))$. A basis of the latter is

$$\{X^{(i-1)}Y^{(j)}\partial_Y - X^{(i)}Y^{(j-1)}\partial_X : 0 \le i,j \le p-1, (i,j) \ne (0,0)\} \cup \{X^{(p-1)}\partial_Y, Y^{(p-1)}\partial_X\}$$

and so we define a linear map from Z to $H(2;(1,1))$ which is an identity on their intersection in $W(2;(1,1))$ and where we send the basis element $Y^{(j-1)}\partial_X - X^{(p-1)}Y^{(j)}\partial_Y \mapsto Y^{(j-1)}\partial_X$. It is clear this descends to an isomorphism of restricted graded Lie algebras $\operatorname{gr}(Z) \to H(2;(1,1))$.

Lemma 2.7. *A simple restricted representation of Z, the minimal p-envelope of either the algebra $H = H(2;(1,1);\Phi(\tau))^{(1)}$ or $H(2;(1,1);\Phi(1))$, is isomorphic to one of $L_Z(0) \cong k$, trivial; $L_Z(1)$, the adjoint module of dimension p^2-1 or p^2, respectively; or $L_Z(r)$ for $2 \le r \le p-1$, the irreducible Verma module of dimension $(r+1)p^2$.*

Proof. A standard argument using Frobenius reciprocity gives every restricted simple module as a quotient of a restricted Verma module. As for the case $H = H(2;(1,1))^{(2)}$ it is straightforward to identify the adjoint module as a quotient of $M(1)$. Thus it suffices to show that the Verma modules $M(r)$ for $2 \le r \le p-1$ are all irreducible. The case where Z is the minimal p-envelope of $H(2;(1,1);\Phi(\tau))^{(1)}$ is given by [FSW14, Theorem 5.3] whose line of argument we follow for the case $H = H(2;(1,1);\Phi(1))$.

From the remarks above, we have $\mathbf{Y} := \operatorname{gr} Z \cong H(2;(1,1))$, which contains the simple graded subalgebra $\mathbf{X} = H(2;(1,1))^{(2)}$ with the cokernel of the map $\mathbf{X} \to \mathbf{Y}$ concentrated in degrees $p-2$ and $2p-4$, i.e. the quotient of $\mathbf{Y}$ by the image of $\mathbf{X}$ is graded with non-trivial components in just these degrees, both of which are contained in $\mathbf{Y}_{(0)}$. Hence the canonical map

$$\operatorname{Ind}_{u(\mathbf{X}_{(0)},0)}^{u(\mathbf{X},0)}(\hat{L}(\lambda)|_{\mathbf{X}}) \to \operatorname{Ind}_{u(\mathbf{Y}_{(0)},0)}^{u(\mathbf{Y},0)}(\hat{L}(\lambda))|_{\mathbf{X}}$$

is an isomorphism. As $\operatorname{Ind}_{u(\mathbf{X}_{(0)},0)}^{u(\mathbf{X},0)}(\hat{L}(\lambda)|_{\mathbf{X}})$ is an irreducible restricted $\mathbf{X}$-module for $\lambda_0 \ne 0,1$ by [Hol98] this implies that $\operatorname{Ind}_{u(\mathbf{Y}_{(0)},0)}^{u(\mathbf{Y},0)}(\hat{L}(\lambda))$ is an irreducible $\mathbf{Y}$-module. Consequently, by Theorem [FSW14, Thm. 4.3], $M(\lambda)$ is an irreducible Z-module unless $\lambda = 0,1$. □

We wish to restrict each simple module to a suitable subalgebra of each non-graded Hamiltonian which is isomorphic to W_1. Thus to play the same game as before, we need in both cases a p-subalgebra W isomorphic to W_1. Such subalgebras do not appear to be well-known. The following lemma gives us a basis of such a subalgebra in each case. The proof (which simply involves checking that the given elements satisfy the commutator relations in W_1) is left to the reader.

Lemma 2.8. *The subalgebra $H(2;(1,1);\Phi(\tau))^{(1)}$ of W_2 contains a p-subalgebra $W = W_1$ having basis*

$$\{(1-X^{(p-1)}Y^{(p-1)})\partial_X, X\partial_X - Y\partial_Y, X^{(2)}\partial_X - XY\partial_Y, X^{(3)}\partial_X - X^{(2)}Y\partial_Y, \ldots, X^{(p-1)}\partial_X - X^{(p-2)}Y\partial_Y\}$$

with these elements playing the roles of $\partial, X\partial, X^{(2)}\partial, \ldots, X^{(p-1)}\partial$, respectively.

The subalgebra $H(2;(1,1);\Phi(1))$ of W_2 contains a p-subalgebra $W = W_1$ having basis

$$\{\partial_Y, Y\partial_Y - X\partial_X, Y^{(2)}\partial_Y - XY\partial_X, Y^{(3)}\partial_Y - XY^{(2)}\partial_X \ldots, Y^{(p-1)}\partial_Y - XY^{(p-2)}\partial_X\}$$

with these elements playing the roles of $\partial, X\partial, X^{(2)}\partial, \ldots, X^{(p-1)}\partial$, respectively.

Finally, the same technique used before yields the following, where again $[V]$ denotes the composition factors of a module V.

Lemma 2.9. *The restrictions of simple restricted modules $L_Z(r)$ for Z the minimal p-envelope of $H = H(2;(1,1);\Phi(\tau))^{(1)}$ or $H(2;(1,1);\Phi(1))$ to the subalgebra W provided by Lemma 2.8 are as follows. We have $[L_Z(0)|W] = L(0)$, $[L_Z(1)|W] = [\bigoplus_{j=0}^{p-2} L(j) \oplus L(p-1)^2]$ for $H = H(2;(1,1);\Phi(\tau))^{(1)}$, $[L_Z(1)|W] = [\bigoplus_{j=1}^{p-2} L(j) \oplus L(p-1)^2 \oplus k^2]$ for $H = H(2;(1,1);\Phi(1))$ and*

$$[L_Z(r)|W] = \left[\left(\bigoplus_{j=1}^{p-2} L(j) \oplus L(0)^2 \oplus L(p-1)^2\right)^{(r+1)}\right].$$

In particular, for a given p-representation of Z, the restriction to W contains the same number of composition factors of type $L(j)$ such that $1 \le j \le p-2$.

Proof. The case $r = 0$ is easy. For $r = 1$, we analyse the action of $h = \pm(X\partial_X - Y\partial_Y)$ on vectors killed by $\operatorname{ad}\partial$ in each case.

In the first case, $\partial = (1 - X^{(p-1)}Y^{(p-1)})\partial_X$. Each of the vectors $Y^{(r)}\partial_X$ for $1 \le r \le p-2$ in H together with ∂ itself are killed by $\operatorname{ad}\partial$ and one gets one of each weight from 0 to $p-2$. Thus there must be at least one composition factor of each type in $L_H(1)|W$. This accounts for $p^2 - 2p + 1$ dimensions of $L_H(1)$ and there remain $2p-2$ to find. However, the 0 weight space for h is accounted for so we must have two copies of $L(p-1)$ remaining.

In the second case, each of the vectors $X^{(r)}\partial_Y$ for $0 \le r \le p-1$ are killed by $\partial = \partial_Y$ with one of each weight from 0 to $p-2$ occurring. Further, one checks that the span of $\{X^{(p-1)}\partial_Y, X^{(p-1)}Y\partial_Y - \partial_X, X^{(p-1)}Y^{(2)}\partial_Y - Y\partial_X, \ldots, X^{(p-1)}Y^{(p-1)}\partial_Y - Y^{(p-2)}\partial_X\}$ is a p-dimensional W-submodule of $\mathfrak{g}$. This contains a further trivial submodule spanned by the vector $X^{(p-1)}\partial_Y$, hence has the structure $L(p-1)/k$, isomorphic to the Verma module $Z^+(p-1)$ as a W_1-module. Counting up the dimensions now found, there is just one left, which must correspond to a trivial composition factor.

The case of the Verma modules $M(r)$ is similar to that in Lemma 2.6; we indicate the changes. Let first $H = H(2;(1,1);\Phi(\tau))^{(1)}$. The elements $x := (1 - X^{(p-1)}Y^{(p-1)})\partial_X$ and $y := (1 - X^{(p-1)}Y^{(p-1)})\partial_Y$ complement $H_{(0)}$, $Z_{(0)}$ and $\mathbf{Y}_{(0)}$ in H, Z and $\mathbf{Y}$ respectively. Thus the induced module $M(r) = \operatorname{Ind}_{u(Z_{(0)},0)}^{u(Z,0)}(\hat{L}(r))$ is still spanned by basis vectors $\{x^a y^b \otimes v_i : 0 \le a \le p-1,\ 0 \le b \le p-1,\ i = r-2c,\ 0 \le c \le r\}$. On this, we still have $x \cdot (x^a y^b \otimes v_i) = x^{a+1}y^b \otimes v_i$ and $y \cdot (x^a y^b \otimes v_i) = x^a y^{b+1} \otimes v_i$. We also have $h := X\partial_X - Y\partial_Y$ satisfying $[h,x] = -x$ and $[h,y] = y$. Thus we may grade $M(r)$ as a vector space as before in such a way that x and h act in the same way as they did in Lemma 2.6. Now by Lemma 2.1 the composition factors of $M(r)|W$ are determined

$\mathfrak{g}$	p	$\mathcal{O}$	$\mathfrak{g}_e(0) \cap \operatorname{im} \operatorname{ad} e$
E_6	5	A_4	$\mathfrak{z}(\mathfrak{l}')$
E_6	5	A_4A_1	$\mathfrak{z}(\mathfrak{l}')$
E_7	5	A_4	$\mathfrak{z}(\mathfrak{l}')$
E_7	5	A_4A_1	$\mathfrak{z}(\mathfrak{l}')$
E_7	5	A_4A_2	$\mathfrak{g}_e(0) \cong A_1$
E_7	7	A_6	$\mathfrak{g}_e(0) \cong A_1$
E_8	7	A_6	$A_1 \subseteq \mathfrak{g}_e(0) \cong A_1^2$
E_8	7	A_6A_1	$\mathfrak{g}_e(0) \cong A_1$

TABLE 1. Non-trivial intersections of $\operatorname{im} \operatorname{ad} e$ with $\mathfrak{g}_e(0)$.

by the action of these elements and since their action is identical to that in Lemma 2.6 the set of composition factors is also identical. For the case $H = H(2; (1,1); \Phi(1))$ the elements x, h, y are actually precisely as in Lemma 2.6 and the same argument guarantees the result. □

3. Finding W_1 subalgebras: Proof of Theorem 1.1.

Let G be an exceptional simple algebraic group over an algebraically closed field of good characteristic p, with $\mathfrak{g}$ its Lie algebra. In this section (together with Appendix A) we prove Theorem 1.1 by means of a series of lemmas. Proposition 3.3 will be quite central in proving the conjugacy statements involved: that is part (iv) of Theorem 1.1. For this we need the following two lemmas.

Lemma 3.1. *Let $0 \neq e \in \mathfrak{g} = \operatorname{Lie}(G)$ be a p-nilpotent element and let $\mathfrak{h} := \langle e \rangle$ be the subspace it generates. Then any torus $\mathfrak{c} \subseteq \mathfrak{g}$ normalising $\mathfrak{h}$ is $\mathfrak{c} = \operatorname{Lie}(C)$ for C a torus of G normalising $\mathfrak{h}$.*

Proof. We first prove that the normaliser $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G is smooth: By the existence of associated cocharacters, there is a one-dimensional torus S which normalises $\mathfrak{h} = \langle e \rangle$ but does not centralise it. Differentiating the cocharacter, we get also a 1-dimensional toral subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ which normalises e but does not centralise e. We may calculate the dimension of $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ by looking at its action on $\mathfrak{h}$. By rank–nullity, we have $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \dim \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}) + 1$. Equally, we may calculate the dimension of $N_G(\mathfrak{h})$ by looking at its action on $\mathfrak{h}$. Again, we have $\dim N_G(\mathfrak{h}) = \dim C_G(\mathfrak{h}) + 1$. But since p is very good, $C_G(\mathfrak{h}) = G_e$ is smooth, so $\dim C_G(\mathfrak{h}) = \dim \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$. Hence the dimensions of the group–theoretic and Lie–theoretic normalisers coincide and the normaliser is smooth as required.

If H is any smooth algebraic group, then [Hum67, Theorem 13.3] shows that any maximal torus $\mathfrak{t}$ of $\mathfrak{h}$ is $\operatorname{Lie}(T)$ for T a maximal torus of H. In particular, this applies to $N_G(\mathfrak{h})$. We may choose an embedding $N_G(\mathfrak{h}) \subseteq \mathrm{GL}_n$ with $\mathfrak{c} \subseteq \mathfrak{t}$ diagonal. Now [Die52, Prop. 2] gives that $\mathfrak{c} = \operatorname{Lie}(C)$ for $C \subseteq T \subseteq N_G(\mathfrak{h})$. □

Lemma 3.2. *Suppose e is a nilpotent element in $\mathfrak{g}$, distinguished in a Levi subalgebra $\mathfrak{l} = \operatorname{Lie}(L)$. Then $\operatorname{im} \operatorname{ad} e \cap \mathfrak{g}_e(0) = 0$ unless L has a factor of type A_{p-1}. For the remaining eight orbits, the intersections are given in Table 1.*

Proof. Unless $\mathcal{O}$ is one of the exceptional orbits, this is stated in [Jan04, p57]. For the remainder, one simply takes a general element in $\mathfrak{g}(-2)$ and applies e to it to get a general element v in $\operatorname{im} \operatorname{ad} e \cap \mathfrak{g}(0)$. Then insisting that the general element satisfies $[e, v] = 0$ gives the data above. See Appendix A on how such calculations can be carried out with the help of GAP. □

We are now in a position to prove the key result about finding suitable cocharacters associated to a nilpotent element e. The toral element J will later be taken to equal $X\partial \in W_1$.

Proposition 3.3. *Suppose $e \in \mathfrak{g}$ is a nilpotent element and let $J = J^{[p]}$ be a toral element of $\mathfrak{g}$ normalising but not centralising $\langle e\rangle$. Let χ be an associated cocharacter to e and let $\mathfrak{g}(i)$ be the associated i-th graded piece of $\mathfrak{g}$. Then the following hold:*

(i) *J is conjugate by an element g of $R_e = R_u(G_e)$ to an element of $\mathfrak{g}(0)$; thus replacing χ by its conjugate by g^{-1} and taking the new associated grading $\mathfrak{g}(i)$, we may assume that J normalises $\mathfrak{g}(i)$ for each $i \in \mathbb{Z}$.*

(ii) *There exists a cocharacter τ associated to e and a toral element H with $\mathrm{Lie}(\tau(\mathbb{G}_m)) = \langle H\rangle$ such that $J = H + H_0$ for some toral element $H_0 \in \mathfrak{g}_e(0)$.*

(iii) *Suppose e is not in an orbit containing a factor of type A_{p-1}, and that J is in the image of $\mathrm{ad}\, e$. Then there is a cocharacter τ associated to e with $\mathrm{Lie}(\tau(\mathbb{G}_m)) = \langle J\rangle$.*

Proof. By Lemma 3.1 there is a torus $T_1 \leq G$ such that $\mathrm{Lie}(T_1) = \langle J\rangle$, with T_1 normalising $\langle e\rangle$. Since J acts non-trivially on e so does T_1. Let $T_2 = \chi(\mathbb{G}_m)$. Now for each $t_1 \in T_1$ there exists $t_2 \in T_2$ such that $t_1.e = t_2.e$ so that $t_1 = t_2.s$ with $s \in G_e$. Thus $T_1G_e = T_2G_e$. Write $G_e = C_eR_e$ with C_e reductive. Then T_2C_e is a subgroup of T_2G_e. Furthermore, T_2C_e is a complement to R_e in the semidirect product $T_2C_e \ltimes R_e$ and so we may take the image $\bar{T}_1$ of T_1 in T_2C_e under the projective homomorphism $T_2C_eR_e \to T_2C_e$. Now $T_1 \cap R_e = \{1\}$ so that $T_1 \subseteq \bar{T}_1R_e$ is a complement to R_e in $\bar{T}_1R_e$. Since R_e is unipotent, T_1 is a maximal torus of the group $\bar{T}_1R_e$ and hence is conjugate to $\bar{T}_1$ by an element of R_e. Thus $\mathrm{Lie}(T_1) = \langle J\rangle$ is conjugate to a subalgebra of $\mathrm{Lie}(T_2C_e) \leq \mathfrak{g}(0)$ as required. For the last part of (i) observe that $J^g \in \mathfrak{g}(0)$ means that $\chi(t)gJg^{-1}\chi(t)^{-1} = gJg^{-1}$. Thus $\chi(t)^{g^{-1}}J\chi(t)^{-g^{-1}} = J$. This proves (i).

For (ii), by part (i) we may assume that J belongs to $\mathfrak{g}(0)$. Take T_2 as above and let $\mathrm{Lie}\, T_2 = \langle H\rangle$ with H chosen so that $[H, e] = [J, e]$. Thus $J - H$ is an element in the zero-grade of the centraliser $\mathfrak{g}_e(0)$, so we may write $J = H + H_0$ with $H_0 \in \mathfrak{g}_e(0)$.

Hence $\tau = \chi$ satisfies the assertions in (ii).

For (iii), we claim that H and H_0 are in the image of $\mathrm{ad}\, e$. For H, we may consider an optimal SL_2-homomorphism $\varphi : \mathrm{SL}_2 \to G$ associated to e, see [McN05, Prop. 33]. By definition, $d\varphi$ maps the nilpotent $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ onto e, and the map $t \mapsto \varphi\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ is a cocharacter associated to e. Temporarily replacing everything by a conjugate by an element of G_e we may assume that this cocharacter is equal to χ. Thus $\langle H\rangle = \mathrm{Lie}(\chi(\mathbb{G}_m))$ and H is in the image of $\mathrm{ad}\, e$. Thus H_0 is in the image of $\mathrm{ad}\, e$ also.

Now, since the orbit type of e does not have a factor of type A_{p-1}, by Lemma 3.2 we must have $H_0 = 0$. Thus $J = H$ and we may take $\tau = \chi$. This proves (iii). ∎

If we are in the situation of part (iii) of the previous proposition, the grading on $\mathfrak{g}$ obtained from any nilpotent element e representing $\partial \in W_1 \subseteq \mathfrak{g}$ is compatible with the action of $X\partial \in W_1$ via $X\partial = -\frac{1}{2}d\tau(1)$, so that the weight of $X\partial$ on $\mathfrak{g}(r)$ is $-\frac{1}{2}r$. This observation will be used in the sequel.

Lemma 3.4. *Let $e \in \mathfrak{g}$ be a nilpotent element, let τ be an associated cocharacter and $\mathfrak{g}(k)$ the kth graded piece of $\mathfrak{g}$ associated to τ. Suppose $\mathfrak{g}$ contains a p-subalgebra $W \cong W_1$ with $e = \partial$. Then the following two conditions must be satisfied:*

(i) *We have* $e^{[p]} = 0$*;*
(ii) *The maximal value of* k *with* $\mathfrak{g}(-k) \neq 0$ *satisfies* $2p-4 \leq k \leq 2p-2$*.*

Proof. The necessity of (i) is clear, since W is assumed to be a p-subalgebra of $\mathfrak{g}$ and the condition $\partial^{[p]} = 0$ holds in W.

For (ii), the upper bound follows from (i) and [McN05, Prop. 30]. For the lower bound, observe that e must be a non-zero vector in the image of $\mathrm{ad}(e)^{p-1}$. This implies that there is a vector f in $\mathfrak{g}(-2p+4)$ for which $\mathrm{ad}(e)^{p-1}(f) = e$. □

The next lemma proves the first two parts of Theorem 1.1. We perform several case by case checks on nilpotent elements satisfying the conclusions of Lemma 3.4 in order to check that the relevant statements hold.

Lemma 3.5. *Suppose* $e \in \mathfrak{g}$ *represents* ∂ *in a* p*-subalgebra* W *of* $\mathfrak{g}$*. Then the statements of Theorem 1.1(i) and (ii) hold.*

Unless the Levi $\mathfrak{l}$ *associated to* e *is of type* A_{p-1}*, there is a unique* 1*-space* $\langle f\rangle \subseteq \mathfrak{g}(-2p+4)$ *such that* $\mathrm{ad}(e)^{p-1}\langle f\rangle = \langle e\rangle$*.*

Proof. This is a case by case check. To start with, we reduce the number of cases we must consider using Lemma 3.4. For each nilpotent orbit e in good characteristic, we may check to see whether it satisfies $e^{[p]} = 0$, for example by looking at the tables in [Law95]. (The validity of these results for nilpotent elements follows from [PS15, Thm. 4.1].) For each of these, we take an associated cocharacter τ. Helpfully, associated cocharacters are listed in the tables in [LT11]. One can then apply τ to each root vector and establish the dimensions of each piece of the associated grading. Since $\mathfrak{g}(-2p+4)$ is assumed to be non-zero, the possible cases for nilpotent orbits through e are given in Table 2.

Let us give an example: Suppose e belongs to the orbit $F_4(a_2)$. Then from [LT11, p78] we see that the associated cocharacter τ can be taken to satisfy $\langle\alpha_2,\tau\rangle = 2 = \langle\alpha_4,\tau\rangle, \langle\alpha_1,\tau\rangle = 0 = \langle\alpha_3,\tau\rangle$.

Applying this to the negative of the highest root with coefficients -2342, we see this is in τ-weight $-2\cdot 3 - 2\cdot 2 = -10$. Running through the remainder of the roots we can establish the possible τ-weights which occur. Indeed -10 is the lowest weight. Now if $2p-4 \leq 10 \leq 2p-2$, we have that $p = 7$. This explains the entry for $F_4(a_2)$ in Table 2.

To complete the first part, one must check for each case in Table 2 to see whether the remaining hypothesis of the lemma is satisfied. This is easily done using GAP, but one can of course do such calculations by hand. For a negative example, let us again consider the orbit $F_4(a_2)$ for $p = 7$. Here we may assume $e = e_{1110} + e_{0001} + e_{0120} + e_{0100}$. The space $\mathfrak{g}(-10)$ is spanned by e_{-1342}, e_{-2342}. Let $f = y_1 e_{-1342} + y_2 e_{-2342}$ be a generic element in $\mathfrak{g}(-10)$. We compute

$$\mathrm{ad}(e)^6(f) = y_1(2\cdot e_{0011} + e_{1100} + 2\cdot e_{0110} + 2\cdot e_{1120}) + y_2(e_{0001} + e_{1110} + e_{0120}),$$

an expression which for no choice of y_1, y_2 is a nonzero multiple of e (for instance it does not involve e_{0100}). So e does not satisfy the required conditions.

For a positive example, let us take the simplest case where $\mathfrak{g}$ is of type G_2, $p = 7$ and e is regular. We may choose $e = e_{10} + e_{01}$. Corresponding to the associated cocharacter τ with weights 2 on 10 and 01, the τ-weight space with weight $-2p+4 = -10$ is occupied just by the span of the root vector which is the negative of the highest root, namely -32. (Thus the uniqueness assertion

G	p	$\mathcal{O}$
G_2	7	G_2
F_4	13	F_4
	7	$F_4(a_2)$
	7	C_3
	7	B_3
	5	$F_4(a_3)$
	5	$C_3(a_1)$
	5	B_2
E_6	13	E_6
	7	$E_6(a_3)$
	7	$D_5(a_1)$
	7	A_5
	7	D_4
	5	A_4A_1
	5	A_4
	5	$D_4(a_1)$
	5	A_3A_1
	5	A_3

G	p	$\mathcal{O}$
E_7	19	E_7
	13	$E_7(a_2)$
	13	E_6
	11	$E_7(a_3)$
	11	D_6
	7	A_6
	7	$E_7(a_5)$
	7	$E_6(a_3)$
	7	$D_6(a_2)$
	7	$D_5(a_1)A_1$
	7	A_5A_1
	7	$(A_5)'$
	7	$D_5(a_1)$
	7	D_4A_1
	7	D_4
	7	$(A_5)''$
	5	A_4A_2
	5	A_4A_1
	5	$A_3A_2A_1$
	5	A_4
	5	A_3A_2
	5	$D_4(a_1)A_1$
	5	$D_4(a_1)$
	5	$A_3A_1^2$
	5	$(A_3A_1)'$
	5	$(A_3A_1)''$
	5	A_3

G	p	$\mathcal{O}$
E_8	31	E_8
	19	$E_8(a_3)$
	19	E_7
	13	$E_8(a_5)$
	13	$E_8(b_5)$
	13	D_7
	13	$E_7(a_2)$
	13	E_6A_1
	13	E_6
	11	$E_8(a_6)$
	11	$D_7(a_1)$
	11	$E_7(a_3)$
	11	D_6
	7	A_6
	7	A_6A_1
	7	$E_8(a_7)$
	7	$E_7(a_5)$
	7	$E_6(a_3)A_1$
	7	$D_6(a_2)$
	7	$D_5(a_1)A_2$
	7	A_5A_1
	7	$E_6(a_3)$
	7	D_4A_2
	7	$D_5(a_1)A_1$
	7	A_5
	7	$D_5(a_1)$
	7	D_4A_1
	7	D_4

TABLE 2. Cases to be checked in Lemma 3.5

follows immediately in this case.) Now apply $(\mathrm{ad}(e_{10}+e_{01}))^6$ to e_{-32}. One finds the answer is $10\cdot e_{10}+(-18)\cdot e_{01}$ which is $3\cdot e_{10}+3\cdot e_{01}$ modulo 7 so that e satisfies the hypotheses of the lemma. Now notice that $7=h+1$ and e is regular in $\mathfrak{g}$ as required.

In case the weight space $\mathfrak{g}(-2p+4)$ is bigger than one-dimensional, say $\mathfrak{g}(-2p+4)$ is spanned by $x_1,\dots,x_r$, then apply $(\mathrm{ad}\,e)^{p-1}$ to $\sum\lambda_i x_i$ and equate this to e. This puts conditions on the λ_i which are either uniquely satisfied, or one is in the case of one of the exceptions when $\mathfrak{l}$ is of type A_{p-1}. □

The following lemma proves the existence assertion of Theorem 1.1 by guaranteeing the existence of at least one p-subalgebra of type W_1 whenever (e,p) satisfy the conditions (i) and (ii) of Theorem 1.1.

Lemma 3.6. *Suppose (e,p) satisfy the conditions (i) and (ii) of Theorem 1.1 and let $\mathfrak{l}$ be a Levi subalgebra in which e is regular. Then there exists a p-subalgebra $W\cong W_1\le\mathfrak{l}'$ with ∂ represented by e.*

Proof. We have $\mathfrak{l}'$ is simple. If $\mathfrak{l}'$ is simple of classical type, then Lemma 2.4 gives the result. The remaining cases occur when e is regular in a Levi of exceptional type and $(p, \mathfrak{l}')$ is one of $(7, G_2)$, $(13, F_4)$, $(13, E_6)$, $(19, E_7)$ or $(31, E_8)$.

In each of these cases we may, without loss of generality, assume $\mathfrak{l}' = \mathfrak{g}$. Now Lemma 3.5 implies that there is, up to scalars, a unique element $f \in \mathfrak{g}(-2p+4)$ such that e is a non-zero multiple of $(\operatorname{ad} e)^{p-1} f$; say $(\operatorname{ad} e)^{p-1} f = \lambda e$. (In each case, we may take f to be the root vector corresponding to the negative of the highest root.) Now according to the standard basis of W_1, we have $(\operatorname{ad} \partial)^{p-1} X^{p-1}\partial = (p-1)!\partial$. Thus, replacing f with $f.(p-1)!/\lambda$ it suffices to check that there is a homomorphism $W_1 \to \mathfrak{g}$ obtained by sending

$$(X^{p-1}\partial, X^{p-2}\partial, \dots, X\partial, \partial) \to \{f, 1/(p-1).[e,f], \dots, 1/(p-2)!(\operatorname{ad} e)^{p-2} f, 1/(p-1)!(\operatorname{ad} e)^{p-1} f = e\}.$$

For this it suffices to check that the latter elements satisfy the commutator and p-th power relations in W_1. This is a straightforward check using the commutator relations amongst basis elements of $\mathfrak{g}$. These were performed in GAP. □

By analogy with the notion of a regular A_1 subalgebra, let us say that a p-subalgebra W_1 of $\mathfrak{g}$ is *regular* if ∂ is represented by a regular element of $\mathfrak{g}$. (Of course, a regular W_1 then contains a regular A_1 subalgebra $\langle \partial, X\partial, X^2\partial \rangle \cong \mathfrak{sl}_2$.) In order to show that non-regular W_1s are not maximal, we will want to show that each normalises a non-trivial abelian subalgebra of $\mathfrak{g}$. For this many cases can be dealt with by showing that all possible modules whose composition factors coincide with those of the restriction $\mathfrak{g}|W$ of the adjoint module $\mathfrak{g}$ to W must contain a trivial submodule. To do this we will compute in the next lemma all the possible composition factors.

Lemma 3.7. *Suppose $W \cong W_1$ is a p-subalgebra of $\mathfrak{g}$ such that the element $\partial \in W$ is represented by the nilpotent element e not of type A_{p-1}. Then the composition factors of W on $\mathfrak{g}$ are given in Table 3.*

Proof. By Proposition 3.3 we may assume that there is a cocharacter τ associated to e such that $X\partial \in \operatorname{Lie}(\tau(\mathbb{G}_m))$. Thus there is a grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ with $e = \partial \in \mathfrak{g}(2)$ acting in degree 2 and with $X\partial$ acting with weight $-i/2$ on each piece. So we may invoke Proposition 2.2. Thus we need only know the dimensions of each $\mathfrak{g}(i)$.

Let us give an example. Suppose e is a nilpotent element of F_4 of type C_3. Then since $e \in W$, we have $p = 7$. The non-zero graded pieces are listed below (note that $\dim \mathfrak{g}(i) = \dim \mathfrak{g}(-i)$):

i	0	1	2	3	4	5	6	7	8	9	10
$\dim \mathfrak{g}(i)$	6	4	3	4	2	2	2	2	1	2	1

Thus $\operatorname{ad}\partial$ must have at least a one-dimensional kernel on $\mathfrak{g}(10)$, $\mathfrak{g}(6)$ and $\mathfrak{g}(2)$ a two-dimensional kernel on $\mathfrak{g}(9)$ and $\mathfrak{g}(3)$. Thus amongst the composition factors of $L(G)|W$ we have must at least find $L(1)$, $L(3)$, $L(5)$, $L(5)^2$ and $L(1)^2$, respectively. Peeling off the weights corresponding to these submodules leaves just three weights in the zero weight space, which must correspond to three trivial composition factors. Thus the composition factors are $[\mathfrak{g}|W] = L(1)^3, L(5)^3, L(3), k^3$. □

Remarks 3.8. (i). Where $\mathcal{O}$ is A_{p-1}, the potential composition factors of a corresponding p-subalgebra of type W_1 will in fact differ according to which toral element represents $X\partial = H + H_0$, since unlike in the case of Proposition 3.3(iii) they are not necessarily unique up to conjugacy. We have computed these also since we need them for the proof of Theorem 1.3, but since they require slightly more in-depth computations in GAP, we leave this to Appendix A, Proposition A.1.

$\mathfrak{g}$	p	$\mathcal{O}$	$[\mathfrak{g}\vert W]$
G_2	7	G_2	$L(1), L(5)$
F_4	13	F_4	$L(1), L(5), L(7), L(11)$
	7	C_3	$L(1)^3, L(3), L(5)^3, k^3$
	7	B_3	$L(1), L(3)^5, L(5), k^3$
	5	B_2	$L(1), L(2)^4, L(3), L(4)^4, k^6$
E_6	13	E_6	$L(1), L(4), L(5), L(7), L(8), L(11)$
	7	A_5	$L(1)^3, L(2), L(3), L(4), L(5)^3, L(6)^2, k^3$
	7	D_4	$L(1), L(3)^8, L(5), k^8$
	5	A_3	$L(1), L(2)^5, L(3), L(4)^8, k^{11}$
E_7	19	E_7	$L(1), L(5), L(7), L(9), L(11), L(13), L(17)$
	13	E_6	$L(1), L(4)^3, L(5), L(7), L(8)^3, L(11), k^3$
	11	D_6	$L(1), L(2)^2, L(3), L(5)^2, L(7), L(8)^2, L(9), L(10)^2, k^3$
	7	$(A_5)'$	$L(1)^3, L(2)^3, L(3), L(4)^3, L(5)^3, L(6)^6, k^6$
	7	D_4	$L(1), L(3)^{14}, L(5), k^{21}$
	7	$(A_5)''$	$L(1), L(2)^7, L(3), L(4)^7, L(5), k^{14}$
	5	A_3	$L(1), L(2)^7, L(3), L(4)^{16}, k^{24}$
E_8	31	E_8	$L(1), L(7), L(11), L(13), L(17), L(19), L(23), L(29)$
	19	E_7	$L(1), L(4)^2, L(5), L(7), L(9), L(11), L(13), L(14)^2, L(17), L(18)^2, k^3$
	13	D_7	$L(1)^3, L(3), L(4)^2, L(5), L(6)^3, L(7), L(8)^2, L(9), L(11)^3, L(12)^2, k^3$
	13	E_6	$L(1), L(4)^7, L(5), L(7), L(8)^7, L(11), k^{14}$
	11	D_6	$L(1), L(2)^4, L(3), L(5)^6, L(7), L(8)^4, L(9), L(10)^4, k^{10}$
	7	A_5	$L(1)^3, L(2)^7, L(3), L(4)^7, L(5)^3, L(6)^{14}, k^{17}$
	7	D_4	$L(1), L(3)^{26}, L(5), k^{52}$

Table 3. Composition factors of subalgebras $W \cong W_1$ containing a nilpotent element not of type A_{p-1}

(ii). Notice that the number of trivial composition factors listed in Table 3 for each case is always $\dim C_e$, the dimension of the reductive part of the centraliser. (This follows for example because one always has $\dim \mathfrak{g}_e(rp) = 0$ for all $r > 0$ in these cases.) This observation does not hold when one considers the cases where e is of type A_{p-1} and $X\partial \notin \mathrm{Lie}(\tau(\mathbb{G}_m))$ for τ an associated cocharacter to e.

To prove part (v) of Theorem 1.1 we will want to see that in many cases, the composition factors listed above can only appear in modules in which there is a fixed vector, hence forcing a corresponding W_1 subalgebra into a parabolic. The following lemma gives a useful bound to ensure this.

Lemma 3.9. *Suppose $p \geq 5$ and V is a W_1-module with $[V : k] = n_0 > 0$, $[V : L(p-1)] = n_{-1}$ and $[V : L(1)] = [V : L(p-2)] = n_1$. Then if V contains no trivial submodule, we have $n_0 \leq 2n_{-1} + n_1$.*

Proof. We proceed by induction on the number of composition factors. We cannot have V irreducible. If V contains 2 composition factors then the module V is uniserial with successive composition factors $k/L(1)$ or $k/L(p-1)$. Thus the result holds in both these cases.

Suppose the number of composition factors of V is r. Then there is an irreducible submodule S, say, with $S = L(t)$, $t \neq 0$. First suppose V/S contains no trivial submodules. Then by induction we have $n_0 \leq 2n_{-1} + n_1$ if $t \neq 1, p-1$; $n_0 \leq 2(n_{-1} - 1) + n_1 \leq 2n_{-1} + n_1$ if $t = p - 1$; and $n_0 \leq 2n_{-1} + n_1 - 1 \leq 2n_{-1} + n_1$ if $t = 1$, which proves the result in all these cases. Now suppose V/S contains a trivial submodule R of dimension l. Then from the exact sequence $0 \to S \to V \to V/S \to 0$, taking the preimage R' of R in V we have an exact sequence $0 \to S \to R' \to R \to 0$ with no non-trivial W_1-module map $R \to R'$. By [BNW09, Theorem A,B] there are no self-extensions of the trivial module, so V/R' contains no trivial submodule and we may appeal to induction. Since there is no non-trivial W_1-module map $R \to R'$, it follows that $\dim \mathrm{Ext}^1_{W_1}(k, S) \geq l$. Thus by [BNW09, Theorem A,B], $l \leq 2$. If $l = 2$ then $S = L(p-1)$ and in the quotient of V by R' we have $n_0 - 2$ trivial composition factors and $n_{-1} - 1$ composition factors isomorphic to $L(p-1)$. Thus by induction, the lemma holds for V/R' and we have $n_0 - 2 \leq 2(n_{-1} - 1) + n_1$ as required. If $l = 1$ then either $S = L(p-1)$ or $S = L(1)$. A similar argument by induction shows that the lemma holds again in each case. □

The next lemma proves part (v) of Theorem 1.1.

Lemma 3.10. *Let W be a p-subalgebra of $\mathfrak{g}$ with ∂ represented by a nilpotent element e. Then if e is not a regular element, W normalises a non-trivial abelian subalgebra of $\mathfrak{g}$, hence is not maximal.*

Proof. We use Lemma 3.9 together with Table 3 when e is not of type A_{p-1}. Assume e is not regular or of type A_{p-1}, yet there is no trivial submodule in $\mathfrak{g}|W$. Inspecting Table 3 together with Lemma 3.9 one sees that this rules out many cases.

We use a special argument in the case $(\mathfrak{g}, p, e)$ is $(E_7, 11, D_6)$ or $(E_8, 19, E_7)$. Since W is assumed to be a subalgebra of $\mathfrak{g}$, its adjoint representation $L(p-2)$ must appear as a submodule of $\mathfrak{g}|W$. Thus its unique composition factor $L(1)$ must appear in the head of the module. Take the quotient by $L(p-2)$ of the largest submodule of $\mathfrak{g}$ not containing the composition factor $L(1)$; call this M. Then M is still self-dual containing no composition factors of the form $L(1)$ or $L(p-2)$. By [BNW09, Theorem A,B] the three trivial composition factors must appear in indecomposable subquotients of the form $k/L(p-1)$. Because M is self-dual, there is a subquotient of the form $L(p-1)/k$. The composition factor k in this subquotient must appear in a (different) subquotient of the form $k/L(p-1)$ and the submodule generated by vectors in $L(p-1)$ must contain both composition factors of this type. Since this applies to all composition factors of type k, we must have a subquotient of the form $L(p-1)/(k \oplus k \oplus k)$, but this is a contradiction by [BNW09, Theorem A,B] as $\mathrm{Ext}^1_{W_1}(L(p-1), k)$ has dimension just 2.

This rules out all but the remaining twelve cases:

$$(\mathfrak{g}, p, e) = (E_6, 5, A_4),\ (E_6, 7, A_5),\ (E_7, 5, A_4),\ (E_7, 7, (A_5)'),\ (E_7, 7, A_6),\ (E_8, 13, D_7),$$
$$(E_8, 7, A_6),\ (E_8, 7, A_5),\ (E_7, 5, A_3),\ (E_6, 5, A_3),\ (F_4, 5, B_2),\ (F_4, 7, C_3).$$

For these we perform an intricate series of direct checks in GAP to find a non-trivial abelian subalgebra normalised by W in $\mathfrak{g}$ in all cases. See Appendix A. □

We now establish the remaining statements of Theorem 1.1.

Lemma 3.11. (i) *There is a unique conjugacy class of regular W_1s.*
(ii) *A regular W_1 in $\mathfrak{g}$ is maximal if and only if $\mathfrak{g}$ is not of type E_6.*

Proof. Suppose the element $X^{p-1}\partial$ is represented by a nilpotent element f. By Proposition 3.3(iii) there is a cocharacter τ associated to $e = \partial$ with $\mathrm{Lie}(\tau(\mathbb{G}_m)) = \langle X\partial \rangle$. We have $X\partial = -\frac{1}{2}d\tau(1)$.

Since $[X\partial, X^{p-1}\partial] = (p-2)X^{p-1}\partial$, we get that f is in the direct sum of the τ-weight spaces congruent to $-2(p-2) = -2p+4$ modulo p. Since e is regular, all τ-weights are even. It follows that $f \in \mathfrak{g}(-2p+4) \oplus \mathfrak{g}(4)$. Now using GAP, relations of the form $[[f,e],f] = 0$, $[[\dots[[f,e],e],\dots,e],f] = 0$ quickly imply that $f \in \mathfrak{g}(-2p+4)$. For example let $\mathfrak{g} = G_2$, so that $p = 7$ and $e = e_{10} + e_{01}$. We have $\mathfrak{g}(-10) = \langle e_{-32} \rangle$ and $\mathfrak{g}(4) = \langle e_{11} \rangle$. Then $f = \lambda_1 e_{-32} + \lambda_2 e_{11}$. Now $[[f,e],f] = -6\lambda_2^2 e_{32} + 2\lambda_1\lambda_2 e_{-11}$. Thus we must have $\lambda_2 = 0$ as required.

Now, since $\dim \mathfrak{g}(-2p+4) = 1$, f is unique up to scalars. Thus W is uniquely determined by e, proving (i).

For (ii), recall F_4 is a subalgebra of E_6. Under this embedding, one checks that a regular element in F_4 is also a regular element in E_6. By (i) there is a unique conjugacy class of subalgebras of type W_1, hence each is in a subalgebra of type F_4 and is not maximal.

It remains to show that a regular W_1 is maximal in the remaining types. Let W be such a p-subalgebra and suppose it is not maximal. Then in the adjoint representation, there is a minimal W-supermodule of W which generates a proper subalgebra. The composition factors $[\mathfrak{g}|W]$ were given in Table 3. One checks that for each type of $\mathfrak{g}$, the dimension of the nullspace of ∂ is equal to the number of composition factors in $[\mathfrak{g}|W]$. Thus there is a basis $\mathcal{B}$ of τ-weight vectors for the nullspace of ∂ corresponding to each composition factor. Since in Table 3 all the composition factors are pairwise distinct, a minimal supermodule of W contains one of the elements in $\mathcal{B}$, v say. It is then a computation in GAP to check that in all cases, $\langle W, v \rangle = \mathfrak{g}$. This proves that W is maximal. □

Proof of Theorem 1.1. We summarise by pointing out where in this section we have established the relevant statements in Theorem 1.1. Parts (i) and (ii) are found in Lemma 3.5; parts (iii) and (iv) are Lemma 3.11; part (v) is Lemma 3.10; the existence assertion is Lemma 3.6. □

4. Other non-classical subalgebras: Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3. We first give a reduction to finding subalgebras of $\mathfrak{g}$ which are less than p^3-3-dimensional.

Lemma 4.1. *There is no proper simple subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of dimension $p^3 - 3$ or higher.*

Proof. Since $5^3 = 125$ and $7^3 = 343$ the only possibility for such a subalgebra would be in E_7 when $p = 5$. We may enlarge $\mathfrak{h}$ to be a maximal p-subalgebra $\mathfrak{h}'$. Now take a Weisfeiler filtration $(\mathfrak{g}_{(n)})_{n\in\mathbb{Z}}$ with $\mathfrak{g}_{(0)} = \mathfrak{h}'$. By [SF88, Proposition 3.1.1], the quotient $\mathfrak{g}_{(-1)}/\mathfrak{g}_{(0)}$ is a faithful irreducible module for $\mathfrak{g}_{(0)}/\mathfrak{g}_{(1)}$. Since $\mathfrak{h}$ is simple, and $\mathfrak{g}_{(1)}$ is a nilpotent ideal of $\mathfrak{g}_{(0)}$ we have $\mathfrak{h} \cap \mathfrak{g}_{(1)} = 0$ and so $\mathfrak{g}_{(-1)}/\mathfrak{g}_{(0)}$ restricts to a faithful module for the image of $\mathfrak{h}$ in $\mathfrak{g}_{(0)}/\mathfrak{g}_{(1)}$. The index of $\mathfrak{h}$ in $\mathfrak{g}$ is ≤ 11; thus $\dim \mathfrak{g}_{(-1)}/\mathfrak{g}_{(0)} \leq 11$. This means that $\dim \mathfrak{h} \leq 11^2 = 121 < 125 - 3 = 122$. But this is a contradiction. □

Using the lemma, it follows from the Premet–Strade Classification [PS06] that the only simple subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ of Cartan type which can possibly appear are as follows:

$\mathfrak{h}$	p
$W(1;1)$	$p < \dim \mathfrak{g}$
$W(1;2)$	$p \leq 13$
$W(2;(1,1))$	$p \leq 11$
$H(2;(1,1))^{(2)}$	$p \leq 13$
$H(2;(1,1);\Phi(\tau))^{(1)}$	$p \leq 13$
$H(2;(1,1);\Phi(1))$	$p \leq 13$

(All dimensions of other simple Lie algebras are at least $p^3 - 1$. For the classification of rank one Hamiltonians, see [Str04, 6.3.10].)

Proof of Theorem 1.3. Let $\mathfrak{h}$ be a simple non-classical subalgebra of $\mathfrak{g}$ not isomorphic to W_1. First we reduce to the case that the p-closure $\mathfrak{h}_p$ is the minimal p-envelope of $\mathfrak{h}$.

We work up inductively through the rank of $\mathfrak{g}$. Since the minimum possible dimension of $\mathfrak{h}$, namely $5^2 - 2$, is bigger than 14, we cannot have $\mathfrak{h} \leq G_2$. Equally, (by considering each element of the table above in turn) since the smallest non-trivial representation of $\mathfrak{h}$ has dimension no more than $p^2 - 2$, $\mathfrak{h}$ cannot be contained in any classical A–D-type algebra of rank less than 12.

Assume we have proved that $\mathfrak{h}$ is not a subalgebra of any simple subalgebra of rank less than that of $\mathfrak{g}$. Now if $\mathfrak{h} \subseteq \mathfrak{h}_p \subseteq \mathfrak{g}$ with $\mathfrak{h}_p$ not semisimple, then the radical $\operatorname{Rad} \mathfrak{h}_p$ is a non-trivial ideal in $\mathfrak{h}_p$. However $\mathfrak{h}$ is also an ideal in $\mathfrak{h}_p$, thus $\mathfrak{h}$ and $\operatorname{Rad} \mathfrak{h}_p$ are mutually normalising and have trivial intersection. Thus $\mathfrak{h}$ centralises an element of the Lie algebra of $\mathfrak{g}$ and hence is in a proper parabolic subalgebra of $\mathfrak{g}$. As $\mathfrak{h}$ can have no intersection with the nilradical of this parabolic, $\mathfrak{h}$ projects to a simple subalgebra of a Levi subalgebra of the parabolic, which is of smaller rank, a contradiction.

Hence we can assume that $\mathfrak{h}_p$ is semisimple. (As $\mathfrak{h}$ is simple, note that the minimal p-subalgebra of $\mathfrak{h}$ is semisimple.) In particular, its trivial centre is contained in $\mathfrak{h}$. By [SF88, 2.5.8(3)], this forces $\mathfrak{h}_p$ to be a minimal p-envelope of $\mathfrak{h}$ as required.

Now we show that there is no p-subalgebra isomorphic to $\mathfrak{h}_p$ where $\mathfrak{h}_p$ is the minimal p-envelope of $\mathfrak{h}$ where $\mathfrak{h}$ is any of $H(2;(1,1))^{(2)}$ (with $\mathfrak{h} = \mathfrak{h}_p$), $H(2;(1,1);\Phi(\tau))^{(1)}$, $H(2;(1,1);\Phi(1))$, or W_2. Since $H(2;(1,1))^{(2)} \leq W_2$ it suffices to deal with the first three cases. So assume, looking for a contradiction, that $\mathfrak{h} \cong H(2;(1,1))^{(2)}$, $\mathfrak{h} \cong H(2;(1,1);\Phi(\tau))^{(1)}$ or $\mathfrak{h} \cong H(2;(1,1);\Phi(1))$ with $\mathfrak{h} \leq \mathfrak{h}_p \leq \mathfrak{g}$.

By Lemma 2.6 and 2.9 we know that there is a p-subalgebra $W \cong W_1$ with $\mathfrak{g}|W$ having composition factors such that every $L(r)$ with $2 \leq r \leq p-1$ appears the same number of times. But inspecting Table 3 (for $\mathcal{O} \neq A_{p-1}$) and Table 6 (when $\mathcal{O} = A_{p-1}$) we find that no such p-subalgebra exists.

It remains to deal with the case $\mathfrak{h} = W(1;(2))$. This is the algebra given by basis $\{e_i : -1 \leq i \leq p^2 - 2\}$ and multiplication

$$[e_i, e_j] = \begin{cases} \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) e_{i+j} & \text{if } -1 \leq i+j \leq p^2-2, \\ 0 & \text{otherwise,} \end{cases}$$

(see for example [Fel07, p4]), where we put $\binom{j}{-1} = 0$ for $j \geq 0$. Thus we check that the nilpotent element e_{-1} is in the image of $(\operatorname{ad} e_{-1})^{p^2-1}$. We can assume again (by induction) that $\mathfrak{h}_p$ is semsimple and that $\mathfrak{h}_p$ is a minimal p-envelope of $\mathfrak{h}$. In particular the element in $\mathfrak{g}$ representing e_{-1} is nilpotent (cf. *loc. cit.*). Now one checks (e.g. the tables in [LT11]) that the largest r for which the space $\mathfrak{g}(-r)$ in the grading associated to e is non-zero is $2h-2$, where h is the Coxeter number of $\mathfrak{g}$. Hence the largest s for which $\mathfrak{g}(2)$ is in the image of $(\operatorname{ad} e)^s$ is $h+1$. Thus $p^2 - 1 \leq h+1$, i.e. $p \leq \sqrt{h+2}$. Now for a root system Φ of type $(G_2, F_4, E_6, E_7, E_8)$ we have $h+2$ is $(8, 14, 14, 20, 32)$. This implies $\mathfrak{p} \leq (2, 3, 3, 3, 5)$, a contradiction as p is a good prime. Thus $\mathfrak{h}$ does not appear as a subalgebra in $\mathfrak{g}$. This finishes the proof of the theorem.

□

Appendix A. GAP Calculations

We have two jobs to perform in this section, both of which use GAP intensively. The first is to find the composition factors of $\mathfrak{g}|W$ in the case that W contains a nilpotent element of type A_{p-1} representing ∂. For the other, recall that there are twelve remaining cases of $(\mathfrak{g}, p, \mathcal{O})$ in Proposition 3.10 for which we must check whether a p-subalgebra isomorphic to W_1 with ∂ represented by a nilpotent element of type $\mathcal{O}$ normalises a non-trivial abelian subalgebra of $\mathfrak{g}$. First we give an introduction to the methods we use here and have used at other points in the paper.

A.1. **Introduction.** We use the routines included in the standard GAP4 distribution for computing with Lie algebras. See [gro14, Manual, Chapter 64] for complete details.

A very straightforward use of GAP we employ is to use its database of root systems to evaluate a cocharacter on a root, thereby finding the weight of the cocharacter on a corresponding root vector for which we have written a function `findweights`. For example, if $\mathfrak{g} = E_6$, $\mathcal{O} = A_4$ then [LT11] notates an associated cocharacter in terms of certain elements of a maximal torus as $\tau : \begin{smallmatrix} 2 & 2 & 2 & -6 & 0 \\ & & 2 & & \end{smallmatrix}$. Thus we may compute

```
gap> T:=[2,2,2,2,-6,0];
[ 2, 2, 2, 2, -6, 0 ]
gap> findweights(T,4);
[ [ 1, 0, 1, 0, 0, 0 ], [ 0, 1, 0, 1, 0, 0 ], [ 0, 0, 1, 1, 0, 0 ],
  [ 1, 1, 1, 2, 1, 0 ], [ 1, 1, 1, 2, 1, 1 ], [ 1, 2, 2, 3, 2, 1 ],
  [ 0, 0, 0, -1, -1, 0 ], [ 0, 0, 0, -1, -1, -1 ], [ 0, -1, -1, -2, -2, -1 ] ]
```

Thus one concludes that $\dim \mathfrak{g}(4) = 9$ and a basis of $\mathfrak{g}(4)$ is

$$\left\{ e_{\substack{11000\\0}}, e_{\substack{00100\\1}}, e_{\substack{01100\\0}}, e_{\substack{11210\\1}}, e_{\substack{11211\\1}}, e_{\substack{12321\\2}}, e_{-\substack{00110\\0}}, e_{-\substack{00111\\0}}, e_{-\substack{01221\\1}} \right\}.$$

Very frequently, we will wish to compute the Lie bracket of two expressions of the form $\sum_\alpha \lambda_\alpha e_\alpha + \sum_\beta \mu_\beta h_\beta$, where the e_α are root vectors in $\mathfrak{g}$, and the h_β are elements from a basis of a maximal torus defining the root system of $\mathfrak{g}$. The λ_α and μ_β are treated as scalar indeterminates. For example, if x is an expression of the above form, and y is some fixed element, then calculating $[x, y]$ and insisting that it is zero will put conditions amongst the λ_α and μ_β. (Such calculations can of course be done by hand, since the bracket of any pair of basis elements is given by structure constants, which can be deduced from the root system but this becomes unwieldy for large expressions.) To do this in GAP we set up a polynomial ring R in a large enough number (e.g. $\dim \mathfrak{g}$) indeterminates and then work with $\mathfrak{g}(R)$. Note that in GAP, the 'canonical' basis B of a simple (classical) Lie algebra $\mathfrak{g}$ in GAP is arranged so that (i) the last $\operatorname{rk} \mathfrak{g}$ elements are a basis for a maximal torus; (ii) the first $|\Phi^+|$ elements are positive root vectors; (iii) the first $\operatorname{rk} \mathfrak{g}$ elements are simple root vectors; (iv) if $r \leq |\Phi^+|$ then the $r + |\Phi^+|$th element of B is a root vector corresponding to the negative of the root corresponding to the rth element of B. The simple root vectors are normally in the Bourbaki ordering; the exception is type F_4, where one needs to apply a permutation.

For example if $\mathfrak{g} = E_8$, and e is a nilpotent element of type A_4, then the following calculates the bracket of e with a general element of the maximal torus $\mu_1 h_1 + \cdots + \mu_8 h_8 \in \mathfrak{t}$.

```
gap> g_rat := SimpleLieAlgebra("E",8,Rationals);;
gap> R := PolynomialRing(Rationals,Dimension(g_rat));;
gap> g := SimpleLieAlgebra("E",8,R);;
gap> x := IndeterminatesOfPolynomialRing(R);;
gap> B := Basis(g);;
gap> e:=B[1]+B[2]+B[3]+B[4];
v.1+v.2+v.3+v.4
gap> y:=x[241]*B[241]+x[242]*B[242]+x[243]*B[243]+x[244]*B[244]+x[245]*B[245]
> +x[246]*B[246]+x[247]*B[247]+x[248]*B[248];
(x_241)*v.241+(x_242)*v.242+(x_243)*v.243+(x_244)*v.244+(x_245)*v.245+(x_246)*\
v.246+(x_247)*v.247+(x_248)*v.248
gap> e*y;
(-2*x_241+x_243)*v.1+(-2*x_242+x_244)*v.2+(x_241-2*x_243+x_244)*v.3+(x_242+x_2\
43-2*x_244+x_245)*v.4
```

We have also implemented a routine which will make substitutions in general elements in order to make a certain expression be zero. For example, if one wanted to calculate $\mathfrak{c}_\mathfrak{t}(e)$, one could insist that the last expression in the above output be zero. Thus we might choose to calculate the substitution into `y` of `x_243` $= 2*$`x_241`, `x_244` $= 2*$`x_242` and so on, a process which this algorithm automates.

A.2. **Composition factors of $\mathfrak{g}|W$ when $\partial \in W$ is nilpotent of type A_{p-1}.** Let us first calculate the composition factors of the restrictions $[\mathfrak{g}|W]$ for W containing a nilpotent element e of type A_{p-1} representing ∂. For this we use the algorithm described in Proposition 2.2. The data required is a grading $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ and the list of the weights ℓ_i with multiplicities of $X\partial$ on each $\mathfrak{g}(i)$. In these cases we have many choices for a toral element h representing $X\partial$. However, by Proposition 3.3(ii) we have that that it is of the form $H + H_0$ where $H \in \mathrm{Lie}(\tau(\mathbb{G}_m))$ and $H_0 \in \mathfrak{g}_e(0) \cap \operatorname{im}\operatorname{ad} e$. We find H by deriving the cocharacter τ given in [LT11, p33] and insisting that it has the correct weight $[\mathtt{H}, \mathtt{e}] = [\mathtt{X}\partial, \mathtt{e}] = -\mathtt{e}$.

In the cases where e is of type A_{p-1}, H and H_0 commute, and H is toral, so that H_0 is also toral. Examining Table 1, $\operatorname{im}\operatorname{ad} e \cap \mathfrak{g}_e(0)$ is the Lie algebra of a connected reductive algebraic group of rank 1 so that H_0 is conjugate to a scalar multiple of some fixed element. We produce this element in GAP by first taking a generic element `w` in the -2 weight space for τ, and then considering $[\mathtt{e}, \mathtt{w}]$. This is a generic element in $\operatorname{im}\operatorname{ad} e \cap \mathfrak{g}(0)$, and insisting that it commutes with `e` and lies in the standard maximal torus fixes a choice of H_0. We may now write $X\partial = H + \lambda H_0$ with some scalar λ. (Note also that if $\{\alpha_1, \ldots, \alpha_{p-1}\}$ are simple roots for the Levi subalgebra of type A_{p-1} then H_0 lies in the centre of the corresponding $\mathfrak{sl}_p$ so one can construct the element H_0 as $h_{\alpha_1} + 2h_{\alpha_2} + \cdots + (p-1)h_{\alpha_{p-1}}$.) As λH_0 is toral we must have $\lambda \in \mathbb{F}_p$.

See Table 4 for our choices of H and H_0.

$(\mathfrak{g}, p, \mathcal{O})$	H	H_0
$(E_6, 5, A_4)$	$3 \cdot h_1 + 3 \cdot h_2 + 2 \cdot h_3 + 2 \cdot h_4$	$2 \cdot h_1 + 3 \cdot h_2 + 4 \cdot h_3 + h_4$
$(E_7, 5, A_4)$	$3 \cdot h_1 + 2 \cdot h_3 + 2 \cdot h_4 + 3 \cdot h_2$	$2 \cdot h_1 + 3 \cdot h_2 + 4 \cdot h_3 + h_4$
$(E_7, 7, A_6)$	$4 \cdot h_1 + 2 \cdot h_3 + h_4 + h_5 + 2 \cdot h_6 + 4 \cdot h_7$	$6 \cdot h_1 + 5 \cdot h_3 + 4 \cdot h_4 + 3 \cdot h_5 + 2 \cdot h_6 + h_7$
$(E_8, 7, A_6)$	$4 \cdot h_2 + 2 \cdot h_4 + h_5 + h_6 + 2 \cdot h_7 + 4 \cdot h_8$	$6 \cdot h_2 + 5 \cdot h_4 + 4 \cdot h_5 + 3 \cdot h_6 + 2 \cdot h_7 + h_8$.

TABLE 4. Choices of H and H_0

$(\mathfrak{g}, p, \mathcal{O})$	e	H
$(F_4, 5, B_2)$	B[3] + B[4]	B[51] + 3 * B[52]
$(F_4, 7, C_3)$	B[1] + B[3] + B[4]	B[49] + 3 * B[51] + 6 * B[52]
$(E_6, 5, A_3)$	B[1] + B[3] + B[4]	B[73] + 3 * B[75] + B[76]
$(E_6, 7, A_5)$	B[1] + B[3] + B[4] + B[5] + B[6]	B[73] + 3 * B[75] + 6 * B[76] + 3 * B[77] + B[78]
$(E_7, 5, A_3)$	B[1] + B[3] + B[4]	B[127] + 3 * B[129] + B[130]
$(E_7, 7, (A_5)')$	B[1] + B[3] + B[4] + B[5] + B[6]	B[127] + 3 * B[129] + 6 * B[130] + 3 * B[131] + B[132]
$(E_8, 7, A_5)$	B[1] + B[3] + B[4] + B[5] + B[6]	B[241] + 3 * B[243] + 6 * B[244] + 3 * B[245] + B[246]
$(E_8, 13, D_7)$	B[2] + B[3] + B[4] + B[5] + B[6] + B[7] + B[8]	7 * B[248] + 2 * B[247] + 11 * B[246] + 8 * B[245] +6 * B[244] + 9 * B[243] + 9 * B[242]
$(E_6, 5, A_4)$	B[1] + B[2] + B[3] + B[4]	3 * B[73] + 2 * B[75] + 2 * B[76] + 3 * B[74]
$(E_7, 5, A_4)$	B[1] + B[2] + B[3] + B[4]	3 * B[127] + 2 * B[129] + 2 * B[130] + 3 * B[128]
$(E_7, 7, A_6)$	B[1] + B[3] + B[4] + B[5] + B[6] + B[7]	4 * B[127] + 2 * B[129] + B[130] + B[131] + 2 * B[132] + 4 * B[133]
$(E_8, 7, A_6)$	B[2] + B[4] + B[5] + B[6] + B[7] + B[8]	4 * B[242] + 2 * B[244] + B[245] + B[246] + 2 * B[247] + 4 * B[248]
$(\mathfrak{g}, p, \mathcal{O})$	H_0	
$(E_6, 5, A_4)$	2 * B[73] + 3 * B[74] + 4 * B[75] + B[76]	
$(E_7, 5, A_4)$	2 * B[127] + 3 * B[128] + 4 * B[129] + B[130]	
$(E_7, 7, A_6)$	6 * B[127] + 5 * B[129] + 4 * B[130] + 3 * B[131] +2 * B[132] + B[133]	
$(E_8, 7, A_6)$	6 * B[242] + 5 * B[244] + 4 * B[245] + 3 * B[246] +2 * B[247] + B[248].	

TABLE 5. Choices of e, H and H_0

Proposition A.1. *For the various choices of $\lambda \in \mathbb{F}_p$, Table 6 lists the possible composition factors of $\mathfrak{g}|W$ where W contains a nilpotent element e of type A_{p-1} representing ∂, and a toral element $H + \lambda H_0$ representing $X\partial$, for H and H_0 in Table 4.*

A.3. **The remaining cases of Lemma 3.10.** The cases are:

$$(\mathfrak{g}, p, e) = (F_4, 5, B_2),\ (F_4, 7, C_3),\ (E_6, 5, A_3),\ (E_6, 7, A_5),\ (E_7, 5, A_3),\ (E_7, 7, (A_5)'),\ (E_8, 7, A_5),\ (E_8, 13, D_7), \tag{1}$$

as well as the cases where $\mathcal{O} = A_{p-1}$:

$$(\mathfrak{g}, p, e) = (E_6, 5, A_4),\ (E_7, 5, A_4),\ (E_7, 7, A_6),\ (E_8, 7, A_6). \tag{2}$$

For the purposes of computation in GAP, we will need elements representing e, H and H_0 for these cases, determined as in A.2. In the canonical basis in GAP, these are given in Table 5.

The rest of this appendix is dedicated to finishing the proof of Theorem 1.1(v). We check directly in GAP to see if a non-regular subalgebra must fix a nonzero vector $v \in \mathfrak{g}$. In most cases, we may find such a v and are done. Choose one of the twelve cases above $(\mathfrak{g}, p, e)$ above. Our strategy is as follows:

- Set up a simple Lie algebra g in GAP of the same type of $\mathfrak{g}$ over the ring of polynomials $\mathbb{Q}[$x_1$, \ldots,$ x_dim $\mathfrak{g}]$ as described in A.1. Let B be its Chevalley basis.

$\mathfrak{g}$	p	$\mathcal{O}$	λ	$[\mathfrak{g}\mid W]$
E_6	5	A_4	0	$L(1)^5, L(2)^3, L(3)^5, L(4)^2, k^5$
			1	$L(1), L(2)^5, L(3), L(4)^8, k^{11}$
			2	$L(1)^4, L(2), L(3)^4, L(4)^6, k^9$
			3	$L(1)^4, L(2), L(3)^4, L(4)^6, k^9$
			4	$L(1), L(2)^5, L(3), L(4)^8, k^{11}$
E_7	7	A_6	0	$L(1), L(2)^3, L(3)^5, L(4)^3, L(5), L(6)^6, k^6$
			1	$L(1)^2, L(2)^3, L(3)^3, L(4)^3, L(5)^2, L(6)^6, k^6$
			2	$L(1)^4, L(2)^2, L(3)^3, L(4)^2, L(5)^4, L(6)^4, k^4$
			3	$L(1)^3, L(2)^3, L(3), L(4)^3, L(5)^3, L(6)^6, k^6$
			4	$L(1)^3, L(2)^3, L(3), L(4)^3, L(5)^3, L(6)^6, k^6$
			5	$L(1)^4, L(2)^2, L(3)^3, L(4)^2, L(5)^4, L(6)^4, k^4$
			6	$L(1)^2, L(2)^3, L(3)^3, L(4)^3, L(5)^2, L(6)^6, k^6$
E_7	5	A_4	0	$L(1)^7, L(2)^9, L(3)^7, L(4)^2, k^{10}$
			1	$L(1), L(2)^7, L(3), L(4)^{16}, k^{24}$
			2	$L(1)^8, L(2), L(3)^8, L(4)^8, k^{16}$
			3	$L(1)^8, L(2), L(3)^8, L(4)^8, k^{16}$
			4	$L(1), L(2)^7, L(3), L(4)^{16}, k^{24}$
E_8	7	A_6	0	$L(1)^5, L(2)^3, L(3)^{13}, L(4)^3, L(5)^5, L(6)^6, k^9$
			1	$L(1)^4, L(2)^5, L(3)^7, L(4)^5, L(5)^4, L(6)^{10}, k^{13}$
			2	$L(1)^8, L(2)^4, L(3)^3, L(4)^4, L(5)^8, L(6)^8, k^{11}$
			3	$L(1)^3, L(2)^7, L(3), L(4)^7, L(5)^3, L(6)^{14}, k^{17}$
			4	$L(1)^3, L(2)^7, L(3), L(4)^7, L(5)^3, L(6)^{14}, k^{17}$
			5	$L(1)^8, L(2)^4, L(3)^3, L(4)^4, L(5)^8, L(6)^8, k^{11}$
			6	$L(1)^4, L(2)^5, L(3)^7, L(4)^5, L(5)^4, L(6)^{10}, k^{13}$

TABLE 6. Composition factors of subalgebras $W \cong W_1$ containing a nilpotent element of type e of type A_{p-1}, where $X\partial = H + \lambda H_0$

- From the tables in [LT11], set `e` to be the nilpotent representative in the orbit $\mathcal{O}_e$ expressed in terms of the elements `B[i]` and set `T` to be an array whose entries are the coefficients of the cocharacter τ associated to e in [LT11]. By the choice of cocharacter in [LT11] we have that each element `B[i]` is a weight vector for τ.
- Organise the set of vectors $\{\mathtt{B[i]} : 1 \leq i \leq \dim \mathfrak{g}\}$ into weight spaces for τ.
- Set `Xd` to be one of the choices $H + \lambda H_0$ as given in Table 5 representing $X\partial$ and corresponding to `e`. Note that, by Proposition 3.3(iii), we have $H_0 = 0$ if e is not of type A_{p-1}.
- Next we produce an element `f` that is a candidate for $\frac{1}{2}X^2\partial$:
  - Let `f` be a generic element in `g`. By a generic element we mean an element of the form $\mathtt{f} := \sum_{\mathtt{i}} \mathtt{x\_i.B[i]}$.
  - We ensure that $[\mathtt{e}, \mathtt{f}] = \mathtt{Xd}$ and $[\mathtt{Xd}, \mathtt{f}] = \mathtt{f}$ by considering linear relations among the `x_i` resulting from these equations and substituting them in the coefficients of `f`.
- The putative subalgebra W contains, additionally, the element $X^3\partial$ and, moreover W is generated by $X^3\partial$ and ∂. We perform a similar routine to the above to find an arbitrary element `ff` representing $\frac{1}{6}X^3\partial$ on which $X\partial$ has the correct weight, i.e. $[\mathtt{Xd}, \mathtt{ff}] = \mathtt{2ff}$. By substituting relations in both `f` and `ff` we force the relation $[\mathtt{e}, \mathtt{ff}] = \mathtt{f}$. (Note that this

puts many constraints on $\texttt{ff}$ but we do not attempt to guarantee that we have $\langle \texttt{ff}, \texttt{e} \rangle \cong W$; indeed this will rarely be true.)

- We look for a vector $\texttt{v} \neq \texttt{0}$ in $\texttt{g}$ which is killed by $\texttt{e}$ and $\texttt{ff}$. Since W is generated by these elements, this will guarantee that $\texttt{v}$ is a fixed vector for W. Specifically:
  - We form a generic element $\texttt{v}$ from the basis vectors.
  - We compute $[\texttt{v}, \texttt{e}]$. Forcing this to be zero puts constraints on the coefficients of $\texttt{v}$.
  - We compute $[\texttt{Xd}, \texttt{v}]$ and set this to be zero, putting more constraints on the coefficients. (This ensures in fact that $\texttt{v} \in \mathfrak{g}_e(0)$.)
  - Now consider the expression $[\texttt{ff}, \texttt{v}] \in \texttt{g}$. Suppose that $\texttt{x\_i}_1, \dots, \texttt{x\_i}_\texttt{r}$ are the indeterminates occuring in $\texttt{v}$. Now it turns out that the coefficients of $[\texttt{ff}, \texttt{v}]$ in the basis $\texttt{B}$ are all linear expressions in the $\texttt{x\_i}_1, \dots, \texttt{x\_i}_\texttt{r}$. Thus there is a matrix A whose entries are polynomials in the coefficients of $\texttt{ff}$ and with $A \cdot (\texttt{x\_i}_1, \dots, \texttt{x\_i}_\texttt{r})^\texttt{t} = \texttt{0}$ if and only if $[\texttt{ff}, \texttt{v}] = \texttt{0}$.
- We proceed with doing row-reductions on A. If the rank of A is strictly smaller than r, we are done: $\texttt{v}$ may be chosen to satisfy $[\texttt{ff}, \texttt{v}] = \texttt{0}$. This deals with all but the following cases:

$$\begin{aligned}(\mathfrak{g}, p, e) =&(E_6, 7, A_5),\ (E_7, 7, (A_5)'),\ (E_8, 7, A_5),\ (E_8, 13, D_7),\\ &(E_7, 7, A_6) \text{ for } \lambda = 1, 2, 3, 4, 5, 6\\ &(E_8, 7, A_6) \text{ for } \lambda = 2, 5.\end{aligned}$$

- Let $a{*}b{*}c$ represent the element $[[a, b], c]$. We go on to consider the elements $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f}$ and $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{ff}$, which both must vanish for $p = 7$. To see this, observe that since $\texttt{ff}$ represents $X^3\partial$ and $\texttt{f}$ represents $X^2\partial$, we must have $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f}$ (resp. $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{ff}$) a multiple of $X^{3+1+1+1+1}\partial$ (resp. a multiple of $X^{3+1+1+2}\partial$), which is zero. (For the $p = 13$ case we consider the element $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{ff}*\texttt{f}*\texttt{f}$.) If there still are linear substitutions that may be read off from the coefficients of these elements, we perform them on both $\texttt{f}$ and $\texttt{ff}$. We next try to show that the remaining relations in $\texttt{ff}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f} = \texttt{0} = \texttt{ff}*\texttt{f}*\texttt{f}*\texttt{ff}$ again force the rank of A to be strictly less than r, by row-reducing A one step at a time and trying to substitute the above relations.
- In the cases $(E_7, 7, (A_5)')$ and $(E_8, 7, A_5)$, we also use the following technique to reduce the number of indeterminates in $\texttt{f}$ and $\texttt{ff}$: We consider root elements $y \in \mathrm{Lie}(C_e)$ with respect to the roots of C_e, the reductive part of the centraliser. As $\mathrm{ad}(y)$ is nilpotent (in fact, $\mathrm{ad}(y)^4 = 0$ in all these cases), we obtain automorphisms $s_y(t) = \exp(t \cdot \mathrm{ad}(y))$ for $t \in k$ of $\mathfrak{g}$, and $s_y(t)$ satisfies $s_y(t)(e) = e$ and $s_y(t)(H) = H$. Thus we may replace the pair $(\texttt{f}, \texttt{ff})$ by any pair $(s_y(t)(\texttt{f}), s_y(t)(\texttt{ff}))$. Choosing t and y suitably, we may use this to kill a number of coefficients in $(\texttt{f}, \texttt{ff})$.
- After these steps, we succeed in finding fixed vectors for all but the following cases:

$$\begin{aligned}&(E_7, 7, A_6), \lambda = 2, \texttt{x\_63} * \texttt{x\_108} = \texttt{0}, \texttt{x\_63} \neq \texttt{0} \text{ or } \texttt{x\_108} \neq \texttt{0} \text{ and}\\ &(E_7, 7, A_6), \lambda = 5, \texttt{x\_41} * \texttt{x\_116} = \texttt{0}, \texttt{x\_41} \neq \texttt{0} \text{ or } \texttt{x\_116} \neq \texttt{0}.\end{aligned}$$

- In the above two cases, we get that $[\texttt{ff}, \texttt{v}] = \texttt{0}$ implies $\texttt{v} = \texttt{0}$. This means that any subalgebras W which arise from these cases do not fix a non-zero vector in $\mathfrak{g}$. However, let us not insist that $[\texttt{ff}, \texttt{v}] = \texttt{0}$ but that $\texttt{v}*\texttt{ff}*\texttt{ff}*\texttt{ff}*\texttt{ff}*\texttt{ff} = \texttt{0}$, by substituting linear equations in v. There still exist nonzero v satisfying this relation. In the cases $(E_7, 7, A_6), \lambda = 2, \texttt{x\_63} = \texttt{0}, \texttt{x\_108} \neq \texttt{0}$ and $\lambda = 5, \texttt{x\_41} \neq \texttt{0}, \texttt{x\_116} = \texttt{0}$ we check with GAP that the subspace spanned by $\texttt{v}*\texttt{f}, \texttt{v}*\texttt{f}*\texttt{f}, \dots, \texttt{v}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f}*\texttt{f}$ is an at most 6-dimensional abelian subalgebra of $\texttt{g}$ normalised by W, and we may choose $\texttt{v}$ such that it is nonzero. In the cases $(E_7, 7, A_6), \lambda = 2, \mathtt{x\_63} \neq 0, \mathtt{x\_108} = 0$ and $\lambda = 5, \mathtt{x\_41} = 0, \mathtt{x\_116} \neq 0$ we consider the subspace spanned by $\mathtt{v} * \mathtt{ff}, \mathtt{v} * \mathtt{ff} * \mathtt{ff} * \mathtt{e}, \mathtt{v} * \mathtt{ff} * \mathtt{ff}, \mathtt{v} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff} * \mathtt{e}, \mathtt{v} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff}, \mathtt{v} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff} * \mathtt{e}, \mathtt{v} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff} * \mathtt{ff}$, which again is an abelian subalgebra of $\mathfrak{g}$, now at most 7-dimensional. It is again normalised by W, and we may assume it nonzero.

From the above calculations it follows that W normalises a non-trivial abelian subalgebra in all cases.

Examples A.2. Let us illustrate our procedure by giving a detailed example of our calculations in the case $(\mathfrak{g}, p, \mathcal{O}) = (E_8, 7, A_6)$. The other cases are similar and easier.

Let us first suppose that $\lambda = 1$. We have:

```
e := B[2]+B[4]+B[5]+B[6]+B[7]+B[8];
T := [0,2,-10,2,2,2,2,2];
p := 7;
H := 4*B[242] + 2*B[244] + B[245] + B[246] + 2*B[247] + 4*B[248];
H0:= 6*B[242] + 5*B[244] + 4*B[245] + 3*B[246] + 2*B[247] + B[248];
Xd := H+H0;
```

We start with completely generic $\mathtt{f}$ and $\mathtt{ff}$ and $\mathtt{v}$. So for instance $\mathtt{f} = \sum_{\mathtt{i}} \mathtt{x\_iB[i]}$. Now to ensure $[\mathtt{e}, \mathtt{f}] = \mathtt{Xd}$, we consider the difference $[\mathtt{e}, \mathtt{f}] - \mathtt{Xd}$.

```
e*f - Xd;
(5*x_242+x_244)*v.2+(x_242+x_243+5*x_244+x_245)*v.4+(x_244+5*x_245+x_246)*v.5+(x_245+5\ ...
```

Since the coefficient of v.2 must vanish, we have for example, that $\mathtt{x\_244} = -5\mathtt{x\_242}$ and may repeat linear substitutions of this form until the whole expression vanishes. Similarly, we manage to ensure after multiple substitutions that $[\mathtt{Xd}, \mathtt{f}] = \mathtt{f}, [\mathtt{Xd}, \mathtt{ff}] = 2\mathtt{ff}, [\mathtt{e}, \mathtt{ff}] = \mathtt{f}, [\mathtt{e}, \mathtt{v}] = 0$ and $[\mathtt{Xd}, \mathtt{v}] = 0$. By way of illustration, we have

```
v =
(x_1)*v.1+(6*x_100)*v.99+(x_100)*v.100+(x_121)*v.121+(x_123)*v.123+(x_129)*v.129+(x_185\
)*v.183+(6*x_185)*v.184+(x_185)*v.185+(x_241)*v.241+(6*x_248)*v.242+(5*x_248)*v.244+(4*\
x_248)*v.245+(3*x_248)*v.246+(2*x_248)*v.247+(x_248)*v.248
```

with similar expressions for ff and v.

Now v features the indeterminates $[1, 100, 121, 123, 129, 185, 241, 248]$ and hence $r = 8$ is the number of indeterminates. From $[\mathtt{ff}, \mathtt{v}]$,

```
ff*v;
(5*x_50*x_123+5*x_54*x_129+4*x_96*x_185+5*x_100*x_198)*v.34+(4*x_50*x_123+4*x_54*x_129+\
6*x_96*x_185+4*x_100*x_198)*v.36+(x_50*x_241+5*x_50*x_248+x_54*x_121)*v.40+(5*x_50*x_24\ ...
```

we calculate the matrix A as described in the steps above to be

```
A :=
[ [ 0, 5*x_198, 0, 5*x_50, 5*x_54, 4*x_96, 0, 0 ],
  [ 0, 4*x_198, 0, 4*x_50, 4*x_54, 6*x_96, 0, 0 ],
  [ 0, 0, x_54, 0, 0, 0, x_50, 5*x_50 ],
  [ 0, 0, 5*x_54, 0, 0, 0, 5*x_50, 4*x_50 ],
  [ 0, 0, 4*x_54, 0, 0, 0, 4*x_50, 6*x_50 ]

...
```

```
  [ 0, 0, 0, 0, 0, 2, 0, x_198 ],
  [ 0, 0, 0, 0, 0, 6, 0, 3*x_198 ],
  [ 0, 0, 0, 0, 0, 1, 0, 4*x_198 ] ].
```

After row reductions we obtain

```
Ared :=
[ [ x_50, 0, 0, 0, 0, 0, 6*x_54, 5*x_54 ],
  [ 0, 3, 0, 0, 0, 0, 0, 3*x_91 ],
  [ 0, 0, x_54, 0, 0, 0, x_50, 5*x_50 ],
  [ 0, 0, 0, x_50, x_54, 5*x_96, 0, 6*x_91*x_198 ],
  [ 0, 0, 0, 0, 3*x_54, 4*x_50*x_54, x_50*x_139+x_54*x_144,
                                       5*x_50*x_139+2*x_54*x_144 ],
  [ 0, 0, 0, 0, 0, 1, 0, 4*x_198 ],
  [ 0, 0, 0, 0, 0, 0, 0, x_91+6*x_96 ],
  [ 0, 0, 0, 0, 0, 0, 0, 0 ],
  ...
  [ 0, 0, 0, 0, 0, 0, 0, 0 ] ]
```

and this clearly has rank less than or equal to 7. As $7 < 8 = r$, we are done.

By contrast to this, for $\lambda = 2$, we have $r = 7$ but we we obtain a matrix A which, even after row-reductions, may have rank 7. But now considering $\texttt{ff} * \texttt{f} * \texttt{f} * \texttt{f} * \texttt{f}$:

```
ff*f*f*f*f;
(2*x_84^2*x_234+4*x_62*x_155+4*x_67*x_159+6*x_84*x_178+6*x_42)*v.2+(2*x_84^2*x_234+4*x_\
62*x_155+4*x_67*x_159+6*x_84*x_178+6*x_42)*v.4+(2*x_84^2*x_234+4*x_62*x_155+4*x_67*x_15\ ...,
```

we see that there still are some linear substitutions in f and ff which are possible, for example, from the coefficient of v.2 we have

$$\texttt{x\_42} = -1/6 * (2 * \texttt{x\_84}^2 * \texttt{x\_234} + 4 * \texttt{x\_62} * \texttt{x\_155} + 4 * \texttt{x\_67} * \texttt{x\_159} + 6 * \texttt{x\_84} * \texttt{x\_178}).$$

After performing these, we obtain

```
ff*f*f*f*f;
(5*x_62*x_84^2*x_234)*v.74+(5*x_67*x_84^2*x_234)*v.79+(3*x_84^3*x_234)*v.89+(4*x_84^3*x\
_234)*v.90+(3*x_84^3*x_234)*v.91+(x_84^4*x_234+x_84*x_112*x_155+6*x_84*x_113*x_159+x_62\
*x_113+x_67*x_112)*v.120+(3*x_84*x_234)*v.200+(4*x_84*x_234)*v.201+(x_84*x_234)*v.202+(\
3*x_84*x_234)*v.203+(4*x_84*x_234)*v.204+(6*x_67*x_234)*v.213+(6*x_62*x_234)*v.217.
```

We know that this must be zero. So let us first suppose that $\texttt{x\_234} \neq 0$. Then the vanishing of the last expression implies $\texttt{x\_62} = \texttt{x\_67} = \texttt{x\_84} = \texttt{x\_112} = \texttt{x\_113} = 0$ and this is enough to ensure that the rank of A is at most 4. Thus we must have $\texttt{x\_234} = 0$. Then the expression for $\texttt{ff} * \texttt{f} * \texttt{f} * \texttt{f} * \texttt{f}$ implies that

(3) $\qquad \texttt{x\_84} * \texttt{x\_112} * \texttt{x\_155} + 6 * \texttt{x\_84} * \texttt{x\_113} * \texttt{x\_159} + \texttt{x\_62} * \texttt{x\_113} + \texttt{x\_67} * \texttt{x\_112} = 0.$

Now we start applying row-reductions to A while distinguishing further subcases. For instance we may consider the subcase $\texttt{x\_159}, \texttt{x\_155}, \texttt{x\_113}, \texttt{x\_112} \neq 0$ and take x_159 as the pivot entry for our first row-reduction in A. After further row-reductions, we end up with the following matrix:

```
Ared :=
[ [ x_159, 0, 5, 0, 0, x_155, 5*x_155 ],
[ 0, x_155, 0, 5, 0, 6*x_159, 5*x_159 ],
[ 0, 0, x_159, 6*x_155, 5, 0, 3*x_178 ],
```

```
[ 0, 0, 0, 5*x_113, 0, x_112*x_155+6*x_113*x_159, x_112*x_155+5*x_113*x_159 ],
[ 0, 0, 0, 0, x_112*x_113, 2*x_112^2*x_155^2+3*x_112*x_113*x_155*x_159+2*x_113^2*x_159^2,
            2*x_112^2*x_155^2+5*x_113^2*x_159^2+2*x_112*x_113*x_178 ],
[ 0, 0, 0, 0, 0, 4*x_84*x_112*x_113^2*x_155*x_159^2+3*x_84*x_113^3*x_159^3+4*x_62*x_113^3*
            x_159^2+4*x_67*x_112*x_113^2*x_159^2, 6*x_84*x_112*x_113^2*x_155*x_159^2+
            4*x_84*x_113^3*x_159^3+3*x_62*x_113^3*x_159^2+x_67*x_112*x_113^2*x_159^2 ],
[ 0, 0, 0, 0, 0, 3*x_84*x_112*x_155^2+4*x_84*x_113*x_155*x_159+3*x_62*x_113*x_155+
            3*x_67*x_112*x_155, 3*x_84*x_112*x_155^2+x_84*x_113*x_155*x_159+x_62*x_113*
            x_155+3*x_67*x_112*x_155 ],
...
```

We may recognise multiples of the left hand side of (3) above in this matrix in its sixth column. This means that `Ared[i][6]` = 0 for all $i \geq 6$ and hence A has rank at most 6 so we are done. Continuing in this way, for each subcase we may row-reduce A to a matrix A', where we may substitute the equation (3) to deduce that the rank of A' is strictly less than $r = 7$. The cases for the remaining choices of λ are dealt with similarly.

Finally we give an example for the use of automorphisms of the form $\exp(t \cdot \mathrm{ad}(y))$, as mentioned in the steps above. We consider the case $(E_8, 7, A_5)$. After the first steps few steps, we have expressions for `f` and `ff`, where for example `f` is given as follows.

```
f = (x_39)*v.39+(x_47)*v.47+(x_52)*v.51+(x_52)*v.52+(x_96)*v.96+(x_97)*v.97+ ...
```

Here C_e is of type G_2A_1. The root y = `B[8]` belongs to C_e. The corresponding automorphisms `s1` $:= s_y(t) = \exp(t \cdot \mathrm{ad}(y))$, `s2` $:= s_z(t) = \exp(t \cdot \mathrm{ad}(z))$ have the following effect on `f`:

```
s1(f) = (x_39)*v.39+(x_39*t+x_47)*v.47+ ...
s2(f) = (x_47*t+x_39)*v.39+ ...
```

We see that if the coefficient `x_39` is nonzero in `f`, we may assume that `x_47` is also nonzero after applying `s1` with `t` chosen such that `x_39 * t + x_47` $\neq$ `0`. But then we may apply `s2` to ensure that `x_39 = 0`. Thus we may assume from the outset that `x_39 = 0`.

We have a total of 14 further root elements $y \in \mathrm{Lie}(C_e)$ to formulate automorphisms to use for this process. Applying them to reduce dependency followed by an application of the methods from before we again succeed in showing that the rank of the corresponding matrix A is less than r in this case.

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