Vortex-induced vibrations of two mechanically coupled circular cylinders with asymmetrical stiffness in side-by-side arrangements

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ABSTRACT

Vortex-induced vibrations of two mechanically coupled circular cylinders with asymmetrical stiffness in side-by-side arrangements are numerically investigated in uniform flows at a low Reynolds number of 100. The oscillation system is restricted to the cross-flow direction, giving rise to a coupled two-degree-of-freedom response. Attention is placed on the two cylinders with a center-to-center gap ratio of 4 and a mass ratio of 10. The flow dynamics is described by the two-dimensional incompressible Navier-Stokes equations and resolved by the Characteristic-Based-Split finite element method. The stiffness of the first spring that connects the lower cylinder to the wall is chosen such that the vortex-induced vibration of the associated single cylinder with the same stiffness undergoes a pre-synchronization (state A), synchronization (state B) and post-synchronization (state C), respectively. For each state, the stiffness of the second spring connecting the two lower/upper cylinders is varied to cover both synchronization and desynchronization regimes. Numerical results show that the coupled system locks on the first-mode natural frequency in state A, while on the second-mode natural frequency in states B and C. In the lock-in regime, the amplitude ratios of the two oscillating and coupled cylinders collapse well onto the corresponding first and second free-vibration modes, respectively. The overall coupling mechanism is further featured in terms of the hydrodynamic coefficients, frequency characteristics, wake patterns and effective added mass, quantifying the distinctive dynamics against the canonical single-degree-of-freedom, single-cylinder system.

1. Introduction

Fluid-structure interaction (FSI) system for cylindrical structures is of great practical significance in engineering applications, such as offshore risers and pipelines conveying gas or oil, suspension bridges, chimney stacks and so on. Such a coupling system is also fundamentally important from the standpoint of understanding the relevant dynamics since a rich complex physical behaviour is involved including wake synchronization and even chaos, etc. For this reason, numerous investigations have been undertaken on the canonical problem of vortex-induced vibration (VIV) of an elastically mounted circular cylinder in the last few decades. A comprehensive review on this research topic has been presented and one can refer to the articles by Thompson et al. (1996), Williamson and Roshko (1988), Sarpkaya (2004), Prasanth and Mittal (2008) and Williamson and Govardhan (2004).

The pioneering study on VIV mechanism of a bluff body was conducted by Feng (1968) about 50 years ago, in which the oscillation was restrained in cross-flow direction at a high mass ratio. He found that, if the vortex-shedding frequency is close to the natural frequency of the structure, it collapses on the natural frequency and remains nearly constant. Similar observation was reported later by Williamson and Roshko (1988) and they referred the synchronization between the shedding vortex and the response of the cylinder as “lock-in” or “frequency synchronization”, where the amplitude of the oscillator attains up to the same order of the cylinder diameter. This lock-in regime occurs...
over a range of reduced velocity, influenced by a number of fluid-structural properties including mass and damping ratios (Sarpkaya, 2004; Williamson and Govardhan, 2004).

Khalak and Williamson (1997) and Govardhan and Williamson (2000) carried out experiments involving the vibration of an elastically mounted cylinder in low mass-damping ratio at high Reynolds number, \( Re \). They identified three branches in the transverse response, namely “initial”, “upper” and “lower” branches. The transition between the initial and upper branches is accompanied by a jump of transverse amplitude and phase difference between the lift force and displacement. In addition, they showed the switching of wake modes at different branches. In terms of flow visualization, the 2S mode of shedding (single vortex is released from each side of cylinder) was found at initial branch; otherwise, the 2P mode of vortex shedding (a pair of vortices are released from each side of cylinder during one circulation) was displayed at upper and lower branches. For the case in which cylinder is allowed to vibrate both in the streamwise and transverse directions, Govardhan and Williamson (2000) discovered a new 2T mode (two triple vortices are shed from the cylinder in each vibration cycle) at the maximum amplitude which can reach \( 1.5D \) and defined this novel response regime as super upper branch. However, for low \( Re \) \((60 \leq Re \leq 200)\), only two branches were found by Mittal et al. (Prasanth and Mittal, 2008; Singh and Mittal, 2005; Prasanth et al., 2006; Mittal and Singh, 2005) for initial/lower branches. They adopted a stabilized finite element method and systematically investigated the response of the cylinder with a mass ratio of 10. Without an upper branch, the maximum amplitude in the transverse direction is only \( 0.6D \) being much lower than that at high \( Re \). The vortex shedding pattern on the initial branch is still the 2S mode; for the lower branch, the wake changes into the C(2S) mode similar to the 2S mode except for the former case having the vortex coalescence in the far wake. A transfer between 2S and C(2S) modes is also accompanied by a jump of the phase difference between the displacement and the fluid force.

Multiple bodies in the vicinity of each other in uniform stream and concomitant flow interference among them are commonly encountered in practical situations. Bearman and Wadcock (1973) and Zhou et al. (2000) demonstrated that the mutual interaction between two stationary cylinders is totally influenced by two non-dimensional parameters: \( Re \) and the gap ratio \( g^* = g/D \), where \( g \) is the gap between the two cylinders. Kang (2003) conducted a series of numerical calculations of two stationary cylinders with a range of \( 1 \leq g^* \leq 5 \) and \( 40 \leq Re \leq 160 \) and summarized that there are totally six different wake patterns. Bao et al. (2013) simulated the flow characteristics of two in-phase oscillating cylinders with four center-to-center cylinder spacings, ranging from 1.2\( D \) to 4.0\( D \). The results showed that there exist five different flow response states according to the phase portraits of the lift force and flow fields visualized through the vorticity contours. The flow interference also has a significant impact on the FSI of elastically mounted cylinders. Zdravkovich (1985) firstly investigated VIV of two cylinders for a variety of arrangements and concluded that the flow interference regime is linked to the observed vortex shedding response. More recently, Qin et al. (2019) conducted extensive measurements to capture the responses of two tandem cylinders and the ambient flow fields using the laser vibrometer, hotwire and PIV techniques. A total of four vibration regimes were identified according to the vibration and wake characteristics. Chen et al. (2018) investigated three tandem cylinders at varied reduced velocities for a fixed \( Re = 100 \) and found two different vibration patterns: wake-induced galloping for the small gap ratio and VIV for the large gap ratio.

Side-by-side arrangement is also an important configuration for the investigation of the flow interference for VIV responses of adjacent cylinders. Zhou et al. (2001) tested the response of two cylinders with three different spacing ratios in the experiment and verified that the structure vibration behaviour is highly relevant to flow characteristics. Chen et al. (2015) also conducted a detailed research on a certain aspect of wake patterns of two elastically supported circular cylinders. Totally six near-wake patterns were observed and plotted in a plane of \( U_r \) and spacing ratio. Huerta-Huarte and Gharib (2011) investigated FSI of two flexible cylinders in side-by-side arrangement and found that the cross-flow motion of two cylinders synchronizes in-phase or anti-phase with each other depending on the gap ratio. The similar consequence was found by Zhao (2013) for two rigidly coupled cylinders of which the gap ratio was in the range of \( 4 \leq g^* \leq 6 \) and the response of the two cylinders is similar to that of a single cylinder. Cui et al. (2014) conducted the numerical simulation of two different cases of elastically coupled cylinders in side-by-side arrangement at \( Re = 5000 \) and found five response regimes in both cases. The two cylinders have distinct amplitudes even though the response frequencies are completely identical, and the phase transfer between the displacement and the lift coefficient is much more complex when compared with the single cylinder.

In this study, VIV responses of two mechanically coupled cylinders in side-by-side arrangements are numerically investigated. A sketch of the two cylinders shown in Fig.1 is similar to the schematic model considered in Cui et al. (2014). Unlike the situation considered in Cui et al. (2014), the differing stiffness of the two springs has an effect on the dynamic responses of the coupled system. The remainder of this paper is organized as follows. The physical model
and governing equations of the FSI coupled system are presented in Section 2. The numerical code is validated against the existing data in the literature in Section 3. Numerical results of two elastically coupled cylinders with asymmetrical stiffness are presented and discussed in Section 4. Key findings and concluding remarks are enclosed in Section 5.

2. Physical model and numerical methodology

2.1. Governing equations

Fig. 1 shows the schematic of two elastically coupled cylinders immersed in a steady uniform flow of velocity \( U_\infty \) in a side-by-side arrangement. The centers of the two cylinders with the same diameter of \( D \) are symmetrically positioned at \((x/D, y/D) = (0.0, \pm 2.0)\), i.e., the gap ratio is fixed at \( g^* = 4 \). The mass ratio of the vibrating body to the displaced fluid is \( m^* = m/0.25\pi\rho_fD^2 \) = 10, where \( m \) is the cylinder mass per unit length and \( \rho_f \) is the fluid density. \( Re = \) is fixed at 100, which is the same in numerical simulations conducted by Singh and Mittal (2005). In order to achieve a maximum response, the structure damping is assigned to be zero. The lower cylinder (hereafter substituted by cylinder1) is mounted on a spring of which the stiffness is assumed as \( K_1 \), while the upper cylinder (hereafter referred to as cylinder2) is linked with the cylinder1 by the second spring of which the stiffness is \( K_2 \). This is the main difference between this model and the one employed in Cui et al. (2014), where \( K_1 = K_2 \).

In the simulation, the flow dynamics of the viscous fluid is governed by the incompressible Navier-Stokes equations, which can be expressed with primary variables in a vector form as

\[
\frac{\partial U}{\partial t} + U \cdot \nabla U = -\frac{1}{\rho_f} \nabla p + \frac{\mu}{\rho_f} \nabla^2 U, \tag{1}
\]

\[
\nabla \cdot U = 0, \tag{2}
\]

where \( U = (u, v) \) is the flow velocity vector in the Cartesian \((x, y)\) coordinate directions; \( p \) and \( t \) are the pressure and time, \( \rho_f \) and \( \mu \) are the density of the fluid and dynamic viscosity, respectively.

The elastic coupling of the spring-mass system is described by the two-degree-of-freedom 2DOF dynamic equations in dimensionless form as

\[
\ddot{Y}_1 + C_1^*(\dot{Y}_1 - \dot{Y}_2) + K_1^*Y_1 + K_2^*(Y_1 - Y_2) = \frac{2C_L1}{\pi m^*}, \tag{3}
\]

\[
\ddot{Y}_2 + C_2^*(Y_2 - \dot{Y}_1) + K_2^*(Y_2 - Y_1) = \frac{2C_L2}{\pi m^*}, \tag{4}
\]
where \( C_i^* = C_i D / m U_\infty \) denotes the reduced damping, which hereafter will be set as 0.0. \( K_i^* = K_i D^2 / m U_\infty^2 \) is the reduced stiffness, and \( C_{Li} = F_{Li} / 0.5 \rho_f U_\infty^2 D \) is the instantaneous lift coefficient with \( F_{Li} \) being the lift force exerting on the two cylinders. \( \ddot{Y}_i, \dot{Y}_i \) and \( Y_i \) denote the cylinder transverse acceleration, velocity and displacement, respectively. The subscripts 1 and 2 stand for the lower and upper cylinders, respectively. The corresponding natural frequencies of the system can be obtained as \((f_{n1}, f_{n2})\)

\[
(f_{n1}, f_{n2}) = \left( \frac{1}{2\pi} \sqrt{\frac{K_1^* + 2K_2^* - \sqrt{K_1^* + 4K_2^*}}{2}}, \frac{1}{2\pi} \sqrt{\frac{K_1^* + 2K_2^* + \sqrt{K_1^* + 4K_2^*}}{2}} \right),
\]

where \( f_{n1} \) and \( f_{n2} \) are the first-mode and second-mode natural frequencies normalized with respect to \( D \) and \( U_\infty \), respectively. In analogy to the definition of the reduced velocity \( U_r \) in single cylinder (Williamson and Roshko, 1988), here we define the reduced velocities associated with the 2DOF system as: \( U_{r1} = 1 / f_{n1} \) and \( U_{r2} = 1 / f_{n2} \). The associated primary and secondary model amplitude of the free vibration can be expressed by

\[
\rho_i = \frac{K_2^*}{-(2\pi f_{n})^2 + K_1^* + K_2^*} \quad (i = 1, 2).
\]

As mentioned above, the structural properties are determined by the combination of the spring stiffness pair in the parameter space \((K_1^*, K_2^*)\). Values of \( K_1^* \) are chosen such that a single cylinder at the same flow velocity would drop into the regimes of pre-synchronization, synchronization and post-synchronization, respectively. Three different values are considered: \( K_1^* = (6, 1.075, 0.1) \) and the associated reduced velocities of the single cylinder are \( U_r = (2.56, 6, 19.86) \). In accordance with the different values of \( K_1^* \), overall simulations will be divided into three following states A, B and C. We thereby focus the attention mainly on the effects of \( K_2^* \) in this work. It is well known that a resonance is normally excited in the situation where the natural frequency of the mass-spring system is close to the vortex shedding frequency. For this reason, we firstly plotted the distribution of normalized natural frequencies of the 2DOF system in a wide range of \( 0.1 \leq K_2^* \leq 10.0 \) for states A, B and C in Fig.2. It can be observed that in a certain range of \( K_2^* \), \( f_{n1} \) is much closer to the associated Strouhal number of side-by-side stationary cylinders (\( St_0 = 0.165 \) (Kang (2003))) in state A, while \( f_{n2} \) approaches \( St_0 \) in states B and C. Therefore, the lock-in is only expected in the situation
where \( f_{n1} \) or \( f_{n2} \) is close to \( St_0 \). Based on this consideration, the range of \( K_2^* \) considered in the simulations is chosen such that the lock-in resonance would be excited either on \( f_{n1} \) or \( f_{n2} \), see the shadow area in Fig.2. For each state, by comparing with the response of the single cylinder, the behaviour of the coupled system is quantified in terms of vibration responses, fluid forces, frequency characteristics as well as effective added mass, emphasising the lock-in region to be discussed in Section 4.

### 2.2. Numerical method

To account for the cylinder motions, an Arbitrary-Lagrangian-Eulerian (ALE) formulation of incompressible Navier-Stokes equations is employed for the solution of the fluid flows with moving boundaries. A field variable of velocity vector, \( \vec{U} = (\vec{u}, \vec{v}) \), is introduced to the convective term in Eq.(1) such that \((U - \vec{U}) \cdot \nabla U\) for describing the flow convection in the ALE reference framework, as proposed originally in Donea et al. (1982) and Hughes et al. (1981). \((\vec{u}, \vec{v})\) are the velocity components of the moving computational domain in \((x, y)\) directions. In the ALE reference, the kinematics of the material point on the moving boundaries is stated by the Lagrangian description, while the Eulerian description can be used at a certain distance from the moving surfaces, as well as all the remaining boundaries. In practice, a modified spring-analogy method is employed in this work to update \( \vec{U} \) and the associated mesh system (Zhang et al., 2010).

The fractional step algorithm in the framework of Characteristic-Based-Split (CBS) method is used to decouple the calculation of the velocity and pressure fields. An elliptic pressure-Poisson equation is derived from the discretized momentum and continuity equations. A stabilized pressure gradient projection method (Codina and Blasco, 2000) is employed to eliminate the artificial oscillation that would occur in the pressure field. The three-node linear triangular element is used to approximate the interpolation functions of both the velocity and pressure over the computational domain. Details of this algorithm can be referred to Bao et al. (2012). The governing equations of the flow field are supplemented with appropriate boundary conditions as follows: a uniform velocity profile is imposed on the inlet boundary, which is specified as \((u, v) = (U_\infty, 0)\); Neumann boundary conditions of \(\partial u / \partial x = 0.0\) and \(\partial v / \partial x = 0.0\) are applied on the far downstream outlet; the lateral boundaries are treated to be slip boundaries, i.e. \(\partial u / \partial y = 0.0\) and \(v = 0.0\); a no-slip boundary condition is prescribed at the surface of the two cylinders such that \((u, v) = (\hat{u}, \hat{v})\).

The equations of motions governing the mechanically coupled system are solved using a Newmark-\(\beta\) method. The integration steps are described as follows:

(i) The displacement of each cylinder is explicitly predicted from the previous time step:

\[
Y_{1}^{n+1} = \frac{A_{4}(\frac{2C_{L1}}{\pi m^*} + \frac{4}{\Delta t^2} U^n + \frac{4}{\Delta t} Y_1^n + \dot{Y}_1^n) - A_{2}(\frac{2C_{L2}}{\pi m^*} + \frac{4}{\Delta t^2} Y_2^n + \frac{4}{\Delta t} \dot{Y}_2^n + \ddot{Y}_2^n)}{A_{1}A_{4} - A_{2}A_{3}},
\]

\[
Y_{2}^{n+1} = \frac{A_{3}(\frac{2C_{L1}}{\pi m^*} + \frac{4}{\Delta t^2} U^n + \frac{4}{\Delta t} \dot{Y}_1^n + \ddot{Y}_1^n) - A_{1}(\frac{2C_{L2}}{\pi m^*} + \frac{4}{\Delta t^2} Y_2^n + \frac{4}{\Delta t} \dot{Y}_2^n + \ddot{Y}_2^n)}{A_{2}A_{3} - A_{1}A_{4}},
\]

here \(A_{1} = \frac{4}{\Delta t^2} + K_1^* + K_2^*\), \(A_2 = -K_2^*\), \(A_3 = -K_1^*\) and \(A_4 = \frac{4}{\Delta t^2} + K_2^*\), respectively.

(ii) The velocity and acceleration of cylinders at the \(n+1\) time step are calculated using a linear approximation

\[
\dot{Y}_1^{n+1} = \frac{4}{\Delta t^2}(Y_{1}^{n+1} - Y_1^n) - \frac{4}{\Delta t} \dot{Y}_1^n - \ddot{Y}_1^n,
\]

\[
\dot{Y}_2^{n+1} = \frac{4}{\Delta t^2}(Y_{2}^{n+1} - Y_2^n) - \frac{4}{\Delta t} \dot{Y}_2^n - \ddot{Y}_2^n,
\]

\[
Y_{1}^{n+1} = Y_1^n + \frac{\Delta t}{2} \dot{Y}_1^n + \frac{\Delta t^2}{2} \ddot{Y}_1^n,
\]

\[
Y_{2}^{n+1} = Y_2^n + \frac{\Delta t}{2} \dot{Y}_2^n + \frac{\Delta t^2}{2} \ddot{Y}_2^n.
\]
Figure 3: (a) Computational mesh used for the two side-by-side cylinders with the gap ratio $g^* = 4.0$; (b) Zoomed view near the cylinders.

In the above solution procedure, the hydrodynamic forces are required a priori, which should be provided from the flow fields. On the other hand, the displacements obtained from the motion equations modify the flow field boundaries as well as the flow-induced forces. A loose coupling strategy is employed for integrating the FSI system. The computational domain is a rectangular with a semi-circle ahead extending $50D$ downstream and a radius of $30D$ in front of the center of the two cylinders. A mesh-refinement study has been carried out, achieving the solution convergence. Three grid systems are adopted as shown in Table 1 for the demonstrated state $C$ with $K_1^* = 0.1$ and $K_2^* = 0.8$ ($U_2 = 4.9$). Grid I, II and III have 160, 200 and 240 nodes on the cylinder, respectively. Table 1 shows that the vibration amplitude and associated hydrodynamic coefficients ($C_{L,rms}$, $C_{L,mean}$, $C_{D,rms}$, $C_{D,mean}$) of both cylinders collapsed well for different grid densities, indicating good convergence properties with small variations ($< 6\%$) when varying mesh densities. Therefore, all the simulations presented hereafter will be based on the mesh characteristics of the selected Grid II. The sketch of the computational domain near the cylinders with mesh discretization and its closed view is given in Fig. 3. The non-dimensional time step size used in the simulation is fixed to be $0.002$.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Elements</th>
<th>Nodes</th>
<th>$A/D$ cylinder1</th>
<th>$A/D$ cylinder2</th>
<th>$C_{L,rms}$ cylinder1</th>
<th>$C_{L,rms}$ cylinder2</th>
<th>$C_{D,rms}$ cylinder1</th>
<th>$C_{D,rms}$ cylinder2</th>
<th>$C_{D,mean}$ cylinder1</th>
<th>$C_{D,mean}$ cylinder2</th>
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</thead>
<tbody>
<tr>
<td>I</td>
<td>41050</td>
<td>20776</td>
<td>0.5729</td>
<td>0.5321</td>
<td>0.5884</td>
<td>0.6034</td>
<td>-0.1344</td>
<td>0.1374</td>
<td>0.5354</td>
<td>0.4944</td>
</tr>
<tr>
<td>II</td>
<td>50140</td>
<td>25361</td>
<td>0.5679(0.8%)</td>
<td>0.5352(0.6%)</td>
<td>0.5897(0.2%)</td>
<td>0.6093(1%)</td>
<td>-0.1370(2.0%)</td>
<td>0.1392(1.3%)</td>
<td>0.5117(2.2%)</td>
<td>0.4684(5.3%)</td>
</tr>
<tr>
<td>III</td>
<td>66554</td>
<td>33608</td>
<td>0.5698(0.3%)</td>
<td>0.5336(0.3%)</td>
<td>0.5927(0.5%)</td>
<td>0.6106(0.2%)</td>
<td>-0.1413(3.2%)</td>
<td>0.1430(2.7%)</td>
<td>0.5252(2.6%)</td>
<td>0.4713(0.6%)</td>
</tr>
</tbody>
</table>

3. Model validation

The accuracy and applicability of the in-house code based on the developed finite element algorithm has been previously verified for different VIV problems in Bao et al. (2010, 2011, 2012). In this section, further validations are presented for the two cases: (i) VIV of an elastically mounted single cylinder at $Re = 100$ and (ii) VIV of two rigidly coupled circular cylinders ($K_2^* \to \infty$) in side-by-side arrangement at $Re = 150$. Apart from examining the applicability of numerical model, the first case also provides a benchmark for comparison in order to understand how the responses of the elastically coupled cylinders are different from that of a single cylinder, which is the focus of our
Figure 4: Variation of (a) the response amplitude and (b) the transverse force coefficient $C_{L,rms}$ with the reduced velocity for a single cylinder at $Re = 100, m^* = 10$.

Figure 5: Variation of the response amplitude with the reduced velocity for two rigidly coupled cylinders at $Re = 150, m^* = 2$.

The second test is selected from Zhao (2013), in which the FSI of two mechanically coupled cylinders arranged in a side-by-side configuration was numerically studied. The two cylinders are positioned in the same condition as that illustrated in Fig.1 at $m^* = 2.0$ with $C^* = 0.0$. The flow boundary conditions are also the same as the previous case as depicted in Fig.1. The comparison between the present results and those in Zhao (2013) is shown in Fig.5. The present results are greatly consistent with those of Zhao (2013), validating the current model for the FSI investigation of two mechanically coupled cylinders in side-by-side arrangement.
4. Numerical results

VIV responses of two elastically coupled circular cylinders in side-by-side arrangements at all three states are examined. In order to quantify the dynamic characteristics of the 2DOF coupled system, the reduced velocity is alternatively normalised in terms of either the first-mode or second-mode natural frequency of the system. In accordance to the variation of $f_n1$ or $f_n2$, the corresponding reduced velocity is varied in the range of $4.3 \leq U_{r1} \leq 10.3$ (State A), $4.1 \leq U_{r2} \leq 5.8$ (State B) and $4.0 \leq U_{r2} \leq 9.7$ (State C).

4.1. Vibration responses

Figs. 6-8 illustrate time histories of displacements of the two cylinders at the representative reduced velocities for both initial and lower branches in states A, B and C, respectively. In state A, the vibration amplitude of cylinder 2 is normally higher than that of cylinder 1 and the oscillations of the two cylinders are nearly in phase with each other, see Fig.6. In contrast, for state B, the vibration of cylinder 1 becomes stronger than that of cylinder 2 as shown in Fig.7. The responses of the two oscillating bodies are nearly anti-phase as the synchronization is excited. For state C, the two cylinders vibrate with nearly the same amplitude in an anti-phase manner, see Fig.8. The first impression from the time histories of displacements indicates that the dynamic responses are greatly affected by the natural properties of the coupled system. A more detailed statistics on the response amplitude are presented in Fig.9. In addition, multiple frequency components are usually implicated from the time traces in all the states depending on the reduced velocities. This normally indicates FSI phenomena with different frequency features. The frequency characteristics will be presented in the following sections.

In Fig.9, results for the three different states are plotted in comparison with those of the single cylinder. For state A of the coupling system, the maximum amplitude of cylinder 2 reaches $0.6D$ at $U_{r1} = 4.7$; in contrast, the amplitude of cylinder 1 is around $0.25D$ at the same $U_{r1}$. As we recall the results for the associated single cylinder at $U_r = 2.56$, the resonance is not excited in state A. Similarly for states B and C, the mechanical coupling of the two cylinders also causes a noticeable change in the system VIV responses. A resonance takes place when $U_{r2} > 4.6$ in state B, which is evidenced by the amplified vibrations of both cylinders. The maximum amplitude nearly reaches $0.6D$ for cylinder 1. The associated single-cylinder response is out of the synchronization regime at $U_r = 19.86$ for state C; however, the coupled cylinders experience the amplified oscillations when $4.6 < U_{r2} < 7.5$. In state C, contrary to states A and B, the vibration amplitudes are nearly the same for the two cylinders, being up to a level of $0.6D$.

It is noted that vibration responses of the single-cylinder vs. two-cylinders configurations reveal some similarities.
In particular, the vibration response of cylinder2 in state A and cylinder1 in state B follows a similar trend to that of the single cylinder, respectively. In state C, both cylinders in the 2DOF system behave like a single cylinder within the synchronization regime. More importantly, the highest vibration amplitude of the 2DOF system nearly reaches 0.6D, comparable to that of a single cylinder. It is reasonable to observe such a similar behavior in terms of the vibration amplitude. The reason is that the synchronization of the two configurations with the same mass ratio is excited by the similar vortex-shedding frequency at the same Re. On the other hand, the two coupled cylinders in the 2DOF system show variable responses at different states owing to the fact that the associated vibration modes are characterized...
distinctively at different states as discussed in detail later on.

It is interesting to note that the vibration mode of the 2DOF system varies, according to the different states, which can be further correlated to the free-vibration characteristics. The system dynamics has two normal modes of vibration corresponding to the two natural frequencies. During the lock-in, the first-mode natural frequency is excited for state A while the second-mode natural frequency is triggered for states B and C, which will be further confirmed by the frequency analysis. The associated mode of vibration for each natural frequency is featured by the amplitude ratio, $\rho_i (i = 1, 2)$, governing the free vibration of the two cylinders as shown in Fig.2. In Fig.11, we further compare the amplitude ratio, $A_1/A_2$, of the two cylinders against the corresponding free vibration mode when synchronization occurs. It can be observed that while $A_1/A_2$ collapses reasonably on $\rho_1$ for state A, $A_1/A_2$ for states B and C collapses on $\rho_2$ of the system. The phase difference between the two cylinder vibrations are also in accordance with the mode of vibration. Hence, the two cylinders are in-phase for state A as $\rho_1$ is positive and anti-phase for states B and C as $\rho_2$ is negative. This comparison further reveals that the 2DOF dynamics during a resonance is strictly dominated by the natural properties including natural frequencies and associated mode vibration. Evidently, the dynamic responses of the 2DOF system become more distinctive and to the author’s knowledge, this is the first time to quantitatively describe such fluid-structure interactions of the mechanically coupled system.

It is worth investigating the effect of gap ratio on the two-cylinder coupled responses as it is well known that the mutual wake interference between two side-by-side stationary cylinders may disappear when $g/D > 5.0$. In so doing, additional simulation with lower ($g/D = 3.0$) and higher ($g/D = 5.0$) gap ratios have been conducted whose results are plotted, in comparison with those of the main $g/D = 4.0$, in Fig.10 and 11 in terms of individual response amplitudes and amplitude ratios, respectively. By varying reduced velocities, Fig.10 reveals that, in each
lock-in at states A, B and C, the overall response patterns for cylinder 1 and cylinder 2 at different gap ratios appear to be similar with comparable maximum amplitudes. For state A, the initial-branch (lower-branch) response is seen to slightly increase (decrease) as \( g/D \) is decreased. For state B or C, the maximum responses around \( 4.4 < U_{r2} < 4.8 \) or \( U_{r2} = 4.5 \) appear to be slightly amplified as \( g/D \) is decreased. Nevertheless, these results suggest a small effect of the gap ratio on the VIV response of this high mass-ratio system in laminar flow with \( Re = 100 \). Fig.11 also shows how the amplitude ratios of individual cylinder responses collapse well onto the corresponding vibration modes in all three states, regardless of \( g/D \). Therefore, these results indicate a generic 2DOF feature for which the free-vibration characteristics of the coupled dynamic system prevail the lock-in fluid-cylinder interaction in this particular small gap range.

### 4.2. Hydrodynamic forces

Figs.12 and 13 illustrate the variations of mean and r.m.s. lift coefficients. Due to the asymmetric stiffness coupling, the mean lift coefficient \( C_{L,\text{mean}} \) are non-trivial, being positive for cylinder 2 and negative for cylinder 1 in all three states. Such behaviour is mainly owing to the flow interference effect between the adjacent bodies as similarly reported by Kang (2003) for two stationary cylinders in side-by-side arrangement.

It can be seen in Fig.13 that the r.m.s. lift coefficients of both cylinders in state A have a similar trend with the increasing reduced velocity; however cylinder 2 experiences stronger lift fluctuation as compared to cylinder 1 over the synchronization regime. For state B, cylinder 2 has an even larger \( C_{L,\text{rms}} \) on the initial branch while the lift on cylinder 1 experiences a significant fluctuation on the lower branch. For state C, the variation curves of \( C_{L,\text{rms}} \) for both cylinders appear similar to those in the single cylinder case, see Fig.13(d). At the transition from initial branch to lower branch (\( U_{r2} = 4.6 \)), cylinder 2 subject to an intensified lift fluctuations in states B and C. The corresponding time histories of the lift coefficients are also displayed in Fig.14, confirming the high-amplitude oscillations.

Figs.15 and 16 display the variations of mean and r.m.s. drag coefficients for comparison with those of a single cylinder. Contrary to the lift coefficients in states A and B, the cylinder with the higher amplitude has a sharp growth of the drag coefficient, which follows the single cylinder case. Specifically, for state A, \( C_{d,\text{mean}} \) of cylinder 2, of which the tendency is nearly the same with that of the single cylinder, suddenly jumps to nearly 2.2 at \( U_{r1} = 4.7 \) while that of cylinder 1 has an imperceptible change with increasing reduced velocity. For state B, it is found in Fig.15(b) that \( C_{d,\text{mean}} \) of cylinder 2, which also has a higher amplitude vibration, has a similar trend with the single cylinder, while that value of cylinder 1 is much lower. In state C, however, \( C_{d,\text{mean}} \) has a significant rise at the branch transition which is similar to the single cylinder case. As shown in Fig.16, it is obvious that the cylinder which has a larger mean drag also has a greater drag fluctuation. In particular, \( C_{d,\text{rms}} \) of cylinder 2 in state A, cylinder 1 in state B and those of both cylinders in state C follow well the curve of the single cylinder.

### 4.3. Frequency characteristics

Figs.17-19 show the power spectral densities (PSDs) of the displacement and lift force for previous representative reduced velocities in states A, B and C, respectively. In state A and at \( U_{r1} = 4.4 \), in which the amplitudes of both cylinders are low, the frequency of \( C_L \) has two adjacent peaks. One of them is nearly 0.17, close to the vortex shedding frequency of the flow over a stationary cylinder, while the other corresponds to the first-mode natural frequency of the system (\( f_{n1} = 0.22 \)). As compared with \( C_L \), the dominant frequency of \( Y \) is more discernible. Fig.17(c)(d) shows a distinct single peak at \( U_{r1} = 5.0 \), matching the lock-in region with a periodic vibration shown in Fig.6(b). The peak frequency is \( fD/U_\infty = 0.2 \), which is equal to \( f_{n1} \), confirming that state A locks on the first-mode oscillation at \( U_{r1} = 5.0 \). At \( U_{r1} = 7.2 \), the main frequency of \( Y \) is \( fD/U_\infty = 0.14 \), also associated with \( f_{n1} \). The corresponding frequency of \( C_L \) is 0.17, close to the vortex shedding frequency, \( S_{t0} \), i.e. frequencies of the cylinder displacement and lift coefficient are not the same even though the synchronization occurs for cylinder 2 as observed in Fig.6(c).

As shown in Fig.18, which illustrates PSDs of the selected reduced velocities in state B, at \( U_{r2} = 4.5 \), the frequency response behaves similarly to that of state A at \( U_{r1} = 4.4 \), showing the characteristics of non-synchronization state. The main reduced frequency of the response at \( U_{r2} = 4.6 \) and 5.5 is nearly 0.217(1/4.6) and 0.182(1/5.5), respectively, indicating that the vibration locks on \( f_{n2} \) in the synchronization regime. For state C, the same situation is revealed in Fig.19(c)-(f) when \( U_{r2} = 4.9 \) and 7.2 where the main frequency is equal to 0.205(1/4.9) and 0.139(1/7.2), respectively, confirming a lock-in state.

Fig.20 shows variations of the main frequencies of displacement and lift coefficient with the reduced velocity for both cylinders in comparison with the respective natural frequencies. It also reveals that in state A the vortex shedding locks on the first-mode natural frequency, while that in states B and state C locks on the second-mode natural frequency.
Figure 10: Variation of the response amplitude with the respective reduced velocity at different gap ratio for two elastically coupled cylinders: (a) State A, Cylinder1; (b) State A, Cylinder2; (c) State B, Cylinder1; (d) State B, Cylinder2; (e) State C, Cylinder1; (f) State C, Cylinder2.

when the synchronization occurs. The synchronization range in terms of the reduced velocity can also be estimated from this figure. For state A the synchronization starts approximately at $U_{r1} = 4.5$ and ends at $U_{r1} = 7.4$; for state B, it ranges between $U_{r2} = 4.5$ and $U_{r2} = 5.8$ while for state C the range is $4.5 \leq U_{r1} \leq 7.7$. The situation is slightly different for state B, since the overall reduced velocity range in this state is very narrow. As $K^*_2$ approaches zero, the system tends to decouple into a system which comprises a single cylinder and an elastically mounted cylinder with $U_r = 6$. We can expect that, even at this extreme condition, the lower cylinder would be synchronized with the vortex shedding of the flow. From the above observation it can be concluded that the lock-in state for the mechanically coupled system is normally narrower than that of the associated single cylinder.
Figure 11: Variation of the response amplitude ratio and the modes of vibration with the respective reduced velocity for two elastically coupled cylinders: (a) State A, $g/D = 3$; (b) State A, $g/D = 4$; (c) State A, $g/D = 5$; (d) State B, $g/D = 3$; (e) State B, $g/D = 4$; (f) State B, $g/D = 5$; (g) State C, $g/D = 3$; (h) State C, $g/D = 4$; (i) State C, $g/D = 5$.

4.4. Wake dynamics

Vortex shedding in the 2S mode was usually found for VIV of a single cylinder as reported, e.g. in Singh and Mittal (2005). However, the existing literature reveals that the interaction between multiple cylinders may lead to an intricate change of wake patterns. In the current work, the vortex shedding dynamics is also examined by scrutinizing the instantaneous vorticity ($\omega = \partial v/\partial x - \partial u/\partial y$) contours around the two cylinders for all three states. In general, the classical 2S mode is observed to be the predominant vortex shedding pattern for both cylinders. Nonetheless, several distinctions in the wake modes still exist at some specific reduced velocities. In this study, wake patterns associated with the peak responses for states A, B and C are presented.

Fig.21-23 show the vorticity contours at $U_{r1} = 4.7$ (state A), $U_{r2} = 4.9$ (state B) and $U_{r2} = 5.2$ (state C), respectively, exemplifying the lock-in responses in the corresponding states. Seven instants are marked in the time histories of the two lift force coefficients and cylinder displacements over a vortex-shedding cycle (Figs.21(h), 22(h) and 23(h)), in which the phase relationships between displacements and forces are also illustrated. In general, the wakes evolve into the parallel von Karman streets behind the cylinders for the states A, B and C.

In state A, the C(2S) pattern, which is featured by the far wake coalescence and common for a high-amplitude vibration (Singh and Mittal (2005)), is observed for cylinder2 whereas the wake behaves in a classical 2S mode behind cylinder1. For the C(2S) wake mode, the vortex formation crosses over the wake centerline, leading to the relatively higher mean (Fig.15(a)) and fluctuating (Fig.16(a)) drag forces acting on cylinder 2. From Fig.21(h), the associated lift force fluctuations of the two cylinders are in-phase, being consistent with the shedding phases of vortices entailing the in-phase oscillating cylinders (Fig.21(a)-(g)). On the contrary, in state B, the wake pattern behind cylinder1 displays the
Figure 12: Variation of the mean lift coefficient with the respective reduced velocity for two elastically coupled cylinders and the associated single cylinder: (a) State A; (b) State B and (c) State C.

C(2S) type associated with its greater response while the 2S mode evolves in the wake of cylinder 2. The associated lift force fluctuations are now anti-phase, leading to the anti-phase oscillations of the two cylinders. The lower cylinder 1 is subject to greater mean (Fig.15(b)) and fluctuating (Fig.16(b)) drag forces. In state C, the two cylinders undergo anti-phase vibrations of comparable responses subject to comparable mean (Fig.15(c)) and fluctuating (Fig.16(c)) drag forces. Therefore, the wake pattern appears in the C(2S) mode in an anti-phase symmetric fashion due to the anti-phase lift fluctuations yielding the anti-phase oscillating cylinders. Overall, results and observations on vorticity contours and hydrodynamic coefficients of the asymmetrically coupled oscillating cylinders in different states A-C suggest the dependence of wake patterns on the predominating response and drag force features regardless of the lift force fluctuations which, nevertheless, govern the phase relationships between the two wakes and the two oscillating cylinders.

4.5. Phase difference and effective added mass

As shown previously, both cylinder vibrations are periodic, sinusoidal in the synchronization region with the same oscillation frequency. A harmonic response hence may be assumed as follows:

\[ Y_i = \bar{Y}_i + \tilde{Y}_i \sin(2\pi f t + \phi_{Y_i}) \quad (i = 1, 2), \]

where \( \bar{Y}_i \) is the mean value, \( \tilde{Y}_i \) is the amplitude, \( \phi_{Y_i} \) is the phase angle of the displacement, respectively and \( f \) denotes the dominant frequency of the cross-flow vibration. Similarly, the lift coefficient for both cylinders can be expressed as

\[ C_{Li} = \bar{C}_{Li} + \tilde{C}_{Li} \sin(2\pi f t + \phi_{C_{Li}}) \quad (i = 1, 2), \]
Figure 13: Variation of the r.m.s. lift coefficient with the respective reduced velocity for two elastically coupled cylinders and the associated single cylinder: (a) State A; (b) State B and (c) State C.

Figure 14: Time trace of lift coefficient: (a) state B at $U_{r2} = 4.6$ and (b) state C at $U_{r2} = 4.6$. 
where $\overline{C_{Li}}$ is the mean value, $C_{Li}$ is the amplitude and $\phi_{C_{Li}}$ is the phase angle of the lift coefficient. The effective added mass coefficient for the two cylinders is obtained, based on the above assumptions, as:

$$C_{EAi} = \frac{\overline{C_{Li}} \cos(\phi_{C_{Li}} - \phi_{Y_i})}{2\pi^3 f_i^2 Y_i} \quad (i = 1, 2).$$ (15)

Fig.24 and Fig.25 displays the variation of phase difference and corresponding effective added mass as a function of respective reduced velocity within the synchronization regime for different states. The phase difference between $Y_2$ and $Y_1$ are also shown as it is $0^\circ$ for state A, and $180^\circ$ for states B and C, except $U_{r2} = 5.8$ in state B. The counterparts for a single cylinder are also included for comparison. As reported by Khalak and Williamson (1997), a single cylinder experiences a phase jump from $0^\circ$ to $180^\circ$ when the response transits from an upper branch to a lower branch. Similar behaviour is also evidenced in the present single cylinder case. However, a significant difference is noticed for the 2DOF system in comparison with the isolated system depending on the states considered.

In state A, the cylinder vibration and the lift oscillation behave nearly in an in-phase manner for lower reduced velocities for both cylinders; however, with increasing reduced velocity the deviation with respect to the in-phase oscillation becomes significant particularly for cylinder2. Specifically, the phase difference increases gradually up to $150^\circ$ as $U_{r1}$ reaches 6.3. Regarding the effective added mass, both cylinders follow the descending trend of the single cylinder as a function of reduced velocity. Note that $C_{EA}$ for cylinder1 is much larger as compared to cylinder2: this is reasonable because of the fact that in state A the vibration amplitude of cylinder1 is much smaller than that of cylinder2.

In contrast to state A, the vibration and lift in state B behave nearly in an anti-phase manner except for $U_{r2} = 5.8$. As a consequence, the sign of the effective added mass becomes negative for both cylinders, with the maximum value
of cylinder 2 is larger than that of cylinder 1. This may correlate to the second-mode vibration, in which the response of cylinder 2 is much weaker than cylinder 1.

As we turn the attention to state C, it is noticed that the phase differences for the two cylinders collapse on a single curve against the reduced velocities. The associated value is close to 180° except the case at $U_{r2} = 6.8$, where it drops down to nearly 0°. Similar collapse is also observed in the effective added mass. Nearly anti-phase oscillation of the lift against the response leads to the fact that the effective added mass is negative except for the case at $U_{r2} = 6.8$, showing a similar trend as displayed in state B. However, the absolute value of the effective added mass, i.e. $|C_{EA}|$, reduces gradually with increasing reduced velocity.

5. Conclusions

VIV responses of two mechanically coupled cylinders with asymmetrical stiffness in side-by-side arrangements in uniform laminar flows are numerically investigated at a low Reynolds number of 100. The lower cylinder is elastically mounted on a wall and the upper cylinder is connected to the lower one through an elastic spring. The stiffness of the lower spring is chosen such that the associated isolated single cylinder undergoes three different VIV scenarios: the presynchronization (state A), synchronization (state B) and post-synchronization (state C). For each state, the stiffness of the spring that connects the two cylinders is varied to capture the effects of asymmetrical coupling mechanism. Overall, key observations on the relationships between the lock-in behaviours, hydrodynamic lift-drag forces and wake modes are summarized in Table 2 and in the following.

Vibration responses of the 2DOF system are first examined. For state A, the upper cylinder exhibits a much higher amplitude vibration, which is similar to the results of a single cylinder. For state B, the maximum amplitude of the upper
cylinder is substantially reduced while the lower cylinder is excited with amplified vibration when the synchronization occurs. For state C, the response diagrams of both cylinders have a similar trend to that of the single cylinder. The amplitude ratio of the two cylinders collapses on the first-mode vibration for state A and the second-mode vibration for states B and C. It is identified that VIV in state A locks on the first-mode natural frequency, while it is synchronized with the second-mode natural frequency in states B and C. This behavior indicates that the dynamic responses during the lock-in are predominantly governed by the associated natural free-vibration properties and modes vibration of the mechanically coupled 2DOF system.

The hydrodynamic forces exerted on the two cylinders are characterized in detail. It is found that the mean lift of the two cylinders is primarily affected by the flow interference, which is similar to that of the stationary side-by-side cylinders. The fluctuations of lift force as a function of reduced velocity follow a trend of a single cylinder, indicating that mechanical coupling has a lesser impact on the lift fluctuations. On the other hand, the drag forces are more sensitive to the dynamic responses of the coupled system, evidenced by a significant deviation of the mean and r.m.s. drag components between the two cylinders at different states.

As a final remark, the mechanical coupling plays an important role in the determination of phase difference between the response and the lift force. The two cylinder vibrations respond nearly in an in-phase manner against the lift force in state A, while they respond nearly in an anti-phase manner against the lift force in states B and C. The effective added mass coefficients are variable in three different states as they are predominantly positive in state A but negative in states B and C. Nevertheless, their absolute values generally decrease with increasing reduced velocities in all the coupled VIV states.

Figure 17: Power spectral density functions of the response displacement and hydrodynamic lift coefficient at representative reduced velocities in state A: (a) $Y, U_{r1} = 4.4$; (b) $C_L, U_{r1} = 4.4$; (c) $Y, U_{r1} = 5.0$; (d) $C_L, U_{r1} = 5.0$; (e) $Y, U_{r1} = 7.2$ and (f) $C_L, U_{r1} = 7.2$. 
Figure 18: Power spectral density functions of the response displacement and hydrodynamic lift coefficient at representative reduced velocities in state B: (a) $Y$, $U_r^2 = 4.5$; (b) $C_L$, $U_r^2 = 4.5$; (c) $Y$, $U_r^2 = 4.6$; (d) $C_L$, $U_r^2 = 4.6$; (e) $Y$, $U_r^2 = 5.5$ and (f) $C_L$, $U_r^2 = 5.5$.

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6. Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

7. Credit authorship contribution statement

Figure 19: Power spectral density functions of the response displacement and hydrodynamic lift coefficient at representative reduced velocities in state C: (a) $Y_1, U_{r2} = 4.5$; (b) $C_{L1}, U_{r2} = 4.5$; (c) $Y_2, U_{r2} = 4.9$; (d) $C_{L1}, U_{r2} = 4.9$; (e) $Y_1, U_{r2} = 7.2$ and (f) $C_{L1}, U_{r2} = 7.2$.

Zhaolong Han: Resources, Supervision.

References


Figure 20: Variation of the non-dimensionalized main response and force frequency as a function of the respective reduced velocity for two elastically coupled cylinders: (a) state A; (b) state B and (c) state C.


Figure 21: (a)-(g) Instantaneous vorticity fields for the fully developed flow past two elastically coupled cylinders at $U_{r1} = 4.7$ in state A.

Figure 22: (a)-(g) Instantaneous vorticity fields for the fully developed flow past two elastically coupled cylinders at \( U_{r2} = 4.9 \) in state B.
Figure 23: (a)-(g) Instantaneous vorticity fields for the fully developed flow past two elastically coupled cylinders at $U_{r2} = 5.2$ in state C.
Figure 24: Variation of the phase between the response displacement and the lift force $\phi_{CL} - \phi_Y$ with respective reduced velocity: (a) State A; (b) State B; (c) State C and (d) Single cylinder.
Figure 25: Variation of the effective added mass coefficient with respective reduced velocity: (a) State A; (b) State B; (c) State C and (d) Single cylinder.
Table 2: Summary of VIV characteristics for side-by-side cylinders with asymmetrically coupled stiffness

<table>
<thead>
<tr>
<th>State</th>
<th>Lock-in Behaviours</th>
<th>Hydrodynamic Forces</th>
<th>Wake Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>• Lock-in of first coupled mode.</td>
<td>• Similar lift force fluctuation for both cylinders.</td>
<td>2S (lower cylinder) and C(2S) (upper cylinder) modes at peak response</td>
</tr>
<tr>
<td></td>
<td>• In-phase responses of side-by-side cylinders.</td>
<td>• Greater mean and fluctuating drag forces for upper cylinder.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Maximum $A/D \approx 0.6D$ for upper cylinder.</td>
<td>• Predominantly positive values of effective added mass coefficients for both cylinders</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Similar lift force fluctuation for both cylinders.</td>
<td>• Greater lift force fluctuation for lower cylinder in initial branch.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Greater mean and fluctuating drag forces for lower cylinder.</td>
<td>• Predominantly negative values of effective added mass coefficients for both cylinders</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>• Lock-in of second coupled mode.</td>
<td>• Similar lift forces fluctuation for both cylinders, except for the initial-to-lower branch transition.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Anti-phase responses of side-by-side cylinders.</td>
<td>• Comparable mean and fluctuating drag forces for both cylinders.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Maximum $A/D \approx 0.6D$ for lower cylinder.</td>
<td>• Predominantly negative values of effective added mass coefficients for both cylinders</td>
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</tr>
<tr>
<td></td>
<td>• Greater lift force fluctuation for lower cylinder in initial branch.</td>
<td>C(2S) mode for both cylinders at peak response</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>• Lock-in of second coupled mode.</td>
<td>• Similar lift forces fluctuation for both cylinders, except for the initial-to-lower branch transition.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Anti-phase responses of side-by-side cylinders.</td>
<td>• Comparable mean and fluctuating drag forces for both cylinders.</td>
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</tr>
<tr>
<td></td>
<td>• Maximum $A/D \approx 0.6D$ for both cylinder.</td>
<td>• Predominantly negative values of effective added mass coefficients for both cylinders</td>
<td></td>
</tr>
<tr>
<td>ALL</td>
<td>Subject to modal amplitude ratios and natural frequencies of mechanically coupled cylinders.</td>
<td>Positive and negative mean lift coefficients for upper and lower cylinders, respectively.</td>
<td>Depending on both cylinder amplitudes and their relative phases.</td>
</tr>
</tbody>
</table>