**Numerical investigation of aspect ratio influences on Rayleigh-Bénard convection of Bingham fluids in vertical cylindrical annuli**

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<td>Manuscript ID</td>
<td>HFF-03-2018-0101</td>
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<tr>
<td>Manuscript Type:</td>
<td>Research Article</td>
</tr>
<tr>
<td>Keywords:</td>
<td>Rayleigh-Bénard convection, Non-Newtonian fluid, Bingham model, Vertical cylindrical annuli, Constant wall temperature, Constant wall heat flux</td>
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ABSTRACT

Numerical simulations have been conducted to investigate steady-state laminar Rayleigh-Bénard convection of yield stress fluids obeying Bingham model in rectangular cross-sectional cylindrical annular enclosures. In this investigation, axisymmetric simulations have been carried out for nominal Rayleigh number range \( Ra = 10^3 - 10^5 \), aspect ratio range \( AR = 0.25 - 4 \) (i.e. \( AR = H/L \) where \( H \) is the enclosure height and \( L \) is the difference between outer and inner radii) and normalised inner radius range \( \eta_i/L = 0 - 16 \) (where \( \eta_i \) is internal cylinder radius) for a nominal representative Prandtl number \( Pr = 500 \). Both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions have been considered for differentially heated horizontal walls to analyse the effects of wall boundary condition. It is found that the convective transport strengthens (weakens) with an increase in \( Ra(AR) \) for both Newtonian (i.e. \( Bn = 0 \)) and Bingham fluids, regardless of the boundary conditions. Moreover, the strength of convection is stronger in the CWT configuration than that is for CWHF boundary condition due to higher temperature difference between horizontal walls for both Newtonian (i.e. \( Bn = 0 \)) and Bingham fluids. The mean Nusselt number \( \bar{Nu}_{cy} \) does not show a monotonic increase with increasing \( Ra \) for \( AR \leq 1 \) and \( \eta_i/L \leq 4 \) because of the change in flow pattern (i.e. number of convection rolls/cells) in the CWT boundary condition, whereas a monotonic increase of \( \bar{Nu}_{cy} \) with increasing \( Ra \) is obtained for the CWHF configuration. Additionally, \( \bar{Nu}_{cy} \) increases with increasing \( \eta_i/L \) and asymptotically approaches the corresponding value obtained for rectangular enclosures \( (\eta_i \to \infty) \) for both CWT and CWHF boundary conditions for large values of \( \eta_i/L \). It is also found that both the flow pattern and the mean Nusselt number \( \bar{Nu}_{cy} \) are dependent on the initial conditions for Bingham fluid cases since hysteresis is evident for \( AR \leq 1 \) for both CWT and CWHF boundary conditions. Finally, the numerical findings have been used to propose a correlation for \( \bar{Nu}_{cy} \) in the range of \( 0.25 \leq \eta_i/L \leq 16, 0.25 \leq AR \leq 2 \) and \( 5 \times 10^4 \leq Ra \leq 10^5 \) for the CWHF configuration.
NOMENCLATURE

\( AR \) [-] Aspect ratio \((H/L)\)

\( Bu \) [-] Bingham number

\( Bu_{\text{max}} \) [-] Bingham number at which or above the mean Nusselt number attains a value of unity

\( c_p \) \( [J/kgK] \) Specific heat at constant pressure

\( erel \) [-] Relative error

\( g \) \( [m/s^2] \) Gravitational acceleration

\( h \) \( [W/m^2K] \) Heat transfer coefficient

\( k \) \( [W/mK] \) Thermal conductivity

\( L \) \( [m] \) height of the enclosure and difference between inner and outer radius

\( m \) [-] Stress growth exponent

\( \overline{Nu} \) [-] Mean Nusselt number

\( \overline{Nu}_{cy} \) [-] Mean Nusselt number for cylindrical annular enclosure

\( P \) \( [Pa] \) Pressure

\( Pr \) [-] Prandtl number

\( q \) \( [W/m^2] \) Heat flux

\( r \) \( [m] \) Radius

\( r_g \) [-] Grid expansion ratio

\( r_i \) \( [m] \) Inner radius

\( r_o \) \( [m] \) Outer radius

\( Ra \) [-] Rayleigh number

\( T \) \( [K] \) Temperature

\( u_i \) \( [m/s] \) \( i^{th} \) velocity component

\( U, W \) [-] Dimensionless radial \((U = uL/\alpha)\) and vertical velocity \((W = wL/\alpha)\)

\( U_{\text{ref}} \) \( [m/s] \) Reference velocity scale

\( \mathcal{G} \) \( [m/s] \) Characteristic velocity

\( x_i \) \( [m] \) Coordinate in \( i^{th} \) direction

\( \alpha \) \( [m^2/s] \) Thermal diffusivity

\( \beta \) \( [1/K] \) Coefficient of thermal expansion

\( \delta_{\text{th}} \) \( [m] \) Velocity and thermal boundary-layer thickness

\( \theta \) [-] Dimensionless temperature

\( \mu \) \( [Ns/m^2] \) Dynamic viscosity

\( \nu \) \( [m^2/s] \) Kinematic viscosity

\( \rho \) \( [kg/m^3] \) Density

\( \tau_{ij} \) \( [Pa] \) Stress tensor (stress)

\( \phi \) [-] Azimuthal co-ordinate

\( \Psi \) [-] Dimensionless stream function

Subscripts

\( C \) Cold wall

\( cen \) Geometrical centre

\( cy \) Cylindrical Annular Enclosure

\( eff \) Effective value

\( H \) Hot wall

\( nom \) Nominal value

\( rec \) Rectangular

\( ref \) Reference value

Special characters

\( \Delta T \) \( [K] \) Temperature difference

\( \Delta_{\text{min,cell}} \) \( [m] \) Minimum cell distance
1. INTRODUCTION

Yield stress fluid is a special type of non-Newtonian fluid, which flows like fluids once critical stress (i.e. yield stress) is exceeded but acts like a solid below this stress. Materials such as mud-slurries, toothpaste, non-drip paint, foams, and mortars are common examples of yield stress fluids. Rayleigh-Bénard convection of yield stress fluids in enclosed spaces has wide applications in chemical and food processing, nuclear waste cooling, cryogenic storages. Therefore, a number of studies concentrated on Rayleigh-Bénard convection of yield stress fluids over the last decade (Zhang et al. 2006; Balmforth and Rust 2009; Vikhansky 2009; Vikhansky 2010; Turan et al. 2012; Darbouli et al. 2013; Hassan et al. 2013; Kebiche et al. 2014; Turan et al. 2014; Yigit et al. 2015a; Yigit et al. 2015b; Hassan et al. 2015; Yigit et al. 2016; Turan et al. 2017; Yigit and Chakraborty 2017b).

The main findings of the existing analyses on Rayleigh-Bénard convection of Bingham fluids are summarised in Table 1. It is shown in Table 1 that several studies focused on the critical condition for the onset of the flow for yield stress fluids (Zhang et al. 2006; Balmforth and Rust 2009; Vikhansky 2009; Vikhansky 2010; Darbouli et al. 2013; Kebiche et al. 2014; Turan et al. 2017), while the others analysed the heat transfer characteristics of yield stress fluids when the flow is well established (Vikhansky 2009; Turan et al. 2012; Hassan et al. 2013; Hassan et al. 2015; Yigit et al. 2015a; Yigit et al. 2015b; Yigit et al. 2016; Yigit and Chakraborty 2017b). Also, most of these analyses were conducted for Bingham fluids (the simplest form of yield stress fluids which shows a linear strain rate dependence of viscous stress). A weakening of natural convection between differentially heated vertical walls has been reported by analytical means (Yang and Ye 1965; Bayazitoglu et al. 2007) with increasing Bingham number (i.e. non-dimensional yield stress) due to the additional flow resistance arising from the yield stress. Additionally, conductive thermal transport has been shown to be dominant for large values of Bingham number using numerical means for the Rayleigh-Bénard convection since fluid flow practically stops under such conditions (Vikhansky 2009; Turan et al. 2012; Hassan et al. 2013; Hassan et al. 2015; Yigit et al. 2015a; Yigit et al. 2015b; Yigit et al. 2016; Yigit and Chakraborty 2017b).

Furthermore, all the analyses except for by Yigit et al. (2016) were carried out for rectangular enclosures and amongst them Yigit et al. (2015a) and Yigit and Chakraborty (2017b) indicated that the convective (diffusive) transport weakens (strengthens) with increasing aspect ratio (height: length) for Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures. Several previous analyses (Bejan 1978; 1980; Ganguli et al. 1980; Bejan et al. 1981; Turan et al. 2011, Turan et al. 2014) indicated that the aspect ratio plays a significant role in natural convection in enclosures with differentially heated vertical walls. It is worth noting that the flow physics in the differentially heated vertical sidewall configuration is fundamentally different from the Rayleigh-Bénard configuration. In the differentially heated vertical wall configuration, convection starts once a temperature differential is created between the vertical walls, whereas convection starts only when a critical Rayleigh number is surpassed in the Rayleigh-
Bénard configuration. Recent findings of Yigit et al. (2016) revealed that the convective thermal transport strengthens with increasing internal cylinder radius for laminar Rayleigh-Bénard convection of Bingham fluids in square cross-sectional cylindrical annular enclosures. Thus, laminar Rayleigh-Bénard convection behaviour is expected to be different in rectangular cross-sectional cylindrical annular enclosures than that in the rectangular enclosures due to the influences of wall curvature. However, the influence of aspect ratio (i.e. \( AR = H/L \) where \( H \) is the enclosure height and \( L \) is the difference between outer and inner radii) (Ganguli et al. 1980; Bejan et al. 1981, Turan et al. 2011, Turan et al. 2014) in vertical cylindrical annuli with differentially heated horizontal walls is yet to be analysed in existing literature. This gap is addressed here by numerically analysing natural convection of Bingham fluids in vertical cylindrical annuli with differentially heated horizontal walls for different values of nominal Rayleigh number (i.e. \( Ra = 10^3 \) – \( 10^5 \)), aspect ratio (i.e. \( AR = 0.25 \) – \( 4 \)) and normalised inner radius (i.e. \( r_i/L = 0 \) – \( 16 \), where \( r_i \) is the internal cylinder radius) for a representative nominal Prandtl number \( Pr = 500 \). Moreover, it was demonstrated in several previous analyses (Yigit et al. 2015a; Yigit and Chakraborty 2017b) that the boundary conditions of the differentially heated horizontal walls significantly affect the aspect ratio \( AR \) dependence of the mean Nusselt number \( \overline{Nu} \) in rectangular enclosures and thus, both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions have been considered for this analysis. In this respect, the main objectives of this analysis are:

1. To analyse the influences of \( AR \) on \( Ra \) and \( r_i/L \) dependences of the mean Nusselt number \( \overline{Nu} \) for natural convection of Bingham fluids in cylindrical annular spaces with differentially heated horizontal walls.
2. To provide physical explanations for the above influences using scaling arguments and numerical results.

2. MATHEMATICAL BACKGROUND

The strain rate dependence of viscous stresses for Bingham fluids (Barnes 1999) can be expressed as:

\[
\begin{align*}
\dot{\gamma} & = 0 \quad \text{for} \quad \tau \leq \tau_y, \\
\tau & = (\mu + \tau_y/\dot{\gamma})\dot{\gamma} \quad \text{for} \quad \tau > \tau_y,
\end{align*}
\]

where \( \dot{\gamma}_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i) \) are the components of the rate of strain tensor \( \dot{\gamma} \), \( \tau \) is the stress tensor, \( \tau_y \) is the yield stress, \( \mu \) is the plastic viscosity, \( \tau \) and \( \dot{\gamma} \) are the second invariants of the stress and the rate of strain tensors in a pure shear flow respectively, which are expressed as:

\[
\begin{align*}
\tau & = \left[ \frac{1}{2} \dot{\gamma} : \dot{\gamma} \right]^{1/2}, \\
\dot{\gamma} & = \left[ \frac{1}{2} \dot{\gamma} : \dot{\gamma} \right]^{1/2}.
\end{align*}
\]
For the current analysis, the bi-viscosity regularisation proposed by O’Donovan and Tanner (1984) has been used to mimic the shear rate dependence of viscous stress for a Bingham fluid:

\[ \tau = \mu_{\text{yield}} \dot{\gamma} \quad \text{for} \quad \dot{\gamma} \leq \tau_y / \mu_{\text{yield}}, \quad (5i) \]

\[ \tau = \tau_y \left( \dot{\gamma} / \dot{\gamma}_y \right) + \mu \dot{\gamma} \quad \text{for} \quad \dot{\gamma} > \tau_y / \mu_{\text{yield}}, \quad (5ii) \]

where \( \tau_y \) is the yield stress tensor, \( \mu_{\text{yield}} \) is the yield viscosity, and \( \mu \) is the plastic viscosity.

O’Donovan and Tanner (1984) indicated that a value of \( \mu_{\text{yield}} \) equal to 1000\( \mu \) mimics the true Bingham model in a satisfactory manner but here \( \mu_{\text{yield}} / \mu = 10^4 \) is taken to ensure higher fidelity of the simulations. Here, a limited number of simulations have also been carried out based on the regularization proposed by Papanastasiou (1987) in order to assess the sensitivity of the simulations on the choice of regularisation. Papanastasiou’s regularisation (Papanastasiou 1987) takes the following form:

\[ \tau = \tau_y \left( 1 - \exp \left( -m \dot{\gamma} \right) \right) + \mu \dot{\gamma} \quad (6) \]

where \( m \) is the stress growth exponent which has the dimension of time. The stress growth exponent has been chosen to be \( m \gg 10^5 L^2 / \alpha \) in this analysis to mimic true Bingham model in a satisfactory manner. Both Eqs. 5 and 6 transform the “unyielded” region to a zone of high viscosity such that the numerical solutions predict negligible magnitudes of velocity in these regions so the heat transfer takes place principally due to conduction. The maximum difference between the mean Nusselt numbers obtained from these two regularisations remains within the typical uncertainties encountered in experimental investigations (~2-3% shown in Table 1 of Yigit and Chakraborty (2017a)).

The schematic diagram of the configuration is shown in Fig. 1, which demonstrates that both CWT and CWHF boundary conditions have been considered for the differentially heated horizontal walls while the vertical walls are assumed to be adiabatic. According to the Buckingham’s pi theorem, it is possible to show that the Nusselt number for natural convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures can be expressed as: \( Nu = f_1(Ra, Pr, Bn, AR, \tau_y / L) \), where the nominal Rayleigh, Prandtl and Bingham numbers for Bingham fluids can be defined for CWT and CWHF boundary conditions in the following manner (Vikhansky 2009; Vikhansky 2010):

For CWT

\[ Ra = \rho g \beta (T_H - T_C) H^3 / \mu \alpha \quad \text{and} \quad Bn = \tau_y H / \mu \sqrt{g \beta (T_H - T_C) H} \quad (7i) \]

For CWHF

\[ Ra = \rho g \beta q H^4 / k \mu \alpha \quad \text{and} \quad Bn = \tau_y / \mu \sqrt{g \beta q / k} \quad (7ii) \]

Here, the Bingham number represents the ratio of yield stress to viscous stress. Moreover, this analysis has been conducted for a single representative value of nominal Prandtl number (i.e. \( Pr = \mu c_p / k = 500 \))
since practical yield stress fluids exhibit Prandtl numbers of the order of 100 (Darbouli et al. 2013; Kebiche et al. 2014; Hassan et al. 2015). For example, a recent experimental analysis (Kebiche et al. 2014) based on Rayleigh-Bénard convection of yield stress fluids in rectangular enclosures reported that 0.05% (by mass) Carbopol solution in water shows yield stress properties with a nominal Prandtl number of $Pr \approx 350$.

Here, the local heat transfer coefficient $h$ is defined as:

$$h = \left[ -k \left( \frac{\partial T}{\partial z} \right)_{z=0} \right] \times \frac{1}{(T_{z=0} - T_{z=H})}$$ (8)

The mean heat transfer coefficient $\bar{h}$ and the mean Nusselt number $\bar{Nu}_{cv}$ are evaluated as:

$$\bar{h} = \int_{r_1}^{r_{eq}} 2\pi rh(r)dr/[\pi (r_{1} + L)^2 - \pi r_{1}^2]$$, $\bar{Nu}_{cv} = \bar{h}H/k$ (9)

The current analysis has been carried out in non-dimensional form for the sake of generalisation. The spatial co-ordinates, velocity components, pressure and temperature are non-dimensionalised in the following manner:

$$r^+ = r/H, z^+ = z/H, u_i^+ = u_i/U_{ref}, P^+ = P/\rho U_{ref}^2, \Theta = (T - T_{ref})/\Delta T_{ref}$$ (10)

where the reference velocity scale $U_{ref}$ is taken to be equal to $\sqrt{\beta\Delta T_{ref} H}$ based on the equilibrium of inertial and the buoyancy forces (Turan et al. 2012; Yigit et al. 2015a; Yigit and Chakraborty 2017b) where $\Delta T_{ref}$ is a reference temperature difference. For the CWT configuration, $\Delta T_{ref}$ can be taken to be $\Delta T = T_h - T_C$ while it is taken to be $\Delta T_{ref} = qH/k$ for the CWHF configuration. Additionally, the reference temperature is taken to be temperature at the centre of the domain $T_{cen}$ for the CWHF boundary condition, whereas it is taken to be the cold wall temperature $T_C$ for the CWT boundary condition. Accordingly, the steady-state non-dimensional forms of the governing equations for mass, momentum and energy for constant temperature-independent thermo-physical properties take the following form under the assumption of axisymmetry:

**Non-dimensional mass conservation equation:**

$$\frac{1}{r^+} \frac{\partial (r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0$$ (11)

**Non-dimensional momentum conservation equations**

Radial direction:

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = - \frac{\partial P^+}{\partial r^+} + \frac{Pr^{1/2}}{Ra^{1/2}} \left( \frac{1}{r^+} \frac{\partial (r^+ u^+)}{\partial r^+} + \frac{\partial (r^+ u^+)}{\partial z^+} \right)$$ (12i)

Vertical direction:

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = - \frac{\partial P^+}{\partial z^+} + \Theta + \frac{Pr^{1/2}}{Ra^{1/2}} \left( \frac{1}{r^+} \frac{\partial (r^+ u^+)}{\partial z^+} + \frac{\partial (r^+ u^+)}{\partial z^+} \right)$$ (12ii)

**Non-dimensional energy conservation equation:**

$$u^+ \frac{\partial \Theta}{\partial r^+} + w^+ \frac{\partial \Theta}{\partial z^+} = \frac{1}{Pr^{1/2}Ra^{1/2}} \left[ \frac{1}{r^+} \frac{\partial}{\partial r^+} \left( r^+ \frac{\partial \Theta}{\partial r^+} \right) + \frac{\partial^2 \Theta}{\partial z^+ \partial z^+} \right]$$ (13)

In Eq. 12 $\tau_{ij}^+$ is the non-dimensional stress tensor which is given by:
\[ \tau_{ij}^+ = \tau_{ij}H/\mu \sqrt{g \beta \Delta T_{\text{ref}}} H \]  \hspace{1cm} (14)

where \( r \) is the radial coordinate, \( z \) axis is taken to align with the vertical direction, and the axisymmetric flow is independent of the azimuthal direction \( \phi \). The components of viscous stress tensor (i.e. \( \tau_{rr}, \tau_{\phi\phi}, \tau_{rz} \) and \( \tau_{zz} \)) are expressed according to Eq. 5. Equations (11-14) are solved in a coupled manner in conjunction with the following boundary conditions. The two vertical walls are kept under adiabatic conditions (i.e. \( \partial \Theta/\partial z^+ = 0 \) at \( z^+ = \eta/H \) and \( r^+ = \eta/H + (1/AR) \), and both velocity components (i.e. \( u^+ \) and \( w^+ \)) are identically zero on each boundary because of the no-slip condition and impenetrability of rigid walls. For the CWHF configuration, the heat fluxes for horizontal hot and cold walls are specified using the Neumann boundary condition (i.e. \( -\partial \Theta/\partial x^+ = 1 \) at \( x^+ = 0.0 \) and \( x^+ = 1.0 \) respectively). By contrast, the temperatures of horizontal walls are specified using the Dirichlet boundary condition (i.e. \( \Theta = 1 \) and \( \Theta = 0 \) at \( z^+ = 0.0 \) and \( z^+ = 1.0 \) respectively) for the CWT configuration.

### 3. NUMERICAL IMPLEMENTATION

The governing equations of mass, momentum and energy have been numerically solved in the context of finite-volume methodology using a commercial package ANSYS-FLUENT. This commercial package was previously used successfully for simulating Bingham fluid flows (Hassan et al. 2013; Yigit et al. 2015a; Yigit and Chakraborty 2017b). A second-order central difference scheme is used for the discretization of the diffusive terms and a second-order up-wind scheme is used for the convective terms. The well-known SIMPLE (Patankar 1980) (Semi-Implicit Method for Pressure-Linked Equations) algorithm is used for coupling of the pressure and velocity components. The convergence criteria were set to \( 10^{-6} \) for all the relative (scaled) residuals.

#### 3.1 Benchmarking and Grid Independence

The mean Nusselt numbers \( \bar{Nu} \) for laminar Rayleigh- Bénard convection of Newtonian fluids in square enclosures for \( 10^3 \leq Ra \leq 10^6 \) and \( Pr = 0.71 \) have been compared to the benchmark data (Ouertatani et al. 2008) in Table 2. It is evident from Table 2 that an excellent agreement has been achieved between the present results and the benchmark data (Ouertatani et al. 2008). The Bingham fluid simulations also have been compared to the benchmark data reported by Vola et al. (2003) for natural convection of Bingham fluids in square enclosures with vertical walls with different uniform temperatures. It is worth noting that Vola et al. (2003) reported only the values of yield stress \( \tau_y \). According to eq. 7, the yield stress values reported in Table 5 of Vola et al. (2003) give rise to \( Bn = 3.0, 0.95 \) and \( 0.3 \) for \( Ra = 10^4, 10^5 \) and \( 10^6 \) respectively. The mean Nusselt number values obtained for the current numerical methodology for the aforementioned values of \( Bn \) and \( Ra \) for \( Pr = 1.0 \) (as used in Vola et al. (2003)) are found to be in good agreement with the corresponding values reported by Vola et al. (2003) (maximum difference in \( \bar{Nu} \) is found to be less than 3%) (Turan et al., 2010). Furthermore, the same
numerical methodology was successfully used in several previous analyses (Yigit et al. 2015a; Yigit and Chakraborty 2017b).

Two different meshes (i.e. M1 and M2) for each AR values have been utilised to ensure grid independence of the results for both Newtonian and Bingham fluids. The details of the non-uniform Cartesian meshes, which have been used in the current analysis, are listed in Table 3. The maximum numerical uncertainty associated with the mean Nusselt number \( \overline{Nu}_{cy} \) for both Newtonian (i.e. \( Bn = 0 \)) and Bingham fluid (i.e. \( Bn = 0.02 \)) cases has been found to be smaller than 1% between M1 and M2 meshes for the range of parameters (i.e. \( 0 \leq \eta/L \leq 16, 0.25 \leq AR \leq 4 \) and \( 10^3 \leq Ra \leq 10^5 \) at \( Pr = 500 \)) considered here. The mesh M1 has been used for each AR for the sake of computational economy.

Interested readers are referred to Lewis et al. (1996), Lewis et al. (2004) and Nithiarasu et al. (2016) for further information on the necessity of grid independence in numerical heat transfer problems and the methodology adopted to establish this.

Zhang et al. (2006) indicated that Bingham fluid flows are unconditionally linearly stable for the quiescent initial condition. Thus, the steady-state solutions for Newtonian (i.e. \( Bn = 0 \) ) fluids for a given set of values of nominal Rayleigh and Prandtl numbers are used as the initial conditions for Bingham fluid simulations. Some magneto- and electro-rheological fluids exhibit yield stress behaviour, and it is possible to modify the yield stress by applying electrical or magnetic fields. Therefore, the steady-state Newtonian solutions can be considered as realistic initial conditions with practical relevance for the Bingham fluid simulations.

4. SCALING ANALYSIS

A detailed scaling analysis is utilised to express the effects of \( Ra, Pr, AR, Bn \) and \( \eta/L \) on the mean Nusselt number \( \overline{Nu}_{cy} \) in the current analysis. The velocity component in the vertical direction (i.e. \( w \)) can be scaled by equating the order of magnitudes of inertial and buoyancy terms as:

\[
\frac{\partial (ru)}{\partial r} \sim \left( \frac{u}{r} + \frac{w}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{\overline{Nu}_{cy}}
\]

which leads to:

\[
u \sim \frac{wr}{(r+L)} \sim \frac{w}{AR(1+L/r)} \sim \frac{1}{L} \frac{\sqrt{RaWCFPr}}{AR^2 (1+L/r)} \quad \text{for CWT} \tag{16i}
\]

\[
u \sim \frac{wr}{(r+L)} \sim \frac{w}{AR(1+L/r)} \sim \frac{1}{L} \frac{\sqrt{RaWCFPr}}{AR^2 (1+L/r)} \sqrt{\frac{\delta_{th}}{H}} \quad \text{for CWHF} \tag{16ii}
\]

Similarly, equating the order of magnitudes of inertial and viscous terms in the radial direction yields:
\( \rho \frac{u^2}{L} \sim \frac{1}{\delta_t} (\tau_y + \mu \frac{u}{\delta_t}) \) \hspace{1cm} (17i)

Using Eqs. 16i and 16ii in Eq. 17i leads to:

\[
\frac{\delta_t}{H} \sim \sqrt{\frac{Pr}{Ra_{CWT}}} \left( \frac{Bn_{CWT}}{2} \frac{AR(1 + L/r_t)^2}{\left( \frac{Bn_{CWT}}{2} \frac{AR(1 + L/r_t)^2}{Pr} + \sqrt{\frac{Ra_{CWT}}{Pr}} \right)} \right) ^{1/2} + \sqrt{\frac{Ra_{CWT}}{Pr}} \left( 1 + L/r_t \right) \text{ for CWT} \hspace{1cm} (17ii)
\]

\[
\frac{1}{f_z^{1/2} H} \sqrt{\frac{Ra_{CWHF}}{Pr}} \sim \left( Bn_{CWHF} AR \right) f_z^{1/2} \left( \frac{\delta_t}{H} \right) ^{1/2} \left( 1 + L/r_t \right) ^2 + AR(1 + L/r_t) \text{ for CWHF} \hspace{1cm} (17iii)
\]

where \( \delta_t \) is the hydrodynamic boundary layer thickness on the horizontal walls and \( f_z (Ra, Pr, Bn, AR, r_t/L) \) depicts the ratio of hydrodynamic and thermal boundary thickness (i.e. \( \delta_t/\delta_{th} \)) for CWT and CWHF boundary conditions, respectively. Eq. 17ii clearly shows that \( \delta_t/H \) increases with increasing \( AR (Bn_{CWT}) \) for a given set values \( Ra \) and \( r_t/L \). Moreover, Eq. 17iii indicates that an exact analytical solution does not exist for the CWHF configuration. However, some behaviours can be obtained based on limiting assumptions. For instances, \( Bn = 0 \) (i.e. Newtonian fluid) Eq. 17iii yields:

\[
\delta_t/H \sim (Pr/Ra_{CWHF})^{0.2} AR^{0.4} (1 + L/r_t)^{0.4} f_z^{0.2}, \text{ whereas for large values of } (BnAR), \text{ one gets:} \hspace{1cm}
\]

\[
\delta_t/H \sim (Bn_{CWHF} AR)^{0.5} ((Pr/Ra_{CWHF})^{0.25} f_z^{0.5}. \hspace{1cm}
\]

This limiting assumptions also suggest that \( \delta_t/H \) increases with increasing \( AR (Bn_{CWHF}) \) for a given set values \( Pr \) and \( r_t/L \). Here, since qualitative trend is expected to be the same for both CWT and CWHF configurations, this scaling analysis is continued for cylindrical annular enclosures in the case of CWT boundary condition.

Based on the scaling estimates in Eqs. 16i and 17i, it is possible to estimate the effective viscosity in horizontal boundary layer (i.e. \( \mu_{eff}^H \)) in the following manner \( (\mu_{eff}^H \sim \mu + \tau_y \delta_t/u) \) for the CWT configuration:

\[
\frac{\mu_{eff}^H}{\mu} \sim \left( 1 + Bn_{CWT} AR(1 + L/r_t) \right) \sqrt{\frac{Pr}{Ra_{CWT}}} \left( \frac{Bn_{CWT}}{2} \frac{AR(1 + L/r_t)^2}{Pr} \right) + \sqrt{\frac{Ra_{CWT}}{Pr}} \left( 1 + L/r_t \right) \right) ^{1/2} \hspace{1cm} (18i)
\]

Similarly, equating order of magnitudes of inertial and viscous terms in the vertical direction (i.e. \( \rho w^2/H \sim 1/\delta (\tau_y + \mu w/\delta) \)) to estimate the effective viscosity in horizontal boundary layer (i.e. \( \mu_{eff}^V \)) in the following manner \( (\mu_{eff}^V \sim \mu + \tau_y \delta/\omega) \):

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where $\delta$ is the hydrodynamic boundary layer thickness on the vertical wall. Using Eq. 18, the effective Rayleigh numbers (i.e. $Ra_{\text{eff}}^V$ and $Ra_{\text{eff}}^H$) in both vertical and horizontal boundary layers can be estimated in the following manner (i.e. $Ra_{\text{eff}} = \rho g \beta (T_H - T_c) L^3 / \mu_{\text{eff}}^2$):

$$
Ra_{\text{eff}}^V \sim Ra_{\text{CWT}} / \left( 1 + Bn_{\text{CWT}} \left( Pr \sqrt{Ra_{\text{CWT}}} \left[ \frac{Bn}{2} + \left( \frac{Bn}{2} \right)^2 + \sqrt{Ra_{\text{CWT}} / Pr} \right] \right) \right)
$$

(18i)

$$
Ra_{\text{eff}}^H \sim Ra_{\text{CWT}} / \left( 1 + Bn_{\text{CWT}} AR(1 + L/\tau_i) \left( Pr \sqrt{Ra_{\text{CWT}}} \left[ \frac{Bn_{\text{CWT}}}{2} AR(1 + L/\tau_i)^2 + \sqrt{\frac{Bn_{\text{CWT}}}{2} AR(1 + L/\tau_i)^2 + \sqrt{Ra_{\text{CWT}} / Pr}(1 + L/\tau_i)} \right] \right) \right)
$$

(18ii)

Equation 19 indicates that the effective Rayleigh number $Ra_{\text{eff}}$ remains smaller than the nominal Rayleigh number for Bingham fluids and $Ra_{\text{eff}}$ decreases with increasing $Bn$. Furthermore, Eq. 18 indicates that $\mu_{\text{eff}}^H$ is expected to be greater than $\mu_{\text{eff}}^V$ in cylindrical enclosures. This further implies that $Ra_{\text{eff}}^H$ is expected to be smaller than $Ra_{\text{eff}}^V$ in cylindrical enclosures as it is shown in Eq. 19.

Finally, the wall heat flux $q$ can be scaled as: $q = h \Delta T \sim k \Delta T / \delta_{th}$ where $\delta_{th}$ is the thermal boundary layer on the horizontal wall. Accordingly, the scaling estimate of the mean Nusselt number can be expressed as $\bar{\bar{Nu}}_{cy} \sim h H / k \Delta T / \delta_{th} \sim (H f_2 / \delta_i)$. It is not possible to obtain an analytical relation for $\delta_{th}$ from Eq. 17iii for CWHF boundary condition but it is possible to obtain a scaling estimate of $\bar{\bar{Nu}}_{cy}$ using Eq. 17ii in the case of CWT boundary condition in the following manner:

$$
\bar{\bar{Nu}}_{cy} \sim \text{Max} \left[ 1, \left( \frac{Ra_{\text{CWT}} \langle Pr \rangle}{Pr} \left( \frac{Bn_{\text{CWT}}}{2} AR(1 + L/\tau_i)^2 + \sqrt{\frac{Bn_{\text{CWT}}}{2} AR(1 + L/\tau_i)^2 + \sqrt{Ra_{\text{CWT}} / Pr}(1 + L/\tau_i)} \right) \right)^2 \right] f_2(Ra_{\text{CWT}}, Pr, Bn_{\text{CWT}}, AR, \tau_i / L)
$$

(20)

Equation 20 shows that $\bar{\bar{Nu}}_{cy}$ is expected to increase with an increase in $Ra$ and $\tau_i / L$, whereas an opposite behaviour is expected with an increase in $Bn$ and $AR$.

5. RESULTS & DISCUSSION
5.1. Effect of varying the Bingham number \(Bn\)

The contours of dimensionless temperature \(\theta_{CWHF} = (T - T_{cen})k/qL\) and stream function \(\Psi = \psi/\alpha\) are shown in Fig. 2 for different values of \(Bn\) for \(r_i/L = 1, AR = 0.5, Ra = 10^5\) and \(Pr = 500\) in the CWHF configuration. Figure 2 indicates that the magnitude of \(\Psi\) decreases and isotherms become progressively parallel to the horizontal walls with increasing \(Bn\). This is an indication of the weakening of convection with an increase in \(Bn\) and heat transfer can only occur purely due to conduction for large values of \(Bn\). The grey regions on the stream function distribution in Fig. 2 indicate the Apparently Unyielded Regions (AURs) (regions where \(|\gamma| \leq \tau_y\) according to Mitsoulis and Zisis (2001)). The AURs can be considered as the regions of slow moving fluids because of the high viscosity (\(\mu_{yield} \geq 10^3\mu\)) in the context of bi-viscosity regularisation. The shapes and sizes of AURs are dependent on the choice of \(\mu_{yield}\) (Hassan et al. 2013; Yigit et al. 2015a) but the qualitative and quantitative distributions of stream function and isotherms remain independent of the value of \(\mu_{yield}\) for \(\mu_{yield} \geq 10^3\mu\), and thus a fixed value \(\mu_{yield} = 10^4\mu\) is used here for all the simulations for the sake of high-fidelity. The precise shapes and sizes of AURs do not affect the mean Nusselt number in the current analysis. As it is shown in Fig. 2, the size of the AURs increases, and thermal transport becomes increasingly conduction-dominated with an increase in \(Bn\). The fluid flow becomes so weak that fluid flow does not contribute to thermal transport for large values of \(Bn\). It is worth noting that the distributions of \(\theta_{CWT} = (T - T_{cen})/(T_H - T_C)\) and stream function \(\Psi\) for the CWT boundary condition remain qualitatively similar to those obtained for the CWHF boundary condition, and thus are not explicitly shown here for the sake of conciseness. These findings are also consistent with previous analyses (Yigit et al. 2016) which dealt with natural convection of Bingham fluids in annular spaces with square cross-sections (i.e. same configuration for \(AR = 1.0\)).

5.2. Effects of varying the nominal Rayleigh number \(Ra\) and normalised cylinder radius \(r_i/L\)

The variations of \(\theta\) and dimensionless vertical (\(W = wL/\alpha\)) (radial (\(U = uL/\alpha\))) velocity components along with the vertical (horizontal) mid-plane for different values of \(Ra\) and \(r_i/L\) at \(AR = 2\) and \(Pr = 500\) are shown in Fig. 3 for Newtonian fluid (i.e.\(Bn = 0\)) and Bingham fluid (i.e.\(Bn = 0.02\)) cases for both CWT and CWHF boundary conditions. Figure 3 shows that the magnitudes of \(W\) and \(U\) increase with an increase in \(Ra\) for both Newtonian (i.e.\(Bn = 0\)) and Bingham fluid (i.e.\(Bn = 0.02\)) cases. This gives an indication of the strengthening of convective thermal transport with increasing \(Ra\) due to enhanced buoyancy effects for both Newtonian (i.e.\(Bn = 0\)) and Bingham fluid (\(Bn = 0.02\)) cases, irrespective of the boundary condition. Furthermore, the temperature difference between hot and cold walls is smaller in the CWHF configuration than that in the case of the CWT boundary condition for a given set of values of \(Ra, Bn, AR\) and \(r_i/L\).
This can be explained by scaling relation as the wall heat flux can be estimated as \( q \sim k \Delta T / \delta_{th} \) for the CWHF boundary condition, where \( \Delta T \) and \( \delta_{th} \) are the characteristic temperature difference and the thickness of thermal boundary layer on the horizontal walls, respectively. This suggests that the dimensionless temperature \( \theta \) in the case of CWHF boundary condition scales as \( \theta \sim \Delta T k / q H / k \sim O(\delta_{th} / H) \), whereas \( \theta \) scales as \( \theta \sim O(1) \) for the CWT boundary condition. The above discussion suggests that \( \theta \sim O(\delta_{th} / H) \) for the CWHF boundary condition is expected to increase with increasing \( \delta_{th} \), as the thermal boundary layer thickness increases with an increase in Bingham number due to the weakening of convection strength (see Fig. 3). The strength of convection also increases with increasing \( Ra \), which is reflected in the thinning of thermal boundary layer thickness. Thus, the temperature difference between horizontal walls also decreases with increasing \( Ra \) for the CWHF boundary condition. A higher temperature difference between the horizontal walls leads to a stronger convective transport in the CWT configuration than in the corresponding CWHF configuration for the same set of values of \( Ra, Bn, AR \) and \( r_i / L \). This can be easily seen from Fig. 3 which shows that the magnitudes of \( U \) and \( W \) are greater in the CWT configuration than in the CWHF configuration for a given set of values of \( Ra, Bn, AR \) and \( r_i / L \).

Additionally, Fig. 3 shows that the magnitude of the radial velocity component \( u \) increases with increasing \( r_i / L \) while the magnitude of the vertical velocity component \( w \) decreases for both Newtonian and Bingham fluid cases, regardless of the boundary condition. This is consistent with the scaling relation given by Eq. 16 (i.e. \( u \sim w / AR (1 + L / r_i) \)). Moreover, Fig. 3 indicates that the temperature difference between horizontal walls decreases with increasing \( r_i / L \) for the CWHF boundary condition, which suggests a stronger convective transport with increasing \( r_i / L \). This can further be confirmed from Fig. 4 where the contours \( \theta \) and \( \Psi \) are shown for different values of \( Ra \) and \( r_i / L \) at \( Bn = 0.02, AR = 2 \) and \( Pr = 500 \) in the CWT configuration. It can be seen from Fig. 4 that the magnitude of \( \Psi \) increases with an increase in \( r_i / L \) in the CWT configuration. This behaviour remains qualitatively similar to those obtained for the CWHF boundary condition, and thus are not explicitly shown here for the sake of conciseness.

**5.3. Effects of varying aspect ratio \( AR \)**

The variations of \( \theta \) and \( W \) along the vertical (horizontal) mid-plane for different \( AR \) and \( Ra \) values at \( r_i / L = 1, Bn = 0.03 \) and \( Pr = 500 \) are shown in Fig. 5 for both CWT and CWHF boundary conditions. The extent of non-linearity in \( \theta \) variation in Fig. 5 provides a measure of convection strength (because a linear variation is indicative of pure-conductive transport) in Fig. 5 for the CWT (CWHF) configuration. Figure 5 shows that the variation of \( \theta \) with \( z / H \) becomes increasingly linear and the magnitude of \( W \) decreases with increasing \( AR \) due to the weakening of convective transport. This behaviour can be confirmed from Fig. 6 where the contours of \( \theta \) and \( \Psi \) in the CWT configuration are
shown for different $AR$ values at $Bn = 0.03, r_i/L = 1, Ra = 10^5$ and $Pr = 500$. It is evident from Fig. 6 that the magnitude of $\Psi$ decreases and the isotherms become increasingly parallel to the horizontal walls with increasing $AR$, indicating conduction-driven thermal transport. Figure 6 also shows the number of rolls within the enclosure changes with $AR$ and thus the qualitative nature of the distribution of $W$ with $(r - r_i)/L$ also changes with the change in $AR$ in Fig. 5.

The weakening (strengthening) of advective (conductive) transport with increasing $AR$ can be explained with the help of the energy flux integral at the vertical mid-plane (Yigit and Chakraborty 2017b):

$$Q = Q_{conv} + Q_{cond} = \int_0^H \rho c_p T u dz - \int_0^H k (\partial T / \partial r) dz$$

(21i)

where the first term on the right hand side represents the effects of convective transport, whereas the second term on the right hand side accounts for the contribution of thermal conduction. The quantities $Q_{conv}$ and $Q_{cond}$ can be scaled in the following manner:

$$Q_{conv,CWT} \sim \rho c_p \Delta T u \delta_{T} \sim Prk\Delta T$$

$$\frac{b_{n,CWT}}{2} (1 + L/r_i) + \frac{b_{n,CWT}^2}{4} (1 + L/r_i)^2 + \frac{Ra_{CWT}}{Pr} \frac{1}{AR^2} (1 + L/r_i)$$

(21ii)

$$Q_{conv,CWHF} \sim \rho c_p u q \delta_{th} k / L \sim -qLf z \sqrt{Ra_{CWHF} Pr} \left( \frac{\delta_{th}}{L} \right)^{5/2}$$

(21iii)

$$Q_{cond,CWT} \sim \frac{\Delta T}{k} / H \sim k\Delta T AR$$

(21iv)

$$Q_{cond,CWHF} \sim kq \frac{\delta_{th} H}{k} \sim LqAR \left( \frac{\delta_{th}}{L} \right)$$

(21v)

Equation (21) indicates that $Q_{conv}$ ($Q_{cond}$) weakens (strengthens) with increasing $AR$ when $k, \Delta T, Ra, q$ and $Pr$ are kept constant for both Newtonian (i.e.$Bn = 0$) and Bingham fluids, which is consistent with the observations made from Figs. 5 and 6.

5.4. Behaviour of mean Nusselt number of cylindrical enclosures $\overline{Nu_{cy}}$

The variation of $\overline{Nu_{cy}}$ with $r_i/L$ for different $Ra$ and $AR$ is shown in Fig. 7 for Newtonian fluids (i.e.$Bn = 0$) at $Pr = 500$ for both CWT and CWHF configurations. It can be seen from Fig. 7 that $\overline{Nu_{cy}}$ increases with increasing $Ra$ and asymptotically approaches the corresponding mean Nusselt number for rectangular enclosures $\overline{Nu_{Rec}}$ in the CWT configuration. However, $\overline{Nu_{cy}}$ increases with increasing $Ra$ but remains insensitive to the changes in $r_i/L$ for $r_i/L > 1$ for the CWHF boundary condition. In order to explain this, the contours of $\Psi$ for different values of $Ra, r_i/L$ and $AR$ at $Pr = 500$ are shown in Fig. 8 in the both CWT and CWHF boundary conditions in the case of Newtonian fluids (i.e.$Bn = 0$). It can be noted from Fig. 8 that the number of rolls within the enclosure changes with $Ra, AR$ and $r_i/L$ in the CWT configuration, whereas changes in $AR$ and $r_i/L$ do not affect the flow pattern in the CWHF boundary condition. The changes in the flow pattern modify the distributions of isotherms between hot and cold walls, which is reflected in the variation of $\overline{Nu_{cy}}$ (see Fig. 7). It is worth
noting that the flow patterns are consistent with previous analyses (Leong 2002), which dealt with
laminar Rayleigh-Bénard convection of Newtonian fluids in a cylinder (i.e. \( \tau_i/L = 0 \)). Lir and Lin
(2001) reported similar flow patterns for Rayleigh-Bénard convection of Newtonian fluid (i.e. air) in a
rectangular shallow cavity based on their experimental flow visualisation. Mueller (1981) also
numerically reported similar flow patterns including rolls stacking over one another based on numerical
simulations. Moreover, it is worth indicating that the flow patterns presented here are not artefacts of
convergence criteria and reducing the convergence tolerance by an order of magnitude did not make
any difference to the magnitudes of the stream functions and also in the values of the mean Nusselt
number (especially for the cases where the rolls are stacked upon one another). It is also worth noting
that the flow patterns in the Rayleigh-Bénard configuration depend significantly on the initial condition
(Yigit and Chakraborty, 2015c) and it is possible to have several steady state solutions based on different
initial conditions for the CWT boundary condition. Thus, the flow patterns presented in this study
(even the ones where the rolls are stacked over one another and it should be noted that the
circulation strength for both rolls are not the same and one is often much weaker than the other) need to
be validated by experimental analyses.

Using Eqs. 17ii and 17iii one obtains the following estimate of mean Nusselt number \( \overline{Nu}_{cy} \approx (H/\delta_i)f_2 \)
for natural convection of Newtonian fluids by putting \( Bn = 0 \) as:

\[
\overline{Nu}_{cy} \approx (Ra_{cwt}/Pr)^{0.25}f_2/(1 + L/\tau_i)^{0.5}
\]

\[
\overline{Nu}_{cy} \approx (Ra_{cwhf}/Pr)^{0.2}f_2^{0.8}/AR^{0.4}(1 + L/\tau_i)^{0.4}
\]

for the CWT (CWHF) boundary condition. The above scaling estimates suggest that \( \overline{Nu}_{cy} \) is expected to increase with increasing \( Ra \)
and \( \tau_i/L \), which are consistent with the observations made from Fig. 7.

The following correlation for the mean Nusselt number was proposed in Yigit et al. (2015a) and Yigit
and Chakraborty (2017b) for natural convection of Newtonian fluids in rectangular enclosures \( \overline{Nu}_{rec} \):

\[
\overline{Nu}_{rec} = \left[ m_0/[1 + \exp((AR - x_0)/n_0)] - y_0 \right] AR - 1 \right|^{0.01} + \overline{Nu}(AR = 1)  \tag{22}
\]

where \( \overline{Nu}(AR = 1) \) is the mean Nusselt number for square enclosures (i.e. \( AR = 1 \)) and
\( m_0, x_0, n_0, y_0 \) are the correlation parameters, which are listed in Table 4 for both CWT and CWHF
boundary conditions. The predictions of Eq. (22) are shown in Fig. 7 (dashed lines), which indicates
that \( \overline{Nu}_{cy} \) for the CWHF configuration can be satisfactorily predicted with the help of Eq. 22 apart
from \( AR = 1, \tau_i/L \leq 0.25 \). Moreover, Eq. (22) is only valid for \( \tau_i/L \geq 4 \) in the CWT configuration
for \( AR \leq 1 \) and \( Ra \geq 5 \times 10^4 \) (see Fig. 7).

The variations of the mean Nusselt number \( \overline{Nu}_{cy} \) with \( Bn \) for different values of \( \tau_i/L \) at \( Ra = 5 \times
10^4, AR = 0.5 \) and \( Pr = 500 \) for both CWT (top) and CWHF (bottom) boundary conditions are shown
in Fig. 9 for Bingham fluids. It can be seen from Fig 9 that the \( \overline{Nu}_{cy} \) decreases with increasing \( Bn \)
which is indicative of the weakening of convective transport as a result of additional flow resistance due to the
yield stress. Furthermore, Fig. 9 shows that $\overline{\text{Nu}}_{cy}$ settles to unity once a threshold value of Bingham number $(Bn_{max})_{cy}$ is surpassed. Here $(Bn_{max})_{cy}$ is taken to be the nominal Bingham number where $\overline{\text{Nu}}_{cy} = 1.01$, so that $\overline{\text{Nu}}_{cy} < 1.01$ for $Bn > (Bn_{max})_{cy}$. The fluid flow becomes so weak for $Bn > (Bn_{max})_{cy}$ that it does not impart any influence on the thermal transport, which takes place purely due to conduction under this situation. Moreover, the simulations have been conducted by increasing (decreasing) $Bn$ until $(Bn_{max})_{cy}$ $(Bn = 0)$ is reached from $Bn = 0$ $(Bn = (Bn_{max})_{cy})$ in order to assess if the variation of $\overline{\text{Nu}}_{cy}$ with $Bn$ shows any hysteresis. It is found that hysteresis occurs (i.e. $\overline{\text{Nu}}_{cy}$ is indeed multi-valued for a given value of $Bn$ and different flow patterns are obtained for a given value of $Bn$) especially for $Ra \geq 5 \times 10^4$, and $AR \leq 1$, depending on the initial condition (see Fig. 9) for both CWT and CWHF boundary conditions.

The variation of $(Bn_{max})_{cy}$ with $r_i/L$ for different values of $Ra$ and $AR$ at $Pr = 500$ for both CWT and CWHF boundary conditions is shown in Fig. 10. It is evident from Fig. 10 that $(Bn_{max})_{cy}$ increases (decreases) with increasing $Ra$ $(AR)$ and thus convection could be sustained up to high values of Bingham number for large (small) values of $Ra$ $(AR)$. Figure 10 also shows that $(Bn_{max})_{cy}$ assumes comparable values in both CWT and CWHF configurations for $Ra \leq 10^4$. However, $(Bn_{max})_{cy}$ with $r_i/L$ in the CWT boundary condition is greater than in the case of CWHF boundary condition for the same set of values of $AR, Pr$ and $r_i/L$ due to stronger convection induced by higher temperature difference between active walls in the CWT configuration (see Fig. 3). Furthermore, it can be seen from Fig. 10 that $(Bn_{max})_{cy}$ increases with increasing $r_i/L$ before approaching the corresponding value for the rectangular enclosure $(Bn_{max})_{Rec}$ in the limit of $r_i/L \rightarrow \infty$. This indicates that convection strengthens with increasing $r_i/L$ for Bingham fluids for both CWT and CWHF boundary conditions.

The quantity $(Bn_{max})_{cy}$ can be estimated for the CWT boundary condition by considering $\overline{\text{Nu}}_{cy} \sim H/\delta_{th} \sim H f_2/\delta_i \sim O(1)$. This along with Eq. 17ii yields a scaling estimate of $(Bn_{max})_{cy}$ for the CWT boundary condition:

$$(Bn_{max})_{cy} \sim AR^{-1} \left[ f_2 \sqrt{Ra/Pr} \left(1+L/r_i\right)^2 - \frac{1}{(1+L/r_i)f_2} \right]$$

(23i)

Using $r_i/L \rightarrow \infty$ in Eq. 23i yields the estimate of the corresponding threshold value of Bingham number for rectangular enclosures for the CWT boundary condition:

$$(Bn_{max})_{Rec} \sim AR^{-1} \left[ f_2 \sqrt{Ra/Pr} - 1/f_2 \right]$$

(23ii)

It can be seen from Eq. 23i that $(Bn_{max})_{cy}$ is expected to increase (decrease) with increasing $Ra$ $(AR)$ for a given set of values of $Pr$ and $r_i/L$, as can be observed from Fig. 10. Moreover, $(Bn_{max})_{cy}$ is expected to increase with increasing $r_i/L$ and asymptotically approaches $(Bn_{max})_{Rec}$ (see Eq. 23ii) in the limit of $r_i/L \rightarrow \infty$, which are consistent with the findings of Fig. 10. The expression for $(Bn_{max})_{cy}$
in the case of CWHF boundary condition cannot be analytically obtained from Eq. 17iii but the same qualitative behaviour as that of the CWT boundary condition can be expected for the CWHF boundary condition.

The quantity \((Bn_{max})_{Rec}\) is parameterised in Yigit and Chakraborty (2017b) in the following manner for \(10 \leq Pr \leq 500\):

\[
(Bn_{max})_{Rec} = \frac{[m_1/[1 + \exp((AR - x_1)/n_1])] - y_1]}{1 + 10^{-0.01} + Bn_{max}(AR = 1)} \]  

(24)

where \(Bn_{max}(AR = 1)\) is \(Bn_{max}\) for square enclosures (i.e. \(AR = 1\)), and \(m_1, x_1, n_1, y_1\) are the correlation parameters, which are listed in Table 5 for both CWT and CWHF boundary conditions.

Equation 24 is consistent with the scaling estimate presented in Eq. 23ii. Using Eq. 24, \((Bn_{max})_{cy}\) is correlated here in the following manner:

\[
(Bn_{max})_{cy} = (Bn_{max})_{sq}/[1 + a_o(L/r_i)^{b_o}] \]  

(25i)

where \(a_o = [0.088lnRa - 0.872]AR^{0.23}\) and \(b_o = \frac{[0.745lnRa - 0.98]}{1 + [1.72lnRa - 1.78]/AR}\) for CWT

(25ii)

\(a_o = [0.042lnRa - 0.406]AR^{0.39}\) and \(b_o = [6.269 - 0.474lnRa]AR^{0.15}\) for CWHF

(25iii)

The predictions of Eq. 25 are shown in Fig. 10 which shows that this correlation satisfactorily predicts \((R^2=0.99)\) \((Bn_{max})_{cy}\) for Bingham fluids in rectangular cross-sectioned cylindrical enclosures for \(10^3 \leq Ra \leq 10^5, 0 \leq r_i/L \leq 16\) at \(Pr = 500\). It is worth noting that \((Bn_{max})_{cy}\) becomes equal to \((Bn_{max})_{Rec}\) in the limit of \(r_i/L \rightarrow \infty\) which ensures that asymptotic condition \((Bn_{max})_{cy} = (Bn_{max})_{Rec}\) for \(r_i/L \rightarrow \infty\) is satisfied.

A correlation for \(\overline{Nu}_{Rec}\) was proposed in Yigit et al. (2015a) and Yigit and Chakraborty (2017b) based on scaling relation of \(\overline{Nu}_{Rec}\) (Eq. 20 in the limit of \(r_i/L \rightarrow \infty\)) for natural convection of Bingham fluids in rectangular enclosures in the range \(0 \leq Bn \leq (Bn_{max})_{Rec}\). The same methodology of Yigit et al. (2015a) and (Yigit and Chakraborty 2017b) has been utilised here to propose a correlation of \(\overline{Nu}_{cy}\) for \(0 \leq Bn \leq (Bn_{max})_{cy}\) in the following manner for the CWT boundary condition:

\[
\frac{\overline{Nu}_{cy}}{(\overline{Nu}_{cy})_{Bn=0}^{-1}} = \frac{2[1 - (Bn^{*}/Bn_{max})^{b_c}]}{Bn^{*4} + Bn^{*2} + 4} \quad \text{when } (\overline{Nu}_{cy})_{Bn=0} > 1 \]  

(26i)

\[
\overline{Nu}_{cy} = 1 \quad \text{when } (\overline{Nu}_{cy})_{Bn=0} = 1 \]  

(26ii)

where \(Bn_{max}^{*} = Bn_{max} AR(Ra/Pr)^{-1/4} (1 + L/r_i)^{1.5}\) and \(b_c\) and \(c\) are the correlation parameters.

Following Refs. (Yigit et al. 2015a; Yigit and Chakraborty 2017b), the functional forms given by Eqs. 26i and 26ii are taken to be valid also for the CWHF boundary condition. Yigit and Chakraborty (2017b) proposed \(b_{Rec}\) and \(c_{Rec}\) in the following manner for both CWT and CWHF boundary conditions.

\[
b_{Rec} = b(AR = 1)\] and \(c_{Rec} = c(AR = 1) - m_2 \exp[-0.5((AR - 0.366)/n_2)^2] AR - 1\]  

(27)
where \( m_2, n_2 \) are the correlation parameters which are listed in Table 6 for both CWT and CWHF boundary conditions. Accordingly, in the current study, \( b \) and \( c \) have been parameterized in the range of \( 0.25 \leq \tau_i/L \leq 16, 0.25 \leq AR \leq 2 \) and \( 5 \times 10^4 \leq Ra \leq 10^5 \) for the CWHF configuration in the following manner:

\[
b_{cy} = b_{Rec} \quad \text{and} \quad c_{cy} = c_{Rec}/[1 + m_3(L/\tau_i)^{n_3}]
\]

\[m_3 = 0.4/[1 + \exp((AR - 0.75)/0.136)]\] and \[n_3 = 0.41\exp(1.4AR)\] for CWHF

This range is determined by the parameter space where the variation of \( \bar{Nu}_{cy} \) with \( Bn \) is monotonic and a single cell convection pattern is observed. It is worth noting that the changes in the flow pattern cause a non-monotonic variation of \( \bar{Nu}_{cy} \) with \( Bn \) in the CWT configuration for \( AR \leq 1, Ra \geq 5 \times 10^4 \). As a result, a correlation for \( \bar{Nu}_{cy} \) would be valid for a limited parameter range. Therefore, a correlation for \( \bar{Nu}_{cy} \) has not been proposed for the CWT boundary condition in this analysis.

The predictions of Eq. 26 are shown in Fig. 11, which shows that this correlation satisfactorily predicts (\( R^2 = 0.99 \) and 3% maximum percentage error) \( \bar{Nu}_{cy} \) for natural convection of Bingham fluids in rectangular-cross sectioned cylindrical enclosures for the CWHF boundary condition. It is also worth noting that \( (c)_{cy} \) becomes equal to \( (c)_{Rec} \) in the limit of \( \tau_i/L \to \infty \), which ensures that asymptotic condition \( \bar{Nu}_{cy} = \bar{Nu}_{Rec} \) for \( \tau_i/L \to \infty \) is satisfied by Eqs. 28i-ii.

5. CONCLUSIONS

Steady-state laminar Rayleigh-Bénard convection of Bingham fluids in rectangular cross-sectional cylindrical annual enclosures has been analysed for the parameter space given by: \( 10^3 \leq Ra \leq 10^5 \), \( 0.25 \leq AR \leq 4, 0 \leq \tau_i/L \leq 16 \) at \( Pr = 500 \) for both CWT and CWHF boundary conditions. It is found that the buoyancy-driven transport strengthens with increasing (decreasing) with \( Ra \) (\( AR \)) for both Newtonian (i.e. \( Bn = 0 \)) and Bingham fluids, regardless of the boundary condition. The mean Nusselt number \( \bar{Nu}_{cy} \) for Bingham fluids has been found to be smaller than the corresponding values obtained for Newtonian fluids (i.e. \( Bn = 0 \)) for both CWT and CWHF boundary conditions due to additional flow resistance arising from the yield stress in the case of Bingham fluids. Moreover, \( \bar{Nu}_{cy} \) decreases with increasing \( Bn \) and it assumes a value equal to unity (i.e. \( \bar{Nu}_{cy} = 1.0 \)) for large values of \( Bn \), which indicates that thermal transport takes place purely due to thermal conduction since the fluid flow becomes extremely weak to influence thermal transport under such conditions. Furthermore, \( \bar{Nu}_{cy} \) increases with increasing \( \tau_i/L \) and approaches the corresponding value obtained for rectangular enclosures (\( \tau_i \to \infty \)) for both CWT and CWHF boundary conditions. It has also been found that the flow pattern (i.e. number of rolls within the enclosure) changes with the variations of \( Ra, AR \) and \( \tau_i/L \) in the CWT configuration, whereas changes in \( AR \) and \( \tau_i/L \) do not significantly affect the flow patterns in the CWHF boundary condition. The changes in the flow pattern give rise to a non-monotonic variation of
$\overline{Nu_{cy}}$ with $Bn$ for the CWT configuration. It has been found that a correlation for the mean Nusselt number for the CWT boundary condition is of limited value because of its non-monotonic variation with $Bn$. Thus, numerical findings have been used here to propose a correlation for $\overline{Nu_{cy}}$ in the range of $0.25 \leq \eta/L \leq 16$, $0.25 \leq AR \leq 2$ and $5 \times 10^4 \leq Ra \leq 10^5$ only for the CWHF configuration.
REFERENCES


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Table 1: Summary of the findings of existing analyses on natural convection of yield stress fluids in enclosed spaces.

<table>
<thead>
<tr>
<th>Ref.</th>
<th>Type</th>
<th>Enclosure</th>
<th>Configuration &amp; Boundary conditions</th>
<th>$AR = H/L$</th>
<th>Model &amp; Fluid</th>
<th>$Ra, Pr$</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zhang et al. (2006)</td>
<td>A, N</td>
<td>Square</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>1</td>
<td>Bingham</td>
<td>$Ra_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>-</td>
</tr>
<tr>
<td>Balmforth and Rust (2009)</td>
<td>A,N,E</td>
<td>-</td>
<td>Diff. heated horizontal layers (CWT)</td>
<td>-</td>
<td>Bingham</td>
<td>$Pr = 1$</td>
<td>$Ra_{crit}$ for $\overline{Nu} &gt; 1$</td>
</tr>
<tr>
<td>Vikhansky (2009)</td>
<td>N</td>
<td>Square</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>1</td>
<td>Bingham</td>
<td>$Ra_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>-</td>
</tr>
<tr>
<td>Vikhansky (2010)</td>
<td>N</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>$0.5 \leq AR \leq 5$</td>
<td>Bingham</td>
<td>$Ra_{crit}$, $Bn_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>$Bn_{crit} = f(Bn, AR)$</td>
</tr>
<tr>
<td>Turan et al. (2017)</td>
<td>N</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall comparison (CWT-CWHF)</td>
<td>$0.25 \leq AR \leq 4$</td>
<td>Bi-viscosity reg.</td>
<td>$Ra_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>$Ra_{crit} = f(Bn, Pr, AR)$</td>
</tr>
<tr>
<td>Darbouli et al. (2013)</td>
<td>E</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>$6 \leq AR \leq 17.9$</td>
<td>Carbopol gel</td>
<td>$Bn_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>-</td>
</tr>
<tr>
<td>Kebiche et al. (2014)</td>
<td>E</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>19.3</td>
<td>Carbopol gel</td>
<td>$Bn_{crit}$ for $\overline{Nu} &gt; 1$</td>
<td>-</td>
</tr>
<tr>
<td>Turan et al. (2012)</td>
<td>N</td>
<td>Square</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>1</td>
<td>Bi-viscosity reg.</td>
<td>$10^3 \leq Ra \leq 10^5$</td>
<td>$\overline{Nu} = f(Ra, Pr, Bn)$</td>
</tr>
</tbody>
</table>

$0.1 \leq Pr \leq 10^2$
<table>
<thead>
<tr>
<th>Authors</th>
<th>Type</th>
<th>Shape</th>
<th>Description</th>
<th>Code</th>
<th>Bi-viscosity Reg.</th>
<th>Pr/Re Range</th>
<th>Nu Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turan et al.</td>
<td>N</td>
<td>Square</td>
<td>Diff. heated horizontal wall comparison (CWT-CWHF)</td>
<td>1</td>
<td>Bi-viscosity reg.</td>
<td>$10^3 \leq Ra \leq 10^5$</td>
<td>$\overline{Nu} = f(Ra, Pr, Bn)$</td>
</tr>
<tr>
<td>Yigit et al.</td>
<td>N</td>
<td>Square</td>
<td>Diff. heated inclined horizontal wall (CWT)</td>
<td>1</td>
<td>Bi-viscosity reg.</td>
<td>$0.1 \leq Pr \leq 10^2$</td>
<td>$10^3 \leq Ra \leq 10^5$</td>
</tr>
<tr>
<td>Yigit et al.</td>
<td>N</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall (CWT)</td>
<td>0.25</td>
<td>Bi-viscosity reg.</td>
<td>$10^3 \leq Ra \leq 10^5$</td>
<td>$\overline{Nu} = f(Ra, Pr, Bn, AR)$</td>
</tr>
<tr>
<td>Yigit and</td>
<td>N</td>
<td>Rectangular</td>
<td>Diff. heated horizontal wall comparison (CWT-CWHF)</td>
<td>0.25</td>
<td>Bi-viscosity reg.</td>
<td>$10^3 \leq Ra \leq 10^5$</td>
<td>$\overline{Nu} = f(Ra, Pr, Bn, AR)$</td>
</tr>
<tr>
<td>Chakraborty</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Pr = 500$</td>
<td></td>
</tr>
<tr>
<td>Hassan et al.</td>
<td>E,N</td>
<td>Square</td>
<td>Diff. heated horizontal wall (CWHF)</td>
<td>1</td>
<td></td>
<td>$10^4 \leq Ra \leq 10^6$</td>
<td>$\overline{Nu} = f(Ra, \eta)$</td>
</tr>
<tr>
<td>Yigit et al.</td>
<td>N</td>
<td>Cylindrical annular</td>
<td>Diff. heated horizontal wall comparison (CWT-CWHF)</td>
<td>1</td>
<td>Bi-viscosity reg.</td>
<td>$Pr = 500$</td>
<td>$\overline{Nu} = f(Ra, Pr, Bn, r/L)$</td>
</tr>
</tbody>
</table>

A: analytical; E: experimental; N: numerical
Table 2: Comparison of the mean Nusselt number $\overline{Nu}$ for Newtonian fluid (i.e. $Bn = 0$) with the benchmark data (Ouertatani et al. 2008) for square enclosure ($AR = 1$) at $Pr = 0.71$.

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>Present study</th>
<th>Ouertatani et al. (2008)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^3$</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>$1 \times 10^4$</td>
<td>2.154</td>
<td>2.158</td>
</tr>
<tr>
<td>$1 \times 10^5$</td>
<td>3.907</td>
<td>3.910</td>
</tr>
<tr>
<td>$1 \times 10^6$</td>
<td>6.363</td>
<td>6.309</td>
</tr>
</tbody>
</table>
Table 3: Summary of the non-uniform Cartesian meshes, which have been used in the current analysis for $0 \leq r_1/L \leq 16$, $0.25 \leq AR \leq 4$ and $10^3 \leq Ra \leq 10^5$ at $Pr = 500$ with non-dimensional minimum cell distance ($\Delta_{\text{min, cell}}/L$) and grid expansion ratio ($r_e$) values.

<table>
<thead>
<tr>
<th>AR</th>
<th>Grid</th>
<th>$\Delta_{\text{min, cell}}/L$</th>
<th>$r_e$</th>
<th>M1 (120 x 160)</th>
<th>M2 (180 x 220)</th>
<th>M1 (140 x 160)</th>
<th>M2 (200 x 240)</th>
<th>M1 (160 x 160)</th>
<th>M2 (240 x 240)</th>
<th>M1 (100 x 200)</th>
<th>M2 (160 x 320)</th>
<th>M1 (100 x 300)</th>
<th>M2 (160 x 480)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>M1 (120 x 160)</td>
<td>8.353 x 10^{-4}</td>
<td>1.013</td>
<td>M1 (140 x 160)</td>
<td>1.433 x 10^{-3}</td>
<td>M1 (160 x 160)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (100 x 200)</td>
<td>4.006 x 10^{-3}</td>
<td>M1 (100 x 300)</td>
<td>4.006 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>M1 (140 x 160)</td>
<td>5.575 x 10^{-4}</td>
<td>1.009</td>
<td>M1 (160 x 160)</td>
<td>1.011</td>
<td>M1 (180 x 160)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (100 x 200)</td>
<td>4.006 x 10^{-3}</td>
<td>M1 (100 x 300)</td>
<td>4.006 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>M1 (160 x 160)</td>
<td>1.013</td>
<td></td>
<td>M1 (200 x 240)</td>
<td>1.003 x 10^{-3}</td>
<td>M1 (200 x 240)</td>
<td>1.673 x 10^{-3}</td>
<td>M1 (160 x 320)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (160 x 480)</td>
<td>2.508 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>M1 (160 x 160)</td>
<td>1.010</td>
<td></td>
<td>M1 (200 x 240)</td>
<td>1.006</td>
<td>M1 (200 x 240)</td>
<td>1.673 x 10^{-3}</td>
<td>M1 (160 x 320)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (160 x 480)</td>
<td>2.508 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>M1 (100 x 200)</td>
<td>2.508 x 10^{-3}</td>
<td>1.010</td>
<td>M1 (160 x 320)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (160 x 320)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (160 x 320)</td>
<td>2.508 x 10^{-3}</td>
<td>M1 (160 x 480)</td>
<td>2.508 x 10^{-3}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4: Summary of $\bar{N}_u_{Rec}$ correlation given by Eq. (22) for both CWT and CWHF boundary conditions.

\[ \bar{N}_u_{Rec} = [(m_0/(1 + \exp((AR - x_0)/n_0)) - y_0)]AR - 1]^{0.01} + \bar{N}_u(AR = 1) \]

<table>
<thead>
<tr>
<th>CWT</th>
<th>$\bar{N}<em>u(AR = 1)</em>{CWT} = 0.178 Ra^{0.269}[Pr/(1 + Pr)]^{0.02}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_0 = 0.55 + 29.25 \exp[-0.5((\ln Ra - 11.12)/0.306)^2]$</td>
</tr>
<tr>
<td></td>
<td>$y_0 = 0.455 + 4097/[1 + \exp((46754 - Ra)/4258)]$</td>
</tr>
<tr>
<td></td>
<td>$m_0 = 0.765 + 4097/[1 + \exp((46743 - Ra)/4270)]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CWHF</th>
<th>$\bar{N}<em>u(AR = 1)</em>{CWHF} = 0.289 Ra^{0.214}[Pr/(1 + Pr)]^{0.017}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_0 = 0.613 + 2.21/[1 + \exp((10.43 - \ln Ra)/0.42)]$</td>
</tr>
<tr>
<td></td>
<td>$y_0 = -33.6 + 40.95/[1 + \exp((-\ln Ra - 4.658)/8.03)]$</td>
</tr>
<tr>
<td></td>
<td>$m_0 = 0.767 + 1.917/[1 + \exp((9.194 - \ln Ra)/0.715)]$</td>
</tr>
</tbody>
</table>
Table 5: Summary of \((Bn_{\text{max}})_{\text{Rec}}\) correlation given by Eq. (24) for both CWT and CWHF boundary conditions.

\[
(Bn_{\text{max}})_{\text{Rec}} = \left[ \left( \frac{\text{m}_1}{1 + \exp((AR - \text{x}_1)/\text{n}_1)} \right) - \text{y}_1 \right]|AR - 1|^{0.01} + Bn_{\text{max}}(AR = 1)
\]

<table>
<thead>
<tr>
<th></th>
<th>CWT</th>
<th>CWHF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Bn_{\text{max}}(AR = 1)_{\text{CWT}} = \frac{((0.00048Pr + 0.132)/(Pr + 18.15))Ra^{0.525}}{[(0.039Pr + 7.687)/(Pr + 13.56)]} ) for (10 \leq Pr \leq 500)</td>
<td>(Bn_{\text{max}}(AR = 1)_{\text{CWT}} = \frac{((Pr + 236)/(2707Pr + 40280))Ra^{0.525}}{[(Pr + 128)/(44.6Pr + 271)]} ) for (10 \leq Pr \leq 500)</td>
</tr>
<tr>
<td></td>
<td>(x_1 = 1.686 - 1.154\exp[-0.5((\ln Ra - 9.04)/0.841)^2])</td>
<td>(x_1 = -34.35 + 35.72/[1 + \exp((0.819 - \ln Ra)/1.733)])</td>
</tr>
<tr>
<td></td>
<td>(y_1 = -0.033 + 1.105/[1 + \exp(13.04 - \ln Ra)/1.417])</td>
<td>(y_1 = -0.0337 + 7.967/[1 + \exp(18.52 - \ln Ra)/1.986])</td>
</tr>
<tr>
<td></td>
<td>(n_1 = 0.354 - 0.43\exp[-0.5((\ln Ra - 9.01)/0.282)^2])</td>
<td>(n_1 = 0.214 + 0.266/[1 + \exp(10.2 - \ln Ra)/0.398])</td>
</tr>
<tr>
<td></td>
<td>(m_1 = -0.01 + 0.804/[1 + \exp((12.07 - \ln Ra)/1.158)])</td>
<td>(m_1 = 0.0014 + 1.034/[1 + \exp(12.7 - \ln Ra)/1.315])</td>
</tr>
</tbody>
</table>
Table 6: Summary of the correlations of $b$ and $c$ given by Eq. (27) for both CWT and CWHF boundary conditions.

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Equation for $b(AR = 1)$</th>
<th>Equation for $c(AR = 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CWT</td>
<td>$b(AR = 1) = 0.6$</td>
<td>$c(AR = 1) = 0.025Ra^{0.171}Pr^{0.095}$</td>
</tr>
<tr>
<td></td>
<td>$m_2 = 0.03 + 0.0334/[1 + \exp((9.48 - \ln Ra)/0.06)]$</td>
<td>$n_2 = 0.09 + 0.032/[1 + \exp((\ln Ra - 9.13)/0.083)]$</td>
</tr>
<tr>
<td>CWHF</td>
<td>$b(AR = 1) = 0.75$</td>
<td>$c(AR = 1) = 0.0818Ra^{0.1019}Pr^{0.054}$</td>
</tr>
<tr>
<td></td>
<td>$m_2 = -2091 + 28070/[1 + \exp((9.419 - \ln Ra)/0.583)]$</td>
<td>$n_2 = 0.024$</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS

Fig. 1: Schematic diagram of the simulation domain for: (a) CWT, (b) CWHF boundary conditions.

Fig. 2: Contours of $\theta$ and $\Psi$ with AURs (shown in grey) for different values of $Br$ at $\eta/L = 1.0, Ra = 10^5, Ar = 0.5$ and $Pr = 500$ in the CWHF configuration.

Fig. 3: Variations of $\theta$ and $W(U)$ along with the vertical (horizontal) mid-plane for different values of $Ra$ and $\eta/L$ at $Ar = 2$ and $Pr = 500$ for both Newtonian (i.e. $Br = 0$) and Bingham (i.e. $Br = 0.02$) fluid cases in the both CWT and CWHF configurations.

Fig. 4: Contours $\theta$ and $\Psi$ with AURs (shown in grey) for $\eta/L = (a) 0, (b) 1.0, (c) 16.0$ at $Br = 0.02, Ar = 2$ and $Pr = 500$ for different values of $Ra$ in the case of CWT boundary condition.

Fig. 5: Variations of $\theta$ ($W$) along the vertical (horizontal) mid-plane for different $Ar$ and $Ra$ at $Br = 0.03, \eta/L = 1.0$ and $Pr = 500$ for both CWT and CWHF boundary conditions.

Fig. 6: Contours $\theta$ and $\Psi$ with AURs (shown in grey) for different values of $Ar$ at $Br = 0.03, \eta/L = 1, Ra = 10^5$ and $Pr = 500$ in the CWT configuration.

Fig. 7: Variation of $\bar{Nu}_{cy}$ with $\eta/L$ for Newtonian fluids (i.e. $Br = 0$) for $Ar = (a) 0.25, (b) 0.5, (c) 1.0, (d) 2.0$ at $Pr = 500$ for different values of $Ra$ in the cases of both CWT and CWHF boundary conditions. The corresponding values of the mean Nusselt number $\bar{Nu}_{Rec}$ for rectangular enclosures is shown by dashed lines (which is correlated earlier by Eq. 22).

Fig. 8: Contours of $\Psi$ for different values of $Ra, \eta/L$ and $Ar$ at $Pr = 500$ in the case of both CWT and CWHF boundary conditions for Newtonian fluids (i.e. $Br = 0$).

Fig. 9: Variation of $\bar{Nu}_{cy}$ (evaluated from simulation data using Eq. 9) with $\eta/L$ for Bingham fluids at $Ra = 5 \times 10^4, Ar = 0.5$ and $Pr = 500$ for a) CWT (top row), b) CWHF (bottom row) boundary conditions.

Fig. 10: Variation of $(Br_{max})_{cy}$ with $\eta/L$ for $Ar = (a) 0.25, (b) 0.5, (c) 1.0, (d) 2.0$ at $Pr = 500$ for different values of $Ra$ in the case of both CWT and CWHF boundary conditions.

Fig. 11: Variation of $\bar{Nu}_{cy}$ with $Br$ for $Ar = (a) \eta/L = 0.25, (b) \eta/L = 1.0, (c) \eta/L = 4.0, (d) \eta/L = 16$ at $Ra = 10^5$ and $Pr = 500$ along with the prediction of Eq. 26 for different values of $Ra$. 
Fig. 1: Schematic diagram of the simulation domain: a) CWT, b) CWHF configurations.
Fig. 2. Contours of \( \Psi \) and \( \theta \) with AURs (shown in grey) for \( r_i/L = 1.0 \), \( Ra = 10^5 \), \( AR = 0.5 \) and \( Pr = 500 \) in the CWHF configuration.
Fig. 3: Variations of $\theta$ and $W(U)$ along with the vertical (horizontal) mid-plane for different $Ra$ and $r_i/L$ at $AR = 2$ and $Pr = 500$ for both Newtonian (i.e. $Bn = 0$) and Bingham (i.e. $Bn = 0.02$) fluid cases in the both CWT and CWHF configurations.
Fig. 4: Contours $\theta$ and $\Psi$ with AURs (shown in grey) for different $Ra$ and $r_i/L$ a) 0, b) 1, c) 16 at $Bn = 0.02$, $AR = 2$ and $Pr = 500$ for the CWT configuration.
Fig. 5: Variations of $\theta$ ($W$) along the vertical (horizontal) mid-plane for different $AR$ and $Ra$ at $Bn = 0.03$, $r_i/L = 1.0$ and $Pr = 500$ for both CWT and CWHF configurations.
Fig. 6: Contours of $\theta$ and $\Psi$ with AURs (shown in grey) for different $AR$ at $Bn = 0.03$, $r_i/L = 1$, $Ra = 10^5$ and $Pr = 500$ in the CWT configuration.
Fig. 7: Variation of $\overline{Nu}_{cy}$ with $r_i/L$ for Newtonian fluids (i.e. $Bn = 0$) for different $Ra$ and $AR$ a) 0.25, b) 0.5, c) 1, d) 2 at $Pr = 500$ for both CWT and CWHF configurations. The corresponding values of the mean Nusselt number $\overline{Nu}_{Rec}$ for rectangular enclosures is given by dashed lines (which is correlated earlier in Eq. 22).
<table>
<thead>
<tr>
<th>CWT</th>
<th>CWHF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_i/L = 0.25 )</td>
<td>( r_i/L = 0.25 )</td>
</tr>
<tr>
<td>( AR )</td>
<td>( Ra = 10^4 )</td>
</tr>
<tr>
<td>0.25</td>
<td>![Image]</td>
</tr>
<tr>
<td>0.5</td>
<td>![Image]</td>
</tr>
<tr>
<td>1</td>
<td>![Image]</td>
</tr>
<tr>
<td>( r_i/L = 1.0 )</td>
<td>( r_i/L = 1.0 )</td>
</tr>
<tr>
<td>( AR )</td>
<td>( Ra = 10^4 )</td>
</tr>
<tr>
<td>0.25</td>
<td>![Image]</td>
</tr>
<tr>
<td>0.5</td>
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</tr>
<tr>
<td>1</td>
<td>![Image]</td>
</tr>
<tr>
<td>( r_i/L = 16 )</td>
<td>( r_i/L = 16 )</td>
</tr>
<tr>
<td>( AR )</td>
<td>( Ra = 10^4 )</td>
</tr>
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<td>0.25</td>
<td>![Image]</td>
</tr>
<tr>
<td>0.5</td>
<td>![Image]</td>
</tr>
<tr>
<td>1</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

Fig. 8: Contours of \( \Psi \) for different values of \( Ra, r_i/L \) and \( AR \) at \( Pr = 500 \) in the case of both CWT and CWHF boundary conditions for Newtonian fluids (i.e. \( Bn = 0 \)).
Fig. 9: Variation of $\overline{Nu}_{cy}$ (evaluated from simulation data using Eq. 9) with $Bn$ for Bingham fluids at $Ra = 5 \times 10^4$, $AR = 0.5$ and $Pr = 500$ for a) CWT (top row), b) CWHF (bottom row) boundary conditions.
Fig. 10: Variation of \( (B_{n_{\text{max}}})_{cy} \) with \( r_i/L \) for \( AR = (a) 0.25, (b) 0.5, (c) 1.0, (d) 2.0 \) at \( Pr = 500 \) for different values of \( Ra \) in the case of both CWT and CWHF boundary conditions.
Fig. 11: Variation of $\bar{N}\bar{u}_c$ with $Bn$ for $AR = (a) r_i/L = 0.25$, (b) $r_i/L = 1.0$, (c) $r_i/L = 4.0$, (d) $r_i/L = 16$ at $Ra = 10^5$ and $Pr = 500$ along with the prediction of Eq. 26 for different values of $Ra$. 