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Analytical solutions of stresses and displacements for deeply buried twin tunnels in viscoelastic rock

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A B S T R A C T

A set of analytical solutions is presented to calculate the stresses and displacements generated when two closely located circular tunnels are sequentially excavated in viscoelastic rock. The solutions are provided for circular tunnels excavated in time dependent rock for any type of linear viscoelastic model, e.g. Burgers and Poyting-Thomson model. In the presented solutions the sequential excavation of the tunnels is also accounted for. The solutions are provided as analytical expressions in integral form. These were obtained by extending the principle of correspondence to solid media with time varying boundaries. A comparison of the stresses and displacements predicted by the analytical solutions and FEM analyses for an example case of twin tunnels excavated in a generalized Kelvin medium shows a good agreement between the two methods. Then, a parametric analysis was performed to investigate the influence of the tunnel spacing on displacements and stresses for various excavation processes. Several dimensionless charts summarizing the result of the parametric analysis are provided for the benefit of practitioners.

1. Introduction

The construction of twin tunnels (or triple tunnels as in the case of the Channel Tunnel) in close proximity to each other is increasingly commonplace. One of the motivations for the presented work is the current absence of normative guidelines for the design of twin tunnels. This paper aims to provide guidance to tunnel engineers on how to estimate the likely degree of interaction between two or more twin tunnels built in viscoelastic rock in the preliminary phase of the design process. In this phase of the design process, numerical analyses are not employed due to time constraints. This paper wants to provide an analytical approach that may be employed to help practitioners taking key decisions in the preliminary design phase (e.g. the distance between the two tunnels).

In the literature several numerical analyses have been presented to determine the rock response for twin tunnels being either deeply buried,1–3 or shallow.4–7 However, although numerical methods such as finite element,1 finite difference, distinct element11 and to a lesser extent boundary element are increasingly used in tunnel design, they require long runtimes especially when a complete parametric analysis needs to be performed to find out which parameters control the problem. Therefore, preliminary design is usually made on the basis of simplified analytical models that allow obtaining a first estimation of the design parameters. Moreover, analytical solutions provide a benchmark against which the overall correctness of numerical analyses in subsequent design stages can be assessed.

Most types of rocks, whether hard or soft, exhibit time-dependent behaviours,1,16 which induce gradual deformations over time occurring even after completion of the excavation process. Elastic and elastoplastic models ignore the effect of time dependency, which may contribute in some cases up to 70% of the total deformation.11 In this paper, rock time-dependent behaviour will be accounted for by linear viscoelastic models. Unlike the case of linear elastic materials with constitutive equations in the form of algebraic equations, linear viscoelastic materials have their constitutive relations expressed by a set of operator equations. In general, it is very difficult to obtain analytical solutions for most of the viscoelastic problems. However, some analytical solutions have been developed for single tunnel problem with circular cross-section excavated in viscoelastic rock,11–13 with the excavation being assumed to take place instantaneously.

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Recently, new analytical derivations have been presented that allow obtaining analytical solutions for single circular and elliptical tunnels excavated in viscoelastic rock accounting for sequential excavations. In this paper, these solutions are extended to the case of two twin tunnels. Note that in practice twin tunnels are never excavated at exactly the same time, so that in any cross-section considered, the steps of sequential excavations take place at different times in the two tunnels and the excavation of one chamber is completed before the other one. The solutions provided in this paper are valid for the general case of sequential excavation of twin tunnels with the excavation processes in the tunnels being different (e.g. a time delay between the excavation of the two tunnels, different excavation speeds in the two tunnels, etc.). The solutions account also for the case of twin tunnels of different final sizes (e.g. the case of an emergency road tunnel beside a high speed train tunnel).

With regard to the literature relative to the analytical methods on which the solutions provided are based, Ling provides a theoretical elastic stress solution for a plate containing two circular holes of equal size. In order to derive the solution for the displacements in an elastic plate with two circular holes, the Schwarz’ alternating method is used. By using this method, Zimmermann gives the second order solution for an elastic plate containing two equally sized circular holes; Kooi and Verruijt provide the analytical solution for twin tunnels in an infinite medium, with an arbitrary load at the tunnel boundary, which is a generalisation of this method. Hoang and Abouseleiman introduced the exact explicit solution for the stress distribution in an infinite plate containing two unequal circular holes and subject to general in-plane stresses at infinity and internal pressure inside the holes by using Airy stress functions, following the approach proposed by Green. By using Schwarz’ alternating method, Ukdagaoener and Patil analyzed the stress state of a plate containing two elliptic holes; Zhang et al. provide the accurate stress solution for two elliptical holes in infinite region by using Schwarz’ alternating method and Fourier series expansions for the resulting stresses on the tunnel boundary. Osman presented solutions for the stability of twin tunnels excavated in soft ground. He derived a compatible displacement field using the principle of superposition for ground deformations around shallow and unlined tunnels embedded in undrained clay. Radi presented an analytical solution for the stresses induced in an infinite elastic plate with two unequal circular holes subjected to uniform loads and arbitrary internal pressures acting on the holes. Spencer and Sinclair employ a sequence of Airy stress functions to derive an analytical solution for twin circular holes under gravitational load in an elastic half-space. An analytical solution for shallow twin tunnels assuming an elastic half space is provided by Fu et al. All the aforementioned works are based on the assumption of elasticity, hence they are unable to account for the time dependent behaviour of the rock and for the tunnel construction processes employed. In this paper instead, a new analytical solution is derived for the calculation of stresses and displacements for two parallel twin tunnels located at a short distance apart accounting for the effect of rock time-dependent behaviour and sequential excavation. They are achieved by using the Muskhelishvili’s complex variable theory, the Laplace transform technique and Schwarz’s alternating method.

2. Formulation of the problem

The excavation of two circular twin tunnels in viscoelastic rock is considered herein. The following assumptions are made:

**Assumption 1.** the two twin tunnels are of circular section. The rock mass is homogeneous, isotropic, and linearly viscoelastic. The tunnels are deeply buried and subject to a hydrostatic state of stress. Although the hypothesis of tunnels deeply buried limits the scope of the paper, deep tunnel issues are still a very active area of research in the rock mechanics community.

**Assumption 2.** the tunnel excavation is sequential. The excavation process is described by two time dependent functions, \( R_1(t) \) and \( R_2(t) \), expressing the growth of the radii of the two tunnels, herein called Tunnel 1 and Tunnel 2, indicating the first and second tunnel to be excavated respectively, i.e. the start time for the construction of Tunnel 2 is equal to or later than the start time of Tunnel 1.

**Assumption 3.** the excavation speed is low enough so that no dynamic stresses are ever induced.

The calculation of stresses and displacements in the vicinity of the face of the tunnel is a genuine three-dimensional (3D) problem. However, show that the problem can be tackled as two dimensional plane-strain accepting the approximation of using a dimensionless parameters \( \lambda(t) \) (0 < \( \lambda \) ≤ 1) to account for the progressive release of the initial stresses cause by the tunnel advancement. Considering the excavation of a single tunnel, the normal and tangential tractions (\( \sigma_{nn} \) and \( \sigma_{nt} \)) acting on the tunnel boundary can be expressed as follows:

\[
\sigma_{nn} = -\lambda(t)\sigma_0^{n}, \quad \sigma_{nt} = -\lambda(t)\sigma_0^{nt}
\]

where \( \sigma_0^{n} \) and \( \sigma_0^{nt} \) represent the normal and tangential stresses at the periphery of the tunnel prior to the start of excavation; and \( \lambda(t) \) is a function of the variation of the radial displacement with the distance \( z \) between the tunnel face and the cross-section considered, that can be determined by in-situ measurements, or calculated numerically. However, in case of twin tunnels, two parameters, \( \lambda_1(t) \) and \( \lambda_2(t) \), need to be introduced to account for the advancement of each tunnel that is likely to take place at different times.

A limitation of the analytical solutions here proposed is due to the lack of consideration of lining. This is because the presence of lining makes the problem mathematically intractable due to structure – ground interactions. Currently in the literature, no analytical solutions exist for lined twin tunnels even in the simplest of the cases of an elastic medium. However, a key benefit of the analytical solutions presented here is that they can be used to predict tunnel convergence to assess whether the presence of a lining would be necessary already in the preliminary phase of the design. Moreover, they allow obtaining a first estimate of the magnitude of the excavation-induced displacements before installation of any lining is carried out. Finally, note that for several tunnels excavated in hard rock, the use of lining is not necessary. Sometimes even if a thin lining is applied, this has no structural significance.

According to the aforementioned assumptions, the equivalent plane-strain problem in the plane of the cross-section of the tunnels, can be formulated as shown in Fig. 1(a). Both Cartesian coordinates \( (x, y) \), \( (x_1, y_1) \) and polar coordinates \( (r, \theta) \), \( (r_1, \theta_1) \) will be employed in the derivation of the analytical solutions. The adopted sign convention is as follows: positive for compression and negative for tension.

Suppose that the rock mass is subject to an initial stress state since time \( t' = 0 \) (before this time the state of stress may have been different due to ongoing geological processes), and the excavation of Tunnel 1 (see Fig. 1) takes place from time \( t' = t'_1 \) until \( t' = t'_{1 \to fin} \). Then Tunnel 2 (the right tunnel in Fig. 1) is excavated from \( t' = t'_2 \) \( \{(t'_2 \geq t'_1) \} \) until \( t' = t'_{2 \to fin} \). The radii of Tunnel 1 \( (R_1 = R_1(t')) \) and Tunnel 2 \( (R_2 = R_2(t')) \) are specified as:

\[
\begin{align*}
R_1(t') &= \begin{cases} 
  a_1 & t'_1 \leq t' < t'_{1 \to fin} \\
  R_{fin}^1 & t' \geq t'_{1 \to fin}
\end{cases} \\
R_2(t') &= \begin{cases} 
  a_2 & t'_2 \leq t' < t'_{2 \to fin} \\
  R_{fin}^2 & t' \geq t'_{2 \to fin}
\end{cases}
\end{align*}
\]

where \( R_{fin}^1 \) and \( R_{fin}^2 \) are the final radius of Tunnel 1 and Tunnel 2 respectively and \( a_1 \) and \( a_2 \) depend on the excavation process employed for the two tunnels: in case drilling and blasting is used, the two functions are time-dependent, so \( a_1 = a_1(t') \) and \( a_2 = a_2(t') \), whereas in case of TBM excavation they are time independent, so \( a_1 = R_{fin}^1 \) and \( a_2 = R_{fin}^2 \) with \( t'_1 \) and \( t'_2 \) the times at which the TBM passes. Note that \( t'_{1 \to fin} \)
may be either smaller, equal to or larger than \( t' \) so that any type of sequential excavation process can be accounted for including the excavation of both twin tunnels occurring at the same time.

In the derivation of the analytical solution presented herein, the excavation process is divided into three stages: The first stage, spanning from time \( t' = 0 \) to \( t' = t_1' \), is prior any excavation takes place. The intact rock mass subject to a hydrostatic initial stress state (see Fig. 1b). The stresses acting on the future boundaries of the tunnel, e.g. \( \sigma_{x1} (1-0) \) and \( \sigma_{y1} (1-0) \) on the boundary of Tunnel 1 and \( \sigma_{x2} (2-0) \) and \( \sigma_{y2} (2-0) \) on the boundary of Tunnel 2, are non-zero and can be calculated from the initial stress state (see Fig. 1b). In the second stage, spanning from \( t' = t_1' \) to \( t' = t_2' \), only Tunnel 1 is excavated with \( \sigma_{x1} (2-1) \) and \( \sigma_{y1} (2-1) \) being the stresses acting on the future boundary of Tunnel 2 (see Fig. 1c). In the third stage, spanning from \( t' = t_2' \) onwards, both twin tunnels are being excavated, as shown in Fig. 1(d). No stresses are applied on the internal boundaries.

Introducing a new reference time \( t \), with \( t' = t - t_1' \), the second stage spans from \( t = 0 \) to \( t = t' - t_2' = t_2 \) and the third stage spans from \( t = t_2 \) onwards. Also \( t_{1-fin} = t' - t_{1-fin} \) and \( t_{2-fin} = t' - t_{2-fin} \).

3. Mathematical formulation for the general viscoelastic problem

The stress-strain behaviour of linear viscoelastic rock can be schematized by a number of springs and dashpots connected either in series or parallel to simulate different viscoelastic characteristics of the rock mass. The constitutive equations for the rock can be expressed in integral form as follows:

\[
\begin{align*}
\sigma_{xx}^v (X, t) &= 2 \int_0^t G(t - \tau) \frac{\partial \sigma_{xx}^v (X, \tau)}{\partial \tau} \, d\tau + \sigma_{xx}^0 (X, t) = 2 \int_0^t G(t - \tau) \frac{\partial \sigma_{xx}^v (X, \tau)}{\partial \tau} \, d\tau + \sigma_{xx}^0 (X, t), \\
\sigma_{yy}^v (X, t) &= 3 \int_0^t K(t - \tau) \frac{\partial \sigma_{yy}^v (X, \tau)}{\partial \tau} \, d\tau + \sigma_{yy}^0 (X, t) = 3 \int_0^t K(t - \tau) \frac{\partial \sigma_{yy}^v (X, \tau)}{\partial \tau} \, d\tau + \sigma_{yy}^0 (X, t).
\end{align*}
\]

where \( X \) is the position vector; \( \sigma_{xx}^0 \) and \( \sigma_{yy}^0 \) are the mean stress and strain respectively for the viscoelastic case; \( \sigma_{xx}^v \) and \( \sigma_{yy}^v \) are the stress and strain deviator tensors respectively (the superscript ‘\( \cdot v \)’ stands for viscoelastic), defined as:

\[
\begin{align*}
s_{xx}^v &= \sigma_{xx}^v - \frac{1}{3} \delta_{ij} \sigma_{yy}^v, \quad s_{yy}^v = \sigma_{yy}^v - \frac{1}{3} \delta_{ij} \sigma_{xx}^v, \\
e_{xx}^v &= \varepsilon_{xx}^v - \frac{1}{3} \delta_{ij} \varepsilon_{yy}^v, \quad e_{yy}^v = \varepsilon_{yy}^v - \frac{1}{3} \delta_{ij} \varepsilon_{xx}^v.
\end{align*}
\]

with \( \sigma_0 \) and \( \varepsilon_0 \) being the stress and strain tensors respectively, and \( \delta_{ij} \) being the unit tensor. \( G(t) \) and \( K(t) \) represent the shear and bulk relaxation moduli, respectively. In Table 1 the expressions of \( G(t) \) are provided for five types of the most common viscoelastic models. The expressions for \( K(t) \) can be obtained by replacing the shear moduli (e.g. \( G_M, G_K, G_P \)) with the bulk ones.

In Wang et al., the methodology for solving a general viscoelastic problem involving time-dependent boundaries is expounded. The Laplace transform, with respect to time, is applied to the governing differential equations of the problem so that the relationship between the general solution for the viscoelastic and the elastic cases is obtained. The general solution for the viscoelastic case is obtained by replacing the shear and bulk moduli \( G \) and \( K \) in the general solution of the elastic case, with \( sG(s) \) and \( sK(s) \) respectively. \( \hat{f}(s) \) is a function of the variable \( s \) defined in the Laplace transform of the function \( f(t) \):

\[
\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty \exp^{-st} f(t) \, dt.
\]
Table 1
Expressions of time-dependent relaxation shear modulus $G(t)$ and two functions, $R(t)$ and $H(t)$ in Eq. (8), for five commonly used linear viscoelastic models.

<table>
<thead>
<tr>
<th>Number</th>
<th>Model</th>
<th>Name</th>
<th>$G(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>①</td>
<td>$E_x(G_x)\eta_x$</td>
<td>Maxwell model</td>
<td>$G_Mt^{-\frac{G_M}{\eta_x}}$</td>
</tr>
<tr>
<td>②</td>
<td>$E_x(G_x)\eta_x$</td>
<td>Kelvin model</td>
<td>$G_K + \eta_K\delta(t)$</td>
</tr>
<tr>
<td>③</td>
<td>$E_x(G_x)\eta_x$</td>
<td>generalized Kelvin model</td>
<td>$\frac{G_K^2}{\eta_K(G_K + \eta_K)^2} + \frac{G_K\eta_K}{G_K + \eta_K}$</td>
</tr>
<tr>
<td>④</td>
<td>$E_x(G_x)\eta_x$</td>
<td>Poyting-Thomson model</td>
<td>$\frac{G_P}{\eta_P^2} + G_R$</td>
</tr>
<tr>
<td>⑤</td>
<td>$E_x(G_x)\eta_x$</td>
<td>Burgers model</td>
<td>$G_B[ae^{-\frac{G_B}{\eta_B}} + ae^{-\frac{G_B}{\eta_B}}]$</td>
</tr>
</tbody>
</table>

The displacements for the viscoelastic case can be obtained by replacing the parameters, $G_x$ and $K_x$ of the elastic case, with $s\hat{G}(s)$ and $s\hat{K}(s)$ respectively, and then applying the inverse Laplace transform:

Substituting the inverse Laplace transformed general solution into the equations expressing the boundary conditions, the set of equations to be satisfied by the particular solution for the viscoelastic case can be found.
\[ \sigma_z (z, t) = \text{Re} \left\{ \frac{\partial \phi (z, t)}{\partial x} + \frac{\partial \psi (z, t)}{\partial y} \right\}, \]
\[ \sigma_y (z, t) = \text{Im} \left\{ \frac{\partial \phi (z, t)}{\partial x} + \frac{\partial \psi (z, t)}{\partial y} \right\}, \]
\[ u_z (z, t) = L^2 \left\{ \frac{1}{2G(z)} \int_0^t \left[ \phi (z, s) - \frac{\partial \phi (z, s)}{\partial t} \right] \right\} \]
(6)

where \( \sigma_z (z, t) \) and \( \sigma_y (z, t) \) are the horizontal and vertical stresses respectively for the viscoelastic case; \( \sigma_z (z, t) \) is the shear stress; \( u_z (z, t) \) and \( u_y (z, t) \) are displacements along the \( x \) and \( y \) direction respectively for the viscoelastic case; \( \psi = \psi (z, t) \) and \( \phi = \phi (z, t) \) are two complex potential functions, with \( z = x + iy \) and \( i = \sqrt{-1} \).

In a sequential excavation, different tractions are induced by the several excavation steps carried out over time. It is important to note that unlike the case of an elastic medium, for a viscoelastic medium the displacements at any point in time depend on the entire previous stress history. Let us assume that \( l \) loads are applied on the structure at different times before the generic time \( t' \), i.e. the \( k \)-th load \( (k=1,2, \ldots, l) \) is applied on the structure at time \( t_0 \) and removed at \( t_{nd} \). According to the principle of superposition, the total displacement induced in the rock at the generic time \( t' \) can be calculated as follows:

\[ u_z (z, t') + i u_y (z, t') = \frac{1}{2} \sum_{k=1}^l \int_{t_0}^{t_k} I (t') \phi (z, t) dt \]
\[ - \frac{1}{2} \int_0^{t_k} H (t') \left[ \frac{\partial \phi (z, t)}{\partial t} + \psi (z, t) \right] dt. \]
(9)

4. Analytical solutions for twin tunnels

In this section, the stresses and displacement increments induced by the excavation of Tunnel 1 and 2 are presented for the three aforementioned stages of excavation. The case with fast longitudinal advancement (i.e. 3D effect being ignored) will be investigated first. Then the solutions will be extended to account for the effect of tunnel face by multiplying the obtained potentials by the dimensionless advancement parameters \( \lambda_1 \) and \( \lambda_2 \).

4.1. Derivation for the first and second stages

According to Eq. (9), the displacements in the first stage, i.e. prior to any excavation (from time \( t' = 0 \) to \( t' = t_1 \)), can be expressed as:

\[ u_z (z, t') + i u_y (z, t') = \frac{1}{2} \int_0^{t_1} I (t') dt \]
\[ - \frac{1}{2} \int_0^{t_1} H (t') \left[ \frac{\partial \psi (z, t)}{\partial t} + \psi (z, t) \right] dt \]
(12)

with \( \psi_1 (z, t) = \phi_1 (z, t) + i \psi_1 (z, t) \), \( \phi_1 (z, t) \) and \( \psi_1 (z, t) \) being potential complex functions for an infinite 2D medium subjected to a hydrostatic (far field) state of stress. Note that with regard to the potentials, the superscript number represents the number of tunnels present in the medium: 0 means no tunnel present, 1 means one tunnel present, etc. The potentials are found to be:

\[ \psi_1^{(0)} (z_1) = \frac{p_0}{2}, \quad \psi_1^{(0)} (z_1) = 0 \]
(13)

where \( p_0 \) is the hydrostatic stress at infinity.

Two loading cases, Case (1) and Case (2), are considered in the following, each with different tractions applied on the domain boundaries. Displacements and stresses for the infinite 2D medium are determined as the sum of the displacements and stresses calculated for the two cases in virtue of the principle of superposition. The two cases are as follows (fast longitudinal advancement is assumed):

Case (1). Infinite 2D medium with a hole, corresponding to the cross-section of Tunnel 1, subject to far-field stresses. The corresponding potentials, denoted as \( \phi_1^{(1)} \) and \( \psi_1^{(1)} \), are given as follows:

\[ \psi_1^{(1)} (z_1, t') = \frac{p_0}{2}, \quad \phi_1^{(1)} (z_1, t') = - \frac{p_0 R (z_1, t')}{z_1} \]
(14)

with the initial stress being applied for the time period \( t' \in [0, \infty) \).

Case (2). Infinite 2D medium with excavation induced tractions acting at the internal boundary of Tunnel 1. Prior to tunnel excavation, due to the interaction between post-excavated rock and outer rock mass, the tractions \( \sigma_z^{(1-0)} \) and \( \sigma_y^{(1-0)} \) act on the time-dependent tunnel boundary (see Fig. 1b). The time period during which the induced tractions are non-zero is \( t' \in [0, t_1] \). In this period, the corresponding potentials are denoted as \( \phi_1^{(1-0)} \) and \( \psi_1^{(1-0)} \). The first superscript number represents the number of tunnels being present whilst the second superscript indicates the cause of the induced tractions; for instance \( 0 \) means that the traction acting on the tunnel boundary is due to the far-field initial stresses (no tunnel present); \( 1 \) represents the traction due to the excavation of Tunnel 1, etc. The notations used for the potentials together with the corresponding boundary conditions are listed in Table 2.

According to Eq. (10), the displacements at time \( t' \) (\( t' \in [t_1, t_2] \)) can be expressed as follows:

\[ u_z (z, t') + i u_y (z, t') = \frac{1}{2} \int_{t_1}^{t_2} I (t') dt \]
\[ - \frac{1}{2} \int_{t_1}^{t_2} H (t') \left[ \frac{\partial \psi (z, t)}{\partial t} + \psi (z, t) \right] dt \]
\[ + \frac{1}{2} \int_{t_1}^{t_2} H (t') \left[ \frac{\partial \psi (z, t)}{\partial t} + \psi (z, t) \right] dt \]
(15)

Applying the principle of superposition, displacements and stresses for the infinite 2D medium are found as the sum of the displacements
and stresses respectively calculated for Cases (1) and (2). Hence, the following relationship between potentials holds true:

\[
\psi(3, r') = \phi(1) (3, r') + \phi^{(1)-0}(3, r'), \\
\psi(0) (3, r') = \phi(3) (3, r') + \psi^{(1)-0}(3, r')
\]

(16)

Substituting Eq. (16) into Eq. (15), yields:

\[
L_{i-0} (z_1, r') = \int_{0}^{z_1} \int_{0}^{r'} I(t' - r) dt' dt
\]

(18)

From Eq. (16) the expression of \(\psi^{(1)-0}\) and \(\psi^{(1)-0}\) can be obtained:

\[
\psi^{(1)-0}(3, r') + \psi^{(1)-0}(3, r') = 0,
\]

(20)

Accounting for the effect of the advancement of Tunnel 1, tractions \(\sigma^{(1)-0}\) and \(\sigma^{(1)-0}\) take the following expressions:

\[
\sigma^{(1)-0}(z_1, r') = \lambda(3) c^{(1)-0} z_1^{2} + \lambda(3) c^{(1)-0} z_1^{2} \quad \text{with } 0 \leq \lambda(3) \leq 1
\]

(21)

and the corresponding potentials become:

\[
\phi^{(1)-0}(3, r') = \lambda(3) c^{(1)-0} z_1^{2} \quad \text{and} \quad \psi^{(1)-0}(3, r') = \lambda(3) c^{(1)-0} z_1^{2}
\]

(22)

with \(\lambda(3)\) being the dimensionless parameter accounting for the 3D effects of the advancement of Tunnel 1. Accordingly, the displacements due to the excavation of Tunnel 1 accounting for the effect of tunnel advancement can be obtained as follows:

\[
\Delta u_{i} (3, r') + i \Delta u_{i} (3, r') = [u_{i} (3, r') - u_{i} (3, r')] + [u_{i} (3, r') - u_{i} (3, r')]
\]

(23)

where

\[
L_{i-0} (z_1, r') = L_{i-0} (z_1, r') = \int_{0}^{z_1} \int_{0}^{r'} I(t' - r) dt' dt
\]

(24)

The total stresses are the summation of the additional stresses and the initial ones (hydrostatic stress state).

4.2. Derivation for the third stage

In the third stage, spanning from \(t' = t'\) onwards, with both twin tunnels being excavated in the rock mass (see Fig. 1(d)), no stresses are applied on the internal boundaries of the tunnels.

Herein the solution will be calculated by superposition of the following three cases (here fast longitudinal advancement is assumed):


Case (3), infinite 2D medium with Tunnels 1 and 2 present and subject to far-field stresses. The corresponding potentials are denoted by $\psi_1^{(2)}$ and $\psi_1^{(2)}$.

Case (4), infinite 2D medium with Tunnels 1 and 2 subjected to the induced tractions acting on the inner boundaries of the tunnels. These tractions are equal to the stresses acting on the boundaries of the tunnels in equilibrium with the initial stresses present prior to tunnel excavation, i.e. $\sigma_1^{(1-0)}$ and $\sigma_1^{(0-1)}$ acting on the boundary of Tunnel 1 and $\sigma_2^{(2-0)}$ and $\sigma_2^{(0-2)}$ on the boundary of Tunnel 2 (see Fig. 1b). The corresponding time dependent potentials are denoted by $\psi_1^{(2,0)}$ and $\psi_1^{(0,2)}$, with the time period of the application of loads being $t' \in [0, t]$.

Case (5), infinite 2D medium subjected to tractions acting on the inner boundary of Tunnel 2. The tractions, denoted by $\sigma_2^{(2-0)}$ and $\sigma_2^{(0-2)}$ in Fig. 1(c), represent the stresses acting on the time-dependent boundary of Tunnel 2 at a generic time after Tunnel 1 is excavated but before any excavation of Tunnel 2 takes place. The corresponding time-dependent potentials are denoted by $\psi_1^{(2-1)}$ and $\psi_1^{(2-1)}$ with the time period of load application being $t' \in [t_1, t_2]$. The notations for the potentials are listed in Table 2.

According to Eq. (30), the displacements occurring in this stage ($t' \in [t_1, t_2]$) can be expressed as follows:

$$u_i'(t_1', t') = \frac{1}{2} \int_{t_1}^{t_2} I(t' - r) [\phi_i^{(2)}(t, r) + \phi_i^{(0)}(t, r)] dr + \frac{1}{2} \int_{t_1}^{t_2} J(t' - r) [\phi_i^{(0)}(t, r) + \phi_i^{(2)}(t, r)] dr$$

$$= \frac{1}{2} \int_{t_1}^{t_2} I(t' - r) [\phi_i^{(2)}(t, r) + \phi_i^{(0)}(t, r)] dr + \frac{1}{2} \int_{t_1}^{t_2} J(t' - r) [\phi_i^{(0)}(t, r) + \phi_i^{(2)}(t, r)] dr$$

(28)

According to the principle of superposition, the potentials in Eq. (13) (corresponding to the case of an infinite 2D medium without any hole subject to the initial stress state) are equal to the sum of the ones corresponding to Cases (3) and (4):

$$\phi_i^{(0)}(t_1', t') = \phi_i^{(2)}(t_1', t') + \phi_i^{(2-0)}(t_1', t')$$

$$\psi_i^{(0)}(t_1', t') = \psi_i^{(2)}(t_1', t') + \psi_i^{(2-0)}(t_1', t')$$

(29)

Analogously, the potentials corresponding to Case (1) can be obtained as the sum of the potentials of Cases (3) and (5):

$$\phi_i^{(1)}(t_1', t') = \phi_i^{(2)}(t_1', t') + \phi_i^{(2-1)}(t_1', t')$$

$$\psi_i^{(1)}(t_1', t') = \psi_i^{(2)}(t_1', t') + \psi_i^{(2-1)}(t_1', t')$$

(30)

Substituting Eqs. (29) and (30) into (28), and rearranging:

$$u_i'(t_1', t') = L_{1-0}(t_1', t') + L_{1-1}(t_1', t') + L_{2-2}(t_1', t')$$

(31)

with the expressions for $L_{1-0}$ and $L_{1-1}$ being available from Eqs. (18) and (19):

$$L_{2-2}(t_1', t') = \frac{1}{2} \int_{t_1}^{t_2} I(t' - r) [-\psi_i^{(2-1)}(t, r)] dr$$

$$= \frac{1}{2} \int_{t_1}^{t_2} H(t' - r) \left[ \frac{\partial [\psi_i^{(2-1)}(t, r)]}{\partial t_1} - \psi_i^{(2-1)}(t, r) \right] dr$$

(32)

According to Eq. (30), the expressions for $\phi_i^{(2-1)}$ and $\psi_i^{(2-1)}$ can be obtained as:

$$\phi_i^{(2-1)}(t_1', t') = \phi_i^{(1)}(t_1', t') - \phi_i^{(2)}(t_1', t') = -\frac{p_0 R_i'(t_1') R_i(t_1')}{\zeta_1} - \psi_i^{(2)}(t_1', t')$$

$$\psi_i^{(2-1)}(t_1', t') = \psi_i^{(1)}(t_1', t') - \psi_i^{(2)}(t_1', t') = -\frac{p_0 R_i'(t_1') R_i(t_1')}{\zeta_1} - \psi_i^{(2)}(t_1', t')$$

(33)

where $\phi_i^{(2)}$ and $\psi_i^{(2)}$ are the potentials for the twin tunnels. Using Schwarz alternating method and Muskhelishvili’s complex variable function techniques, the two potentials were found following the method expounded in Ref. 26 as:

$$\phi_i^{(2)}(t_1', t') = \frac{1}{2} \left[ \frac{p_0 R_i'(t_1') R_i(t_1')}{\zeta_1 d} - \sum_{k=1}^{n} (D_{i+k-1}) \right]$$

$$\psi_i^{(2)}(t_1', t') = -\frac{p_0 R_i'(t_1') R_i(t_1')}{\zeta_1 d} + \frac{R_i'(t_1')}{\zeta_1 d} \sum_{k=1}^{n} (D_{i+k-1})$$

(34)

(35)

where $d$ is the distance between the two centres of the tunnels (see Fig. 1a), $D_i$ and $D_{i+k}$, with $k=1$ to $n$ being the number of terms, are the complex coefficients of the Laurent series used to approximate the non-zero redundant surface tractions acting on the tunnel boundary that are progressively eliminated in Schwarz alternating method. The values of $D_i$ and $D_{i+k}$ for the three cases considered in the next section (Section 5) to validate the proposed analytical solution by comparing displacements and stresses calculated from the analytical solution with displacements and stresses calculated from Finite Element are provided in Table 3.

If the advancements of Tunnel 1 and 2 are both accounted for, the tractions $\sigma_i^{(2-1)}$ and $\sigma_i^{(2-1)}$ on the boundary of Tunnel 2 take the following expressions:

$$\sigma_i^{(2-1)} = \lambda_i(t) \lambda_i(t) \sigma_i^{(2-1)}$$

$$\sigma_i^{(2-1)} = \lambda_i(t) \lambda_i(t) \sigma_i^{(2-1)}$$

(36)

and the corresponding potentials become:

$$\psi_i^{(2-1)}(t_1', t') = \lambda_i(t) \lambda_i(t) \psi_i^{(2-1)}(t_1', t')$$

$$\psi_i^{(2-1)}(t_1', t') = \lambda_i(t) \lambda_i(t) \psi_i^{(2-1)}(t_1', t')$$

(37)

with $\lambda_i$ and $\lambda_i$ being the dimensionless parameters accounting for the advancement of Tunnels 1 and 2 respectively.

Subtracting the displacements occurred prior to the excavation of Tunnel 1 from the total displacements in this time period (see Eq. (31)), the component of displacements considering the longitudinal advancements of the two tunnels occurred after the excavation of Tunnel 1 are as follows:

<table>
<thead>
<tr>
<th>$D_{i+k}$</th>
<th>$D_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real part</td>
<td>Imaginary part</td>
</tr>
<tr>
<td>Real part</td>
<td>Imaginary part</td>
</tr>
<tr>
<td>Real part</td>
<td>Imaginary part</td>
</tr>
<tr>
<td>Real part</td>
<td>Imaginary part</td>
</tr>
</tbody>
</table>

| Case (1) | 1.053 | 0.0 | 0.217 | 0.0 | 0.045 | 0.0 | 0.001 | 0.0 | 0.004 | 0.0 |
| Case (2) | 9.487 | 0.0 | 4.266 | 0.0 | 1.969 | 0.0 | 0.026 | 0.0 | 0.001 | 0.0 |
| Case (3) | 37.961 | 0.0 | 16.958 | 0.0 | 7.770 | 0.0 | 0.123 | 0.0 | 0.004 | 0.0 |

Table 3

Values of $D_{i+k}$ in Eqs. (34) and (35) for the three cases employed for comparison with FEM analysis.
\( \Delta u^*_{1}(t_i, t') + i \Delta u^*_{2}(t_i, t') = [u^*_{1}(t_i, t') - u^*_{1}(t_i, t'_0)] \\
\quad + i[u^*_{2}(t_i, t') - u^*_{2}(t_i, t'_0)] \right) \\
= L_{1-\Delta}(t_i, t'_0) + L_{2-\Delta}(t_i, t') \\
\right) \\
(38) \\
\right)

with \( L_{1-\Delta} \) being given in Eq. (24). \( L_{2-\Delta}^{(1)}(t_i, t'_0) \) can be obtained by replacing the potentials \( \phi^{(1-0)}_{1} \) and \( \psi^{(1-0)}_{1} \) in Eq. (19) with \( \phi^{(2-1)}_{1} \) and \( \psi^{(2-1)}_{1} \) in Eq. (32) with \( \phi^{(2-1)}_{1} \) and \( \psi^{(2-1)}_{1} \). Subtracting the displacements occurred prior to the excavation of Tunnel 2 from the total displacements taken place in this time period, the displacements due to the excavation of Tunnel 2 can be obtained:

\[ \Delta u_{1}(t_i, t') + i \Delta u_{2}(t_i, t') = [u^*_{1}(t_i, t') - u^*_{1}(t_i, t'_0)] \\\n\quad + i[u^*_{2}(t_i, t') - u^*_{2}(t_i, t'_0)] \right) \\
= L_{2-\Delta}(t_i, t'_0) + L_{2-\Delta'}(t_i, t') \\
\right) \\
(39) \\
\right)

where:

\[ L_{2-\Delta}(t_i, t'_0) = L_{1-\Delta}(t_i, t'_0) - L_{1-\Delta}(t_i, t'_0) \]

\[ + \frac{1}{2} \int_{t_i}^{t'_0} H(t' - t) \lambda_1(t) \left[ \psi^{(1-0)}_{1}(t'_0, t) - \psi^{(1-0)}_{1}(t_i, t) \right] dt \]

\[ - \frac{1}{2} \int_{t_i}^{t'_0} H(t' - t) \lambda_2(t) \left[ \phi^{(2-1)}_{1}(t'_0, t) - \phi^{(2-1)}_{1}(t_i, t) \right] dt \]

(40)

and

\[ L_{2-\Delta'}^{(2)}(t_i, t'_0) = L_{2-\Delta'}^{(1)}(t_i, t'_0) - L_{2-\Delta'}^{(1)}(t_i, t'_0) \]

\[ + \frac{1}{2} \int_{t_i}^{t'_0} H(t' - t) \lambda_1(t) \left[ \psi^{(1-0)}_{1}(t'_0, t) - \psi^{(1-0)}_{1}(t_i, t) \right] dt \]

\[ - \frac{1}{2} \int_{t_i}^{t'_0} H(t' - t) \lambda_2(t) \left[ \phi^{(2-1)}_{1}(t'_0, t) - \phi^{(2-1)}_{1}(t_i, t) \right] dt \]

(41)

In case of the generalized Kelvin, Kelvin and the Poynting-Thomson viscoelastic models, it can be demonstrated that \( L_{2-\Delta'} \) is zero. Introducing the new reference time \( t = t' - t'_0 \) (hence \( t'_2 = t'_2 - t'_0 \)), the calculated excavation induced displacements (see Eqs. (23), (38) and (39)) can be expressed as: 

\[ \Delta u_{1}(t_i, t') + i \Delta u_{2}(t_i, t') = [u^*_{1}(t_i, t') - u^*_{1}(t_i, t'_0)] \\
\quad + i[u^*_{2}(t_i, t') - u^*_{2}(t_i, t'_0)] \right) \\
= L_{2-\Delta}(t_i, t'_0) + L_{2-\Delta'}(t_i, t') \\
\right) \\
(39) \\
\right)

Fig. 2. Boundary conditions and external loading applied in the numerical simulations.

Fig. 3. FEM meshes adopted. Partito VI are sequentially excavated.
\[ \Delta (t) + \Delta (t') = (t) + (t) + (t) \]

with \( t \geq t_0 \) (or \( t' \geq t' \))

\[ \Delta_t u_t^e (t_0, t) + i \Delta_{x_t} u_t^e (t_0, t) = L_{t-x} (t_0, t) + L_{t-x_0}^e (t_0, t) + L_{t-x_0}^e (t_0, t) \]

\[ \Delta_{x_t} u_t^e (t_0, t) + i \Delta_{x_t} u_t^e (t_0, t) = L_{x_t-x_0} (t_0, t) + L_{x_t-x_0}^e (t_0, t) + L_{x_t-x_0}^e (t_0, t) \]

with \( t \geq t_0 \) (or \( t' \geq t' \))

and in the new reference time as:

\[ L_{x_t-x_0} (t_0, t) = -\frac{1}{2} \int_{t_0}^{t} I(t - \tau) \Delta_{x_t} (t) \Delta (t) \, d\tau + \frac{1}{2} \int_{0}^{t} I(t_0 - \tau) \Delta (t_0) \Delta (t) \, d\tau \]

\[ + \frac{1}{2} \int_{0}^{t} H(t - \tau) \Delta (t) \Delta (t) \, d\tau \]

\[ - \frac{1}{2} \int_{0}^{t} H(t_0 - \tau) \Delta (t_0) \Delta (t) \, d\tau \]

\[ - \frac{1}{2} \int_{0}^{t} H(t_0 - \tau) \Delta (t_0) \Delta (t) \, d\tau \]

and in the new reference time as:

\[ L_{x_t-x_0}^e (t_0, t) = -\frac{1}{2} \int_{t_0}^{t} I(t - \tau) \Delta_{x_t} (t) \Delta (t) \, d\tau + \frac{1}{2} \int_{0}^{t} I(t_0 - \tau) \Delta (t_0) \Delta (t) \, d\tau \]

\[ + \frac{1}{2} \int_{0}^{t} H(t - \tau) \Delta (t) \Delta (t) \, d\tau \]

\[ - \frac{1}{2} \int_{0}^{t} H(t_0 - \tau) \Delta (t_0) \Delta (t) \, d\tau \]

The additional stresses induced by the excavation of Tunnel 2 are as follows:

\[ \Delta \sigma_{x_t}^e (t_0, t) = -\arg \left( \int \Delta_{x_t} (t) \sigma_{y_t} (t) \, d\tau \right) \]

\[ \Delta \sigma_{y_t}^e (t_0, t) = -\arg \left( \int \Delta_{y_t} (t) \sigma_{x_t} (t) \, d\tau \right) \]

\[ \Delta \sigma_{z_t}^e (t_0, t) = -\arg \left( \int \Delta_{z_t} (t) \sigma_{z_t} (t) \, d\tau \right) \]

\[ \Delta \sigma_{xy}^e (t_0, t) = -\arg \left( \int \Delta_{xy} (t) \sigma_{xy} (t) \, d\tau \right) \]

\[ \Delta \sigma_{xz}^e (t_0, t) = -\arg \left( \int \Delta_{xz} (t) \sigma_{xz} (t) \, d\tau \right) \]

\[ \Delta \sigma_{yz}^e (t_0, t) = -\arg \left( \int \Delta_{yz} (t) \sigma_{yz} (t) \, d\tau \right) \]

\[ \Delta \sigma_{xx}^e (t_0, t) = -\arg \left( \int \Delta_{xx} (t) \sigma_{xx} (t) \, d\tau \right) \]

\[ \Delta \sigma_{yy}^e (t_0, t) = -\arg \left( \int \Delta_{yy} (t) \sigma_{yy} (t) \, d\tau \right) \]

\[ \Delta \sigma_{zz}^e (t_0, t) = -\arg \left( \int \Delta_{zz} (t) \sigma_{zz} (t) \, d\tau \right) \]
The additional stresses occurred after excavation of Tunnel 1 are provided by the summation of Eqs. (46) and (27). Superposing the hydrostatic initial stresses to them yields the total stresses of the ground.

4.3. Applicability of the obtained solution

With regard to rock rheology, investigated which viscoelastic model and which range of values for the relative parameters are best suited for various rock types. For weak, soft or highly jointed rock masses and/or subjected to high stresses, which are prone to excavation induced continuous viscous flows, the Maxwell or Burgers viscoelastic model (see Table 1) are suitable to simulate their rheology since they account for both primary and secondary rock creep. Instead for rock of good mechanical properties or subject to low stresses, limited viscosity is present. For this type of rocks, Kelvin, generalized Kelvin or Poynting-Thomson viscoelastic models are commonly employed.

Also more complicated viscoelastic models are listed. Back analysis is often used to identify the most suited rheological model for a specific rock and its constitutive parameters. In rheological model and constitutive parameters were determined from in-situ measurements during the excavation whereas in they were determined employing genetic algorithms applied on measurements from creep tests.

With regard to consideration of 3D effects for cross sections located near the tunnel face, expressions for the advancement parameters, and are not available in the literature since tunnel face effects have been investigated only in case of single tunnels. The variation of and with time is more complicated than in the case of single tunnel, and the longitudinal displacement profiles for two tunnels advancement is not available in current references to determine the two parameters. Therefore, in the following of the paper and is assumed. This means that we consider only cross-sections located at a distance from both tunnel faces such that three dimensional effects are not felt.
5. Comparison between analytical solution and FEM predictions

Here, in order to validate the analytical solution presented, a comparison is provided between the displacements and stresses calculated according to the determined analytical solution and numerical FEM simulations carried out using the code ANSYS (version 11.0, employing the so called module 'structure mechanics'). To maintain consistency with the derivation of the analytical solution, all the FEM analyses were carried out under plane-strain conditions and assuming small displacements. Note that the sign convention adopted herein is as follows: compressive stresses and strains are positive while tensile stresses and strains are negative; displacements along the direction of the axes of the coordinate system are negative.

Let us consider an infinite viscoelastic 2D medium with two circular holes subject to a hydrostatic stress \( p_0 = 20 \text{MPa} \) (representing the construction of twin tunnels at depth of approximately 800 m). The final radii of hole 1 and hole 2, \( R_{1f}^m \) and \( R_{2f}^m \) in Eq. (2), are 3 and 6 m, respectively. The two holes are excavated according to the following:

\[
R_{t_{mt}}(t) = \begin{cases} 
1 & 0 \leq t < 2 \\
2 & 2 \leq t < 4 \\
3 & t \geq 4 
\end{cases} \quad \text{(Unit: 'Day' for time.)} \tag{47}
\]

According to the experimental data available\(^{39}\), the following values for the constitutive parameters of the generalized Kelvin model (see Table 1) were adopted: \( G_M = 2000 \text{ MPa} \), \( G_K = 1000 \text{ MPa} \), \( \eta_K = 10000 \text{ MPa-day} \) and \( K(t) = \infty \) (volume incompressible).

To exploit the symmetry of the problem with respect to \( Ox_1 \)-axis, only half of the domain has been modelled. A rectangular region 400 m long and 200 m wide was employed in the FEM model. In Fig. 2 the geometry of the numerical domain adopted together with the boundary conditions employed are provided. In Fig. 3, the mesh in the vicinity of the holes together with the parts to be sequentially excavated are presented. In the numerical simulations, the initial stresses are first applied on the rectangular domain without any hole to apply to the medium the initial stresses prior to any excavation. Since the material is assumed incompressible and the medium is subjected to hydrostatic stresses, the displacements before tunnel excavation are zero. Part I-III (see Fig. 3) are excavated instantaneously at \( t=0 \) day, \( t=2 \)th day and \( t=4 \)th day respectively. The excavation of Tunnel 2 starts on the 2th day, and part IV-VI (see Fig. 3) are excavated instantaneously at \( t=2 \)th day, 4th day and \( t=6 \)th day respectively. Consequently, the induced displacements and stresses can be obtained by subtracting the stresses and displacements present before the excavation takes place from the total ones. In the FEM analysis, the elements were deleted from the mesh at the corresponding excavation times. To delete elements of the mesh, the stiffness of the elements to be deleted is set to zero (the stiffness matrix of the elements was multiplied by \( 1.0^{-6} \)).

The comparison of the time dependent displacements predicted by the analytical solution with the ones predicted by the FEM simulations is illustrated in Fig. 4. Let us consider points A, B and C (see Fig. 3). In Fig. 4 the displacements of points A and B occurred after the excavation of Tunnel 1 and the displacement of point C occurred after the excavation of Tunnel 2 are plotted over time. The numerical results exhibit a close agreement with the analytical solution. An analogous comparison in terms of stresses is shown in Fig. 5. A good match between numerical results and analytical predictions is apparent for all the stresses considered.

Fig. 7. Distribution of the normalized displacements and stresses in the pillar between the two tunnels as a function of the distance from the centre of Tunnel 1, \( x_1 \), calculated for various pillar widths: (a) distribution of displacements at time \( t = 0.5 T_5 + 2 \) corresponding to the end of Tunnel 2 excavation; (b) distribution of displacements at time \( t = 10 T_5 \) (long term displacements); (c) and (d) distributions of normal stresses along the \( x \) and \( y \) direction respectively after the excavation of both tunnels has been completed.
6. Parametric investigation

A parametric investigation on the effect of various parameters ruling the mutual influence between the two twin tunnels was carried out. The parametric investigation is here illustrated for the case of a rock mass whose rheology is described by the generalized Kelvin viscoelastic model. For sake of generality, all the variables considered in the analysis, e.g. displacements, stresses and time, will be provided in normalized form. For a single circular tunnel of radius $R_{\text{fin}}$ excavated in an infinite medium and subject to a hydrostatic initial stress $p_0$ and shear modulus $G = G_{\text{K}} + G_{\text{M}}$ (permanent shear modulus of the generalized Kelvin model), the induced final radial displacement at the inner boundary of the tunnel is:

$$u_R \Delta = 2\frac{r_{\text{fin}}}{R_{\text{fin}}} p_0 G \Delta$$

(48)

The excavated induced displacements will be normalized by $u_R \Delta$, the stresses by $p_0$, tunnel radii and tunnel spacing by $R_{\text{fin}}$ and the generic time $t$ by $T_K$ with $T_K = \frac{G}{G_{\text{M}}}$ denoting the retardation time. In the following analysis, the displacements at the boundary of Tunnel 1 are the component of displacements taking place after the excavation of Tunnel 1 ($\Delta u_1$), and those at the boundary of Tunnel 2 are the component of displacements taking place after the excavation of Tunnel 2 ($\Delta u_2$), here simply called “displacements”. Instead, all the stresses will be current stresses (no stress increments).

6.1. Influence of tunnel spacing

The influence of the spacing between the two tunnels on displacements and stresses is here investigated. For sake of simplicity, it is assumed that the final radii of the two tunnels are equal, i.e. $R_{\text{fin}} = R_{\text{fin}}^1$ and that Tunnel 1 is excavated instantaneously at $t=0$, while the excavation of Tunnel 2 starts at $t = t_2$. The radius of Tunnel 2 grows from zero to its final value in a step-like fashion:

$$R_{\text{t}} = \begin{cases} 
0 & 0 \leq t < t_2 \\
0.5 & 0.5 \leq t < t_2 + 2 \\
1 & t \geq t_2 + 2 
\end{cases} \text{(Unit for time: Day)}$$

(49)

The displacements of Tunnel 2 are analyzed at two different times: $t = 0.5T_K + 2$ (Day), when the excavation of Tunnel 2 is complete and $t = 10T_K$ (Day), when displacements no longer increase. Defined that the tunnel spacing is $w = d - R_{\text{fin}}^1 - R_{\text{fin}}^2$, Fig. 6(a) and (b) present the variation of radial displacements along the boundary of Tunnel 1 at $\theta = 0^\circ$ and that along the boundary of Tunnel 2 at $\theta = 180^\circ$ plotted versus tunnel spacing. In Fig. 6(c) instead, the hoop stresses acting on the tunnel boundaries are plotted. From the plots it emerges that the radial displacements at the boundaries of the tunnels grow larger with tunnel spacing increasing, whereas the hoop stresses become smaller. However, the displacements along the boundary of Tunnel 2 are only marginally influenced by the spacing between the two tunnels, i.e. they are similar to the values exhibited in the case of a single tunnel being excavated (Tunnel 2 being excavated).

A way to quantitatively assess the mutual influence of one tunnel on the other is by comparing the fields of stresses and displacements generated in case of a single tunnel with the fields generated in the presence of the two twin tunnels. To this end, the radial displacements and the hoop stresses along the boundaries of the tunnels were selected as the variables to be used in the comparison (the radial stresses are obviously nil and the circumferential displacements are of less inter-

Fig. 8. Normalized displacements versus time along the final boundary of Tunnel 1 for various excavation times of Tunnel 2 (generalized Kelvin model with $G_K/G_M = 0.5$); (a) normal displacement at point $D_1$ ($\theta = 0^\circ$); (b) normal displacement at point $F_1$ ($\theta = 180^\circ$); (c) and (d) normal and tangential displacements respectively at point $E_1$ ($\theta = 90^\circ$).
For sake of comparison, it is convenient to define a function, \( C_u(t) \), expressing the dimensionless radial displacement occurring at the tunnel boundary for the case of a single tunnel with radius \( R = R_1^{\text{in}} \) subject to a hydrostatic far-field stress:

\[
C_u(t) = \frac{\Delta u_r^e(R, t)}{\rho_0}\frac{1}{R_1^{\text{in}}}
\]

where \( \Delta u_r^e(R, t) \) is the excavation induced convergence. \( C_u(t) \) expresses the dimensionless hoop stress along the boundary of the single tunnel:

\[
C_\sigma(t) = \frac{\sigma_{\theta 0}^e(R, t)}{\sigma_{\theta 0}^0}
\]

where \( \sigma_{\theta 0}^e(R, t) \) is the hoop stress at tunnel boundary. The following dimensionless ratios are here proposed as quantitative measures of how strongly the presence of a second twin tunnel modifies the convergence and stresses that would occur in case the tunnel was single:

\[
f_{zt}^{(1)}(\zeta_1, t) = \frac{\Delta u_r^e(\zeta_1, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(2)}(\zeta_2, t) = \frac{\Delta u_r^e(\zeta_2, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(3)}(\zeta_3, t) = \frac{\sigma_{\theta 0}^e(\zeta_3, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(4)}(\zeta_4, t) = \frac{\Delta u_r^e(\zeta_4, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(5)}(\zeta_5, t) = \frac{\sigma_{\theta 0}^e(\zeta_5, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(6)}(\zeta_6, t) = \frac{\Delta u_r^e(\zeta_6, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(7)}(\zeta_7, t) = \frac{\sigma_{\theta 0}^e(\zeta_7, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(8)}(\zeta_8, t) = \frac{\Delta u_r^e(\zeta_8, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(9)}(\zeta_9, t) = \frac{\sigma_{\theta 0}^e(\zeta_9, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

For sake of comparison, it is convenient to define a function, \( C_u(t) \), expressing the dimensionless radial displacement occurring at the tunnel boundary for the case of a single tunnel with radius \( R = R_1^{\text{in}} \) subject to a hydrostatic far-field stress:

\[
C_u(t) = \frac{\Delta u_r^e(R, t)}{\rho_0}\frac{1}{R_1^{\text{in}}}
\]

where \( \Delta u_r^e(R, t) \) is the excavation induced convergence. \( C_u(t) \) expresses the dimensionless hoop stress along the boundary of the single tunnel:

\[
C_\sigma(t) = \frac{\sigma_{\theta 0}^e(R, t)}{\sigma_{\theta 0}^0}
\]

where \( \sigma_{\theta 0}^e(R, t) \) is the hoop stress at tunnel boundary. The following dimensionless ratios are here proposed as quantitative measures of how strongly the presence of a second twin tunnel modifies the convergence and stresses that would occur in case the tunnel was single:

\[
f_{zt}^{(1)}(\zeta_1, t) = \frac{\Delta u_r^e(\zeta_1, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(2)}(\zeta_2, t) = \frac{\Delta u_r^e(\zeta_2, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(3)}(\zeta_3, t) = \frac{\sigma_{\theta 0}^e(\zeta_3, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(4)}(\zeta_4, t) = \frac{\Delta u_r^e(\zeta_4, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(5)}(\zeta_5, t) = \frac{\sigma_{\theta 0}^e(\zeta_5, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(6)}(\zeta_6, t) = \frac{\Delta u_r^e(\zeta_6, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(7)}(\zeta_7, t) = \frac{\sigma_{\theta 0}^e(\zeta_7, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

\[
f_{zt}^{(8)}(\zeta_8, t) = \frac{\Delta u_r^e(\zeta_8, t)/\Delta u_r^n - C_u(t)}{C_u(t)}
\]

\[
f_{zt}^{(9)}(\zeta_9, t) = \frac{\sigma_{\theta 0}^e(\zeta_9, t)/\sigma_{\theta 0}^0 - C_\sigma(t)}{C_\sigma(t)}
\]

In Table 4, some values for the introduced dimensionless ratios are provided for various pillar widths. It emerges that the pillar width affects displacements significantly more than stresses especially in the long term. The stress influence ratio is lower than 10% for \( w=R_1^{\text{in}} \) and becomes lower than 5% for \( w=6R_1^{\text{in}} \). This suggests that the pillar width threshold for which the twin tunnel interaction may be ignored could be taken as \( w_R^{\text{in}} \geq 6R_1^{\text{in}} \). However, the maximum influence ratio for the long-term displacements of Tunnel 1 is 22% and 16% for pillar width of \( R_1^{\text{in}} \) and \( 6R_1^{\text{in}} \), respectively. This shows a strong interaction. The influence ratio of Tunnel 1 displacements remains at about 10% for \( w=R_1^{\text{in}} \).

The normalized components of horizontal displacements along the line \( O_1O_2 \) are plotted in Fig. 7(a) and (b) for various pillar widths. It can be noted that the displacements at the points close to tunnel boundaries change more significantly with \( x_1 \). As expected, when tunnel spacing is larger, the variation of displacement is smaller and the displacements at the points near the Tunnel 2 boundary become larger.

Point 1 at the stable time is basically located at the middle of the pillar, which means that the displacements are symmetrically distributed.

Fig. 7(c) and (d) show the distribution of stresses in the pillar between the two twin tunnels. Stresses in areas close to the boundaries of two tunnels are subject to a greater change with \( x_1 \). For a tunnel pillar larger than \( 6R_1^{\text{in}} \), the stresses on the tunnel boundaries, as well as in the middle of the pillar are very little affected. This means that the interaction between tunnels from a static point of view is marginal and can be

![Fig. 9. Normalized displacements versus time at points along the final boundary of Tunnel 2 for various excavation times of Tunnel 2 (generalized Kelvin model with \( G_K/G_M=0.5 \)); (a) normal displacement at a point \( F_2 \) (\( \theta_2=0^\circ \)); (b) normal displacement at a point \( D_2 \) (\( \theta_2=180^\circ \)); (c) and (d) normal and tangential displacements at a point \( E_2 \) (\( \theta_2=90^\circ \)), respectively.](image-url)
In this section, the influence of the starting time of excavation of the second tunnel on displacements

In this section, twin tunnels with different radii are examined. Four cases are considered: Case (1) \( R_{2}^{m} = 0.5R_{1}^{m} \); Case (2) \( R_{2}^{m} = R_{1}^{m} \); Case (3) \( R_{2}^{m} = 1.5R_{1}^{m} \); Case (4) \( R_{2}^{m} = 2.5R_{1}^{m} \). In all the cases, the same distance between tunnels, \( w=2R_{1}^{m} \), is assumed. Also we assume that Tunnels 1 and 2 are instantaneously excavated at time \( t_{0} = 0 \) and \( t_{2} = 0.5T_{K} \) respectively. In Figs. 10 and 11, the distribution of displacements and stresses along the boundaries of the tunnels are plotted at two significant times: when the excavation of Tunnel 2 is completed, \( t = 0.5T_{K} \), and when displacements no longer increase, \( t = 10T_{K} \). For obvious reasons of symmetry, stresses and displacements along the tunnel boundaries are plotted for \( \theta_{1} = \theta_{2} \in [0^\circ, 180^\circ] \) only.

The distributions of radial and circumferential displacements along the boundary of Tunnel 1 at time \( t = 0.5T_{K} \) and at \( t = 10T_{K} \) are plotted in Figs. 10(a,b) and in Figs. 10(c,d), respectively. The figures show that the larger the radius of Tunnel 2 is, the more the radial displacements are the same for all the four cases considered. Also it emerges that the curves of radial displacements versus \( \theta_{1} \) maintain the same shape over time. Finally Fig. 10(b) and (d) show that the circumferential displacements exhibit a peak for \( 50^\circ < \theta_{1} < 60^\circ \) and that the larger the radius of Tunnel 2 is, the larger...
From Fig. 10, it emerges that the size of Tunnel 2 significantly influences the distribution and value of displacement of Tunnel 1, and the maximum difference of radial displacement among these cases is around $0.6 \Delta u^\infty$.

The distributions of the displacements along the boundary of Tunnel 2 are plotted in Fig. 11. From the figure the following emerges: first, little variation of the radial displacement with the radial direction, $\theta_2$, is exhibited; second, the radial displacement is almost the same as the one of single hole problem. For example, displacement in Fig. 11(c) for the case with $R_2^\infty = R_1^\infty$ is approximately equal to $\Delta u^\infty$ which is the long-term radial displacement experienced by a single circular tunnel ($R = R_1^\infty$); third, looking at circumferential displacements along $\theta_2$ two peaks are visible. However, comparing with radial displacement, circumferential displacements are quite small. These three characteristics show that the field of displacements originated by the construction of Tunnel 2 is very similar to the case of a single tunnel, i.e. the convergence experienced by Tunnel 2 is affected only by the presence of Tunnel 2.

The distribution of hoop stresses along the boundary of Tunnel 2 are plotted in Fig. 12(a) and (b) for various sizes of Tunnel 2 and for different times. The hoop stress of Tunnel 1 changes significantly with angle $\theta_1$ when the radius of Tunnel 2 is large. The larger the Tunnel 2 is, the larger the maximum hoop stress of Tunnel 1 turns out to be and the smaller the minimum hoop stress is. The stress concentration factor is about 3.14 and 2.45 for Tunnel 1 and 2, respectively, when the radius of Tunnel 2 equals to 2.5 times that of Tunnel 1, while concentration factor is 2 for single tunnel.
tunnel problem. Fig. 12(b) emerges that the maximum hoop stress around Tunnel 2 appears at $\theta_2 = 180^\circ$. The bigger the Tunnel 2, the larger the minimum hoop stress is. However, the maximum hoop stress of Tunnel 2 is almost insensitive to the size of Tunnel 2. The obtained field of hoop stresses for twin-tunnel problem is greatly different from that of the single tunnel problem.

In Fig. 13, the distributions of vertical and horizontal normal stresses along the horizontal distance between the two tunnels ($D_1-D_2$) are plotted for various ratios of the tunnel radii. With regard to the distribution of horizontal normal stresses, from Fig. 13(a) it emerges that $\sigma_{y}^\prime$ (minor principle stress), increases along the horizontal direction from zero to a peak value taking place not in the middle of the pillar but at a point which is closer to the smaller tunnel. With regard to the distribution of vertical normal stresses, in Fig. 13(b) we observe that $\sigma_{y}^\prime$ (major principle stress), takes different values at the boundary of Tunnel 1 (point $D_1$) depending on the relative size of the two tunnels. Moving along the horizontal direction, $\sigma_{y}^\prime$ decreases from the value taken at the boundary of tunnel 1 until a minimum and then increases until a maximum value is reached at the boundary of tunnel 2. This maximum value is independent of the relative size of the two tunnels. When the radii of the two tunnels are equal ($R_1^m = R_2^m$), obviously the stresses show symmetrical distributions.

7. Conclusions

A general analytical solution for rock stresses and displacements accounting for rock rheology and sequential excavation has been derived, for the first time for closely located parallel circular twin tunnels excavated at large depth.

To verify the analytical solution obtained, a FEM analysis for an example case was run. A good agreement between analytical solutions and FEM results was exhibited. Then a parametric study was performed to investigate the influence of tunnel spacing, start time of excavation, and the relative size of the two tunnels. Also, dimensionless (here called influence ratios) coefficients (here called influence ratios) were proposed to quantify how strongly the presence of a second twin tunnel modifies the fields of displacements and stresses that would occur in case the tunnel were single (axisymmetric case). From the parametric analysis, the following conclusions can be drawn:

- Tunnel spacing affects displacements significantly more than stresses. For instance, for a tunnel spacing $w \geq 6R$, the influence ratio for stresses is less than 5% but the influence ratio for long-term displacements is 16%.
- The later the excavation of the second tunnel starts, the less displacements take place at the boundary of the first tunnel.
- The larger the radius of the second tunnel is, the more the displacement experienced by the first tunnel varies spatially. However, the final displacement fields around the two tunnels are qualitatively very similar.
- The field of displacements taking place along the boundary of the second tunnel after its excavation is completed, is very similar to that of a single tunnel.
- In the future related research, the applicability of the solutions in real projects, as well as the determination of advancement parameters for twin tunnel will be carried out.

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