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Hyperovals arising from a Singer group action on $H(3, q^2)$, $q$ even

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Abstract

The action of a Singer cyclic group of order $q^2 + 1$ on the Hermitian surface $\mathcal{H}(3, q^2)$, $q$ even, is investigated. Infinite families of hyperovals of size $2(q^2 + 1)$, $q$ even, are then constructed.

Keywords: Hermitian surface, Hyperoval, Symplectic subquadrangle
MSC: 51E12 (51E21)

1 Introduction

A hyperoval (or local subspace) in a polar space is defined to be a set of points with the property that each line of the polar space meets the set in either 0 or 2 points. Hyperovals of polar spaces are significant in the study of locally polar spaces, as discussed in [5], which renders the classification of hyperovals of polar spaces an interesting objective.

De Bruyn [10] has established significant numerical properties of hyperovals of generalized quadrangles.

Theorem 1.1. Let $S$ be a generalized quadrangle of order $(s, t)$ and let $H$ be a hyperoval of $S$. Then

(a) $2$ is a divisor of $|H|$.

(b) We have $|H| \geq 2(t + 1)$, with equality if and only if there exists a regular pair $\{x, y\}$ of non-collinear points of $S$ such that $H = \{x, y\}^\perp \cup \{x, y\}^{\perp\perp}$.

(c) We have $|H| \geq (t - s + 2)(s + 1)$. If equality holds then every point outside $H$ is incident with exactly $(t - s)/2 + 1$ lines which meet $H$.

(d) We have $|H| \leq 2(st + 1)$, with equality if and only if $H$ is a 2-ovoid.

In this paper we shall focus on the generalized quadrangle $\mathcal{H}(3, q^2)$, $q$ even, the incidence structure of all points and lines (generators) of a nonsingular Hermitian surface in $\text{PG}(3, q^2)$, a generalized quadrangle of order $(q^2, q)$, with automorphism group $\text{PΓU}(4, q^2)$, see [18] for more details.

The hyperovals of $\mathcal{H}(3, 4)$ have been classified by Makhnev [15]. He found hyperovals of size 6, 8, 10, 12, 14, 16, 18.

In this paper we construct an infinite family of hyperovals of $\mathcal{H}(3, q^2)$, $q$ even, of size $2(q^2 + 1)$ admitting the normalizer of a Singer cyclic subgroup of
$\Omega^-(4, q)$ as an automorphism group. We show that there are, in fact, three subfamilies, distinguished by their full automorphism groups in $\text{PU}(4, q^2)$.

## 2 The geometric setting and some preliminary results

For the reader’s convenience, some facts about the Hermitian surface are summarized below.

In $\text{PG}(3, q^2)$ a non–singular Hermitian surface is defined to be the set of all absolute points of a non–degenerate unitary polarity, and is denoted by $\mathcal{H}(3, q^2)$.

A Hermitian surface $\mathcal{H} = \mathcal{H}(3, q^2)$ has the following properties.

(a) The number of points on $\mathcal{H}$ is $(q^2 + 1)(q^3 + 1)$.

(b) Any line of $\text{PG}(3, q^2)$ meets $\mathcal{H}$ in one of the following: 1 point (a tangent line), $q + 1$ points (a hyperbolic line) or $q^2 + 1$ points (a generator or line of $\mathcal{H}$).

(c) Through every point $P$ of $\mathcal{H}$ there pass exactly $q + 1$ generators, and these generators are coplanar. The plane containing these generators, say $\pi_P$, is the polar plane of $P$ with respect to the unitary polarity defining $\mathcal{H}$. The tangent lines through $P$ are precisely the remaining $q^2 - q$ lines of $\pi_P$ incident with $P$, and $\pi_P$ is called the tangent plane to $\mathcal{H}$ at $P$.

(d) Every plane of $\text{PG}(3, q^2)$ which is not a tangent plane to $\mathcal{H}$ meets $\mathcal{H}$ in a non-degenerate Hermitian curve, and is called a secant plane.

We assume throughout that $q$ is even and $q > 2$.

Let $\mathcal{W} = \mathcal{W}(3, q)$ be a symplectic subquadrangle of $\mathcal{H}(3, q^2)$ and let $Q = Q^-(3, q)$ be an elliptic quadric embedded in $\mathcal{W}$ with group $\text{PO}^-(4, q) < \text{PSp}(4, q)$ (these groups are isomorphic to $O^-(4, q)$ and $Sp(4, q)$ respectively since $q$ is even). Then $Q$ is an ovoid of $\mathcal{W}$, i.e., the quadratic form associated with $Q$ polarizes to the symplectic polarity $\mathcal{A}$ fixed by $\text{PSp}(4)$.

A line $\ell$ of $\mathcal{W}$ (in other words, a totally isotropic line with respect to $\mathcal{A}$) has a $\text{GF}(q^2)$-extension $\bar{\ell}$ that lies on $\mathcal{H}$: we say that $\bar{\ell}$ is the extension of $\ell$ to $\mathcal{H}$, or that
ℓ is the restriction of ℓ to W. Each generator of \( \mathcal{H} \) either extends a line of \( W \) or is disjoint from \( W \). This means that a point of \( \mathcal{H} \setminus W \) lies on a unique generator arising from a line of \( W \). Since \( q \) is even, the lines of \( W \) are tangents to the elliptic quadric \( Q \) (see [17]).

Let \( \Sigma = \text{PG}(3,q) \) be the 3–dimensional projective space underlying \( W \) and let \( G \) be a Singer cyclic group of \( PO^-(4,q) < PSp(4,q) \). Then \( G \) has order \( q^2 + 1 \) and \( G \leq P\Omega^-(4,q) \).

**Proposition 2.1.** ([14]) The \( G \)-orbits partition the points of \( \Sigma = \text{PG}(3,q) \) into \( q + 1 \) elliptic quadrics, one of which is \( Q \).

**Remark 2.2.** From [2], no two disjoint ovoids of \( W \) exist. This means that \( Q \) is the only one of the \( q + 1 \) elliptic quadrics above whose associated quadratic form polarizes to \( A \) and hence the other \( G \)-orbits on \( \Sigma \) cannot be ovoids of \( W \). Denote the \( G \)-orbits on \( \Sigma \) by \( \Omega_0 = Q, \Omega_1, \Omega_2, \ldots, \Omega_q \). Then each line of \( W \) is tangent to \( Q \), but intersects \( \Omega_i, i > 0 \), either in 0 or 2 points.

**Proposition 2.3.** ([17]) The group \( G \) has \( q + 1 \) orbits on lines of \( W \), each of size \( q^2 + 1 \), one of which is a Desarguesian spread of \( \Sigma \).

**Remark 2.4.** From the previous proposition there exists a line \( \ell \) of \( W \) that is tangent to \( Q \) such that \( S = \ell^G \) is a Desarguesian spread of \( W \) and none of the remaining \( G \)-orbits can be a spread [3].

### 2.1 The classical span of \( \mathcal{H}(3,q^2) \)

Consider the \( G \)-invariant Desarguesian spread \( S \) of \( W \) mentioned above. The \( GF(q^2) \)-extensions of the lines of \( S \) form a complete span \( \mathcal{S} \) of \( \mathcal{H}(3,q^2) \) [1]. The point set of this span can be partitioned into two skew generators \( r \) and \( r' \) of \( \mathcal{H}(3,q^2) \) and \( q - 1 \) mutually disjoint symplectic subquadrangles \( \mathcal{W}_1, \ldots, \mathcal{W}_{q-1} \) of \( \mathcal{H}(3,q^2) \) (of which \( W \) is one) and such a partition is stabilized by \( G \). This can be seen by consideration of the Klein map, under which the generators of \( \mathcal{H} \) correspond to points of an elliptic quadric \( Q^-(5,q) \), and points of \( \mathcal{H} \) correspond to lines of \( Q^-(5,q) \). The lines in \( \mathcal{S} \) correspond to points of an elliptic quadric \( \mathcal{E} \), a solid section of \( Q^-(5,q) \), and \( r, r' \) correspond to points \( R, R' \) of the secant line \( L \) polar to \( \mathcal{E} \); the subgroup of \( PO^-(6,q) \) corresponding to \( G \) permutes the points of \( \mathcal{E} \) and fixes each point of \( L \). The subquadrangles \( \mathcal{W}_i \) then correspond to parabolic quadrics \( \mathcal{P}_i = Q^-(5,q) \cap P_i^\perp \), where \( P_1, P_2, \ldots, P_{(q-1)} \) are the points of \( L \) not on \( Q^-(5,q) \) and \( \perp \) is the polarity induced by \( Q^-(5,q) \).
We prove the following proposition.

**Proposition 2.5.** The group $G$ has the following orbits on generators of $\mathcal{H}(3, q^2)$:

(a) Two orbits of length 1, consisting of the transversals $r$ and $r'$ of the complete span $\mathcal{S}$.

(b) An orbit of length $q^2 + 1$ consisting of the lines of the complete span $\mathcal{S}$.

(c) $2(q - 1)$ orbits of length $q^2 + 1$, each consisting of lines not in $\mathcal{S}$ and intersecting either $r$ or $r'$.

(d) $q(q - 1)$ orbits of length $q^2 + 1$, each consisting of lines not in $\mathcal{S}$ and arising from lines of the symplectic subgeometries $\mathcal{W}_1, \ldots, \mathcal{W}_{(q-1)}$.

**Proof.** Taking into account the proof of [16, Theorem 5] and Proposition 2.3, (a), (b) and (d) immediately follow. Let $P \in r$. Then, the stabilizer of $P$ in $G$ is trivial and the $q - 1$ generators on $P$ not in $\mathcal{S}$ and distinct from $r$ produce $q - 1$ $G$-orbits all of size $q^2 + 1$. Since the lines of $\mathcal{S}$ are the only generators meeting both $r$ and $r'$, (c) holds true. □

3 The construction of the hyperovals

In this section we construct an infinite family of hyperovals of $\mathcal{H}(3, q^2)$ admitting the Singer cyclic group $G$ as an automorphism group. In the following section, our consideration of the full automorphism groups shows that there are actually three subfamilies having different automorphism groups.

In the geometric setting of the previous section let us choose two symplectic subquadrangles $\mathcal{W}$ and $\mathcal{W}'$ among the subquadrangles $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{(q-1)}$.

Notice that any generator $g$ of $\mathcal{H}(3, q^2)$ not belonging to $\mathcal{S}$ either is disjoint from $\mathcal{W}_1, \ldots, \mathcal{W}_{(q-1)}$ (and this happens when $g$ meets $r$ or $r'$) or extends a line of exactly one of $\mathcal{W}_1, \ldots, \mathcal{W}_{(q-1)}$ and is disjoint from the others.

Let now $\Omega$ and $\Omega'$ be two elliptic quadrics in $\mathcal{W}$ and $\mathcal{W}'$, respectively, invariant under $G$, whose associated quadratic forms do not polarize to the symplectic polarities $\mathcal{A}$ and $\mathcal{A}'$ induced by $\mathcal{W}$ and $\mathcal{W}'$, respectively. Let $\mathcal{I} = \Omega \cup \Omega'$. Any line of the span $\mathcal{S}$ meets each of $\Omega$ and $\Omega'$ in exactly one point. Any other generator $g$ of $\mathcal{H}(3, q^2)$ is either disjoint from both $\mathcal{W}$ and $\mathcal{W}'$ or extends a line of just one of $\mathcal{W}$ and $\mathcal{W}'$. Assume that $g$ extends a line
of $\mathcal{W}$. Then $g$ is clearly disjoint from $\mathcal{W}'$ and intersects $\Omega$ in either 0 or 2 points. Hence $\mathcal{I}$ is a hyperoval of $\mathcal{H}(3,q^2)$ of size $2(q^2 + 1)$.

We have proved the following result.

**Theorem 3.1.** There exists an infinite family of hyperovals of $\mathcal{H}(3,q^2)$, $q$ even, admitting a Singer cyclic group of order $q^2 + 1$. \hfill \Box

### 4 Automorphism groups

In this section we identify the full automorphism group $F$ of $\mathcal{I} = \Omega \cup \Omega'$, in other words $F$ is the stabilizer of $\mathcal{I}$ in $P\Gamma U(4,q^2)$, where $\mathcal{I}, \Omega, \Omega'$ are as in the previous section; clearly $G \leq F$. In the process we find that three cases arise, depending in part on whether or not $P\Gamma U(4,q^2)$ contains elements interchanging $\Omega, \Omega'$.

**Notation.** Recall that each line of $\mathcal{W}$ extends to a unique line of $\mathcal{H}$. If $X$ is a set of lines of $\mathcal{W}$ or a set of lines of $\mathcal{W}'$, we write $\tilde{X}$ for the corresponding set of lines of $\mathcal{H}$. Notice that each line of $\tilde{S}$ also extends a line of $\mathcal{W}'$ (although the intersections with $\mathcal{W}$ and $\mathcal{W}'$ are disjoint) and so $\tilde{S} = \tilde{S}'$.

Recall the Klein map under which lines of $\mathcal{H}$ are represented as points of $O^{-}(5,q)$ and points of $\mathcal{H}$ as lines of $O^{-}(5,q)$ and, in particular, the lines of $\tilde{S}$ correspond to points of an elliptic quadric $\mathcal{E}$. Via the Klein map there are group isomorphisms $P\Gamma U(4,q^2) \cong PO^{-}(6,q)$ and $PSU(4,q^2) \cong PO^{-}(6,q)$, the latter subgroups having index 2 in the former.

**Notation.** Whenever we have a subgroup $X$ of $P\Gamma U(4,q^2)$, we write $\tilde{X}$ for the corresponding (isomorphic) subgroup of $PO^{-}(6,q)$ (and vice-versa). We deduce properties of $X$ from those of $\tilde{X}$ and vice-versa. Thus, for example, $\tilde{G}$ is a cyclic subgroup of $PO^{-}(6,q)$ permuting the points of $\mathcal{E}$ (and fixing each point on the line $L$ polar to $\mathcal{E}$).

Our starting point is a result giving more details on the $G$-orbits on points of $\mathcal{W}$.

**Proposition 4.1.** [7, 13] There is a second $G$-orbit $\Delta$ of points of $\mathcal{W}$ such that $\Omega \cup \Delta$ is a 2-ovoid of $\mathcal{W}$, i.e., each line of $\mathcal{W}$ meets $\Omega \cup \Delta$ in 2 points.

**Proposition 4.2.** Let $\mathcal{T}$ and $\mathcal{U}$ be the sets of lines of $\mathcal{W}$ meeting $\Omega$ and $\Delta$ respectively in 2 points, let $\Omega', \mathcal{T}', \mathcal{U}'$ be the corresponding sets with respect to $\Omega'$ and $\mathcal{W}'$. Then the following hold:

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(a) Let \( \mathcal{R} = \mathcal{T} \cup \mathcal{T}' \cup \mathcal{S} \). Then \( F \) stabilizes \( \mathcal{R} \).

(b) (i) Each point of \( \mathcal{I} \) lies on \( q + 1 \) lines of \( \mathcal{R} \).

(ii) Each point of \( \Delta \cup \Delta' \) lies on 1 line of \( \mathcal{R} \) and \( q \) lines of \( \mathcal{U} \cup \mathcal{U}' \).

(iii) Each point of \( (\mathcal{W} \cup \mathcal{W}') \setminus (\Omega \cup \Delta \cup \Omega' \cup \Delta') \) lies on \( 1 + \frac{q}{2} \) lines of \( \mathcal{R} \) and \( \frac{q}{2} \) lines of \( \mathcal{U} \cup \mathcal{U}' \).

(iv) A point of \( \mathcal{H} \setminus (\mathcal{W} \cup \mathcal{W}') \) lies on at most 2 lines of \( \mathcal{R} \).

(c) Let \( \mathcal{Z} \) be the set of points of \( (\mathcal{W} \cup \mathcal{W}') \setminus (\Delta \cup \Delta') \). Then \( F \) stabilizes \( \mathcal{Z} \).

(d) \( F \) stabilizes each of the line sets \( \mathcal{S}, \mathcal{T} \cup \mathcal{T}', \mathcal{U} \cup \mathcal{U}' \) and each of the point sets \( \Delta \cup \Delta' \) and \( \mathcal{W} \cup \mathcal{W}' \).

Proof.

(a) \( \mathcal{R} \) is the complete set of lines of \( \mathcal{H} \) meeting \( \mathcal{I} \) in 2 points, and hence is fixed (globally) by \( F \).

(b) We begin with properties of \( \Omega \) and \( \Delta \). Given that the lines of \( \mathcal{W} \) meeting each of \( \Omega \) and \( \Delta \) in one point are precisely the lines of \( \mathcal{S} \), the set of lines of \( \mathcal{W} \) is the disjoint union of \( \mathcal{S}, \mathcal{T} \) and \( \mathcal{U} \). If \( X \in \Omega \), then every line through \( X \) lies in \( \mathcal{S} \cup \mathcal{T} \), so \( X \) lies on 1 line of \( \mathcal{S} \) and \( q \) lines of \( \mathcal{T} \) (and no lines of \( \mathcal{T}', \mathcal{U} \) or \( \mathcal{U}' \)). Thus \( X \) lies on \( q + 1 \) lines of \( \mathcal{R} \) and hence on \( q + 1 \) lines of \( \mathcal{R} \). The argument for \( X \in \Omega' \) is analogous and therefore (i) is established.

Similarly, if \( X \in \Delta \), then \( X \) lies on 1 line of \( \mathcal{S} \) and \( q \) lines of \( \mathcal{U} \) (and no lines of \( \mathcal{T}, \mathcal{T}' \) or \( \mathcal{U}' \)). Hence \( X \) lies on 1 line of \( \mathcal{R} \) and \( q \) lines of \( \mathcal{U} \cup \mathcal{U}' \), with \( X \in \Delta' \) analogous, and therefore (ii) is established.

Suppose that \( X \) is a point of \( \mathcal{W} \setminus (\Omega \cup \Delta) \). Each line of \( \mathcal{W} \) through \( X \) lies in the polar plane \( X^A \) in \( \mathcal{W} \). Let \( \mathcal{C}_\Omega = X^A \cap \Omega \) and \( \mathcal{C}_\Delta = X^A \cap \Delta \). Then each of \( \mathcal{C}_\Omega \), \( \mathcal{C}_\Delta \) is either a non-degenerate conic (with \( q + 1 \) points) or a single point. The line of \( \mathcal{S} \) through \( X \) meets each of \( \mathcal{C}_\Omega, \mathcal{C}_\Delta \) in 1 point. Each other line of \( \mathcal{W} \) through \( X \) either meets \( \mathcal{C}_\Omega \) in 2 points or \( \mathcal{C}_\Delta \) in 2 points. At most \( \frac{q}{2} \) lines through \( X \) can meet \( \mathcal{C}_\Omega \) in 2 points and at most \( \frac{q}{2} \) can meet \( \mathcal{C}_\Delta \) in 2 points, so exactly \( \frac{q}{2} \) lines through \( X \)
lie in each of $\mathcal{T}, \mathcal{U}$ (and no lines of $\mathcal{T}'$ or $\mathcal{U}'$). Hence $X$ lies on $1 + \frac{q}{2}$ lines of $\mathcal{R}$ and $\frac{q}{2}$ lines of $\mathcal{U} \cup \mathcal{U}'$, with $X \in \mathcal{W}' \setminus (\Omega' \cup \Delta')$ analogous, and therefore (iii) is established.

For (iv), we note that a point of $\mathcal{H} \setminus (\mathcal{W} \cup \mathcal{W}')$ lies on at most one (extended) line of $\mathcal{W}$ and at most one (extended) line of $\mathcal{W}'$, and therefore at most 2 lines of $\mathcal{R}$.

(c) The points of $\mathcal{Z}$ each lie on at least 3 lines of $\mathcal{R}$, and any other point of $\mathcal{H}$ lies on at most 2 lines of $\mathcal{R}$. Therefore $F$ stabilizes $\mathcal{Z}$.

(d) Observe that lines of $\mathcal{S}$ meet $\mathcal{Z}$ in $2q$ points, lines of $\mathcal{T} \cup \mathcal{T}'$ meet $\mathcal{Z}$ in $q + 1$ points, lines of $\mathcal{U} \cup \mathcal{U}'$ meet $\mathcal{Z}$ in $q - 1$ points, and every other line of $\mathcal{H}$ is skew to $\mathcal{Z}$. It follows that $F$ stabilizes each of $\mathcal{S}, \mathcal{T} \cup \mathcal{T}', \mathcal{U} \cup \mathcal{U}'$. As $\Delta \cup \Delta'$ is the set of points of $\mathcal{H}$ lying on $q$ lines of $\mathcal{U} \cup \mathcal{U}'$, it follows that $F$ stabilizes $\Delta \cup \Delta'$. Given that $F$ also stabilizes $\mathcal{Z}$ it follows that $F$ stabilizes the point set $\mathcal{W} \cup \mathcal{W}'$.

**Proposition 4.3.** The group $F$ stabilizes setwise the pair $\{\mathcal{W}, \mathcal{W}'\}$ and also the pair $\{\Omega, \Omega'\}$.

**Proof.** In Proposition 4.2 above, we established that $F$ stabilizes the set of points of $\mathcal{W} \cup \mathcal{W}'$. We show here that the point-line incidence graph formed by the points of $\mathcal{W}$ and the lines in $\mathcal{T} \cup \mathcal{U}$ is connected (using the lines in $\mathcal{T} \cup \mathcal{U}$ would be equivalent). Choose any two points $U, V \in \mathcal{W}$. If the line $UV$ is not in $\mathcal{T} \cup \mathcal{U}$, then it is either a line of $\mathcal{S}$ or a non-isotropic line of $PG(3,q)$ with respect to the polarity $A$ of $\mathcal{W}$. If $UV$ is in $\mathcal{S}$, then each other line of $\mathcal{W}$ through $V$ lies in $\mathcal{T} \cup \mathcal{U}$ so we can replace $V$ by a point on such a line that does not lie in $UA$. If $UV$ is non-isotropic, then $ UV )^A$ is a non-isotropic line of $\mathcal{W}$ skew to $UV$, with each point $X$ (in $\mathcal{W}$) of $ UV )^A$ collinear with each of $U, V$ (i.e., $UX, VX$ are lines of $\mathcal{W}$). We can choose $X$ so that $UX, VX$ are not lines of $\mathcal{S}$ (since there is only one line of $\mathcal{S}$ through each of $U$ and $V$), and then $UX, VX \in \mathcal{T} \cup \mathcal{U}$. Hence the point-line incidence graph is connected, as claimed.

Clearly the corresponding property holds for $\mathcal{W}'$. It follows that the point-line incidence graph formed by the points of $\mathcal{W} \cup \mathcal{W}'$ and the lines in $\mathcal{T} \cup \mathcal{T}' \cup \mathcal{U} \cup \mathcal{U}'$ has two connected components. This structure is preserved by $F$ and therefore $F$ stabilizes $\{\mathcal{W}, \mathcal{W}'\}$. Although this is established for
point sets of \(W, W'\), it necessarily applies to line sets as well. Since \(F\) stabilizes \(I\) it now follows that it is the stabilizer of the pair \(\{\Omega, \Omega'\}\).

\[\Box\]

**Theorem 4.4.** Let \(F_0\) be the subgroup of \(F\) stabilizing each of \(\Omega, \Omega'\). Then \(F_0\) is isomorphic to \(D_{2(q^2+1)}\).

**Proof.** In [7], the stabilizer of \(\{\Omega, \Delta\}\) in \(PSp(4, q)\) was shown to contain a semidirect product \(G \rtimes C_4\) of \(G\) with a cyclic group of order 4, \(C_4\) being generated by a collineation \(\theta\) that interchanges \(\Omega, \Delta\); in fact it was shown that \(G \rtimes C_4\) stabilizes \(S\) and \(Q\) and \(\langle G, \theta^2 \rangle\) stabilizes each of the \(G\)-orbits of points in \(W\); note that \(\langle G, \theta^2 \rangle\) normalizes \(G\).

We have shown that \(F\) stabilizes \(S\). The stabilizer of \(\bar{S}\) in \(PGU(4, q^2)\) is isomorphic to the stabilizer of \(E\) in \(PO^-(6, q)\), which has the structure \(PO^-(4, q) \times PO^+(2, q)\). The subgroup \(F_0\) stabilizes each of \(W\) and \(W'\). In particular this means that \(F_0\) stabilizes \(E\) and each of the parabolic quadrics \(P, P'\) corresponding to \(W, W'\), so fixes each of the points \(P, P'\) of \(L\) that are polar to \(P, P'\) respectively. Note that \(PO^-(6, q)\) contains elations centred on points of \(PG(5, q) \setminus Q^-(5, q)\), the elation on a point \(X\) being an involution \(\sigma_X\) fixing each point of \(X^\perp\); these elations are sometimes called reflections (or symmetries in the vector space context), and they lie in \(PO^-(6, q) \setminus PO^-(6, q)\); we denote by \(E_X\) the cyclic group of order 2 generated by \(\sigma_X\) (an ‘elation subgroup’). Note that \(\sigma_P\) corresponds to the unique involutory semi-linear element \(\tau\) of \(PGU(4, q^2)\) fixing each point of \(W\).

Let us write \(\tilde{J}_1 = PO^-(4, q) \times E_P\), \(\tilde{J}_2 = J_1 \cap PO^-(6, q)\), \(\tilde{J}_3 = PO^-(4, q) \times 1\) and \(\tilde{J}_4 = PO^-(4, q) \times 1\). Then \(J_1\) is the stabilizer of both \(W, S\) in \(PGU(4, q^2)\), \(J_2 = \tilde{J}_2 \cap PSp(4, q^2)\) (isomorphic to \(PSL(2, q^2)\); 2) and \(J_4\) is the unique subgroup of \(J_2\) isomorphic to \(PSL(2, q^2)\). Note that \(\tilde{J}_3\) is the stabilizer in \(PO^-(6, q)\) of each of \(E, P, P'\) so contains \(F_0\). Note also that \(|J_1 : J_2| = |J_1 : J_3| = |J_2 : J_4| = |J_3 : J_4| = 2\) and each of \(J_2, J_3, J_4\) is normal in \(J_1\).

For \(i = 1, 2, 3, 4\), let \(H_i\) be the stabilizer of \(Q\) in \(J_i\). We make the following observations:

- \(J_4\) is transitive on points of \(W\) (because \(\tilde{J}_4\) is transitive on the lines of \(P\), and so \(H_4\) is a proper subgroup of \(J_4\) containing \(\langle G, \theta^2 \rangle\).
- The normalizer of \(G\) in \(J_4\) is a group isomorphic to \(D_{2(q^2+1)}\) which contains any proper subgroup of \(J_4\) containing \(G\) (as follows from consulting a list of the subgroups of \(PSL(2, q^2)\), such as found in [12]). Hence \(H_4 = \langle G, \theta^2 \rangle \cong D_{2(q^2+1)}\).
The normalizer of $G$ in $J_1$ has order at most $8(q^2 + 1)$. Noting that $	heta$ and $	au$ commute, both normalize $G$ and both lie in $H_1$ we see that $H_1 = \langle G, \theta, \tau \rangle \cong (G \rtimes C_4) \times C_2$. Any proper subgroup of $J_1$ containing $G$ but not containing $J_4$ is a subgroup of $H_1$. In particular $F_0 \leq H_1$; indeed $F_0 \leq H_3$.

We note that $\theta \in H_2 \setminus H_3$ and $\theta \tau \in H_3 \setminus H_2$, with $\theta \tau$ having order 4. Thus $H_2, H_3$ are distinct but are both isomorphic to $G \rtimes C_4$. In particular $H_3 = \langle G, \theta \tau \rangle$.

The element $(\theta \tau)^2 = \theta^2$ fixes $\Omega$ but $\theta \tau$ does not. Therefore $F_0 \leq \langle G, \theta^2 \rangle = H_4$. Recall that $H_4$ fixes each $G$-orbit of points of $W$, and $\Omega$ in particular.

The group $J_4$ stabilizes both of $W', \bar{S}$, and applying the above analysis to $W'$ we find that $H_4$ (being the normalizer of $G$ in $J_4$) fixes each $G$-orbit of points of $W'$, and $\Omega'$ in particular.

Hence $F_0 = H_4 \cong D_2(q^2 + 1)$.

\[ \square \]

**Theorem 4.5.** The infinite family of hyperovals described in Theorem 3.1 falls into three subfamilies distinguished by their stabilizers in $P\Gamma U(4, q^2)$: $D_2(q^2 + 1)$, $D_2(q^2 + 1) \times C_2$ or $C(q^2 + 1) \rtimes C_4$.

**Proof.** Having shown $F_0 = H_4$, we know that $H_3$ is the normalizer of $F_0$ in $J_3$. In $PO^{-}(6, q)$, the stabilizer of both $E$ and the pair $\{P, P'\}$ is the group $\tilde{J} = \langle \tilde{J}_3, \sigma_R \rangle$, where $R$ is a point of $L$ and $\sigma_R$ is the unique elation switching $P, P'$. We write $\tau_R$ for the semilinear element of $P\Gamma U(4, q^2)$ corresponding to $\sigma_R$, noting that $\tau_R$ fixes $\bar{S}$ but switches $W, W'$. Then the normalizer of $F_0$ in $J$ is the group $H = \langle H_3, \tau_R \rangle$ and $F \leq H$ with $|H : F_0| = 4$.

In order to determine $F$, it suffices to consider $\tau_R, \theta \tau \tau R$ as representatives of cosets of $F_0$ in $H$ that do not fix $W$. Note that $\tau_R, \theta \tau \tau R$ both have order 2. Hence we have three possibilities:

(a) $\Omega' = \tau_R \Omega$ in which case $F = \langle G, \theta^2, \tau_R \rangle \cong D_2(q^2 + 1) \times C_2$.

(b) $\Omega' = \theta \tau \tau R \Omega$ in which case $F = \langle G, \theta \tau \tau R \rangle \cong G \rtimes C_4$.

(c) $\Omega' \neq \tau_R \Omega, \theta \tau \tau R \Omega$ in which case $F = F_0 \cong D_2(q^2 + 1)$.

\[ \square \]
5 The isomorphism problem

Up to date, the known infinite families of hyperovals on the Hermitian surface are:

- **Del Fra-Ghinelli-Payne’s family**: a member in this family has size $2(q^3 - q)$ and it is the union of two ovoids of $H(3, q^2)$ sharing a chord, [11].

- **Symplectic hyperovals**: Any 2-ovoid of $W(3, q)$ embedded in $H$ gives rise to a hyperoval of size $2(q^2 + 1)$ of $H(3, q^2)$. Such hyperovals have been called *symplectic hyperovals* in [6], and examples of 2-ovoids of $W(3, q)$ have been constructed in [4] and [7].

- **$PSL(2, q)$-hyperovals**: a member in this family has size $2(q^3 - q)$ and its automorphism group contains $PSL(2, q)$ fixing a Baer subconic, [9].

A hyperoval on $H(3, q^2)$ constructed in Theorem 3.1 has size $2(q^2 + 1)$ and hence it cannot be isomorphic to a hyperoval belonging to the first and third family. On the other hand, it cannot be isomorphic to a symplectic hyperoval due to the fact that it cannot be completely contained in a symplectic subquadangle embedded in $H(3, q^2)$. Indeed, two symplectic subquadangles embedded in $H(3, q^2)$ either are disjoint, or meet in a totally isotropic line or in a hyperbolic pair [8].

References


