Motion of micron–size particles in turbulent helium II

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Abstract

The surprising result of a recent experiment in turbulent helium II is that micron–size tracer particles move with about half the speed of the imposed normal fluid. We develop a theory of the interaction of small spheres with quantized vortices and predict that the particles slip velocity (resulting from the balance of buoyancy and Stokes drag forces) must be corrected by an amount which is proportional to the normal fluid velocity, in quantitative agreement with the observations.

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I. INTRODUCTION

The recent success in applying the Particle Image Velocimetry technique in liquid helium [1, 2] has opened the way to better visualization and more detailed studies of superfluid turbulence [3], a subject which is receiving increasing theoretical [4] and experimental [5] attention both in ^{4}He and in ^{3}He-B [6, 7]. It is apparent from these initial visualization experiments that progress will depend on properly understanding the interaction of the micron–size tracer particles with the normal fluid, the superfluid and the quantized vortices. The aim of this brief report is to shed light into a striking effect which has been recently discovered by Zhang and Van Sciver [2], that in counterflow turbulence the tracer particles move with half the average speed of the normal fluid. Counterflow turbulence is perhaps the most studied form of turbulence in helium II (although there are still many problems which need to be solved) and has important applications of cryogenics engineering, so it is a useful testing ground.

The experimental set up consists of a vertical channel which is closed at the bottom end and opened to the helium bath at the top end. At the bottom end a resistor dissipates a known heat flux q. Let v_n, ρ_n and v_s, ρ_s be respectively the velocity and the density of the normal and superfluid components, where $\rho = \rho_n + \rho_s$ is the total density, T the absolute temperature and s the specific entropy. The heat is carried away from the resistor by the normal fluid, $v_n = q/(\rho s T)$, and the total mass flux is zero, $\rho_n v_n + \rho_s v_s = 0$, so a superflow is induced towards the resistor, $v_s = -\rho_n v_n/\rho_s$. In this way a relative counterflow $v_{ns} = v_n - v_s$ of the two fluid components is set up which is proportional to the applied heat flux, $v_{ns} = q/(\rho_s s T)$, and to the normal fluid velocity, $v_{ns} = (\rho/\rho_s)v_n$. It is well known [8] that if q exceeds a small critical value the heat transfer becomes turbulent and a tangle of superfluid vortex lines is created. The tangle is characterized by the superfluid vortex line density L (length of vortex line per unit volume), which is usually measured by detecting the extra attenuation of a second sound wave across the channel. Experiments [9] and numerical simulations [10] show that $L = \gamma^2 v_{ns}^2$ where γ is a temperature dependent parameter.

Zhang and Van Sciver [2] performed experiments in a wide channel of 4.3×1.95 cm^2 cross section and 20cm length in the temperature range from 1.62 to 2.0 K; the applied heat flux ranged from 0.11×10^7 to 1.37×10^7 erg/(cm^2 s) (110 to 1370 milliwatts per square centimeter). Small polymer microspheres with mean radius $a = 0.85 \times 10^{-4}$ cm and density $\rho_p = 1.1$ g/cm^3 were visualized by lasers using the PIV technique. Since $\rho_p > \rho \approx 0.1415$ g/cm^3 the particles sedimented under gravity. As expected, if $q = 0$ and the normal fluid was at rest, after an initial transient the particles fell with terminal speed v_{slip} resulting from the balance of Stokes drag and gravity. If $q \neq 0$, Zhang and Van Sciver expected the particle velocity, v_p to be $v_p = v_n - v_{slip}$; surprisingly, they observed that the particles' velocity was less, $v_p = v_n - v_{slip} - v_{add}$, by an amount v_{add}. Thus, calling $v_{pa} = v_p + v_{slip}$ the adjusted particle velocity, their main result was

$$v_{pa} = v_n - v_{add}, \tag{1}$$

rather than the expected $v_{pa} = v_n$. Furthermore, Zhang and VanSciver found that v_{add} was proportional to the heat flux (hence to v_n) and that the ratio v_p/v_n was approximately 1/2 independently of temperature, as shown in Fig. 1. It is this effect which we set out to explain.

II. MODEL

In a previous paper [11] we determined that, under a number of reasonable assumptions which are justified in most experimental conditions, the equations of motion of a small sphere of radius a, density ρ_p, position $\mathbf{r}_p(t)$ and velocity $\mathbf{v}_p(t)$ are:

$$\begin{aligned}\rho_p \vartheta \frac{d\mathbf{v}_p}{dt} &= 6\pi a \mu (\mathbf{v}_n - \mathbf{v}_p) + \vartheta(\rho_p - \rho)\mathbf{g} \\ &+ \rho_n \vartheta \frac{D\mathbf{v}_n}{Dt} + C\rho_n \vartheta \left(\frac{D\mathbf{v}_n}{Dt} - \frac{d\mathbf{v}_p}{dt}\right) \\ &+ \rho_s \vartheta \frac{D\mathbf{v}_s}{Dt} + C\rho_s \vartheta \left(\frac{D\mathbf{v}_s}{Dt} - \frac{d\mathbf{v}_p}{dt}\right),\end{aligned} \tag{2}$$

and $d\mathbf{r}_p/dt = \mathbf{v}_p$, where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal fluid and superfluid velocities, $C = 1/2$, $\vartheta = 4\pi a^3/3$, μ is the viscosity of liquid helium II, $\mathbf{g}$ the gravitational acceleration and t time.

If the counterflow is in the vertical direction and we assume uniform, time–independent normal fluid and superfluid velocity profiles, the acceleration of the sphere is simply

$$\begin{aligned}\rho_p \vartheta \frac{dv_p}{dt} &= 6\pi a \mu (v_n - v_p) - \vartheta(\rho_p - \rho) g \\ &\quad - C\rho_n \vartheta \frac{dv_p}{dt} - C\rho_s \vartheta \frac{dv_p}{dt}.\end{aligned} \tag{3}$$

Before we proceed we seek justification for the approximations which we have introduced. Firstly, the form of Stokes drag which we use (the first term at the right hand side of Eq. 2) assumes that the normal flow around the sphere is laminar. Fig. 2 shows that this is indeed the case in in the $(T,\, q)$ experimental range of Zhang and Van Sciver [2] because the Reynolds number of the normal fluid based on the sphere's radius, $Re = v_n a \rho_n/\mu$, is of order unity. Secondly, the vortex tangle is in a statistical steady state, so $D\mathbf{v}_s/Dt$ is not zero; we estimate $|D\mathbf{v}_s/Dt| \approx v_\delta/\tau_\delta$ where $v_\delta \approx \kappa/(2\pi\delta)$ and $\tau_\delta \approx 2\pi\delta/v_\delta$ are respectively the superflow velocity at the typical distance $\delta \approx L^{-1/2}$ from one vortex line to another and the time taken by a vortex line to rotate around another line. Since we know from the experiment [2] that $v_p \approx v_n/2$, the ratio, c, of the superfluid inertial force, $|\rho_s \vartheta D\mathbf{v}_s/Dt|$ and the Stokes drag force, $|6\pi a\mu(\mathbf{v}_n - \mathbf{v}_p)|$, is approximately $c \approx \rho_s a^2 \kappa^2/(18\pi^3 \mu v_n \delta^3)$. In the temperature and heat flux range of the experiment, we find $10^{-6} < c < 10^{-3}$, so we can safely neglect the term $D\mathbf{v}_s/Dt$ in our analysis. Thirdly, we neglect $D\mathbf{v}_n/Dt$ because there is no direct evidence that the normal fluid is turbulent, although this possibility has been raised theoretically [12].

The terminal velocity of the sphere is obtained by letting $dv_p/dt = 0$ in (3):

$$6\pi a\mu(v_n - v_p) - \vartheta(\rho_p - \rho)g = 0, \tag{4}$$

hence the sphere's velocity differs from the normal fluid velocity by the slip velocity, defined as

$$v_n - v_p = v_{slip} = \frac{2a^2}{9\mu}(\rho_p - \rho)g. \tag{5}$$

We now consider the possibility that, as the sphere moves through the tangle, it reconnects with vortex lines, as envisaged in [11]. We argue that although a vortex line which attaches to the sphere may later disconnect from it, on the average the sphere is likely to have one or more vortex loops attached. It is also reasonable to expect that, in the experiment of Zhang and VanSciver, vortex reconnections with the sphere are not very frequent and happen one at the time, because the radius of the sphere is much smaller than the typical intervortex spacing δ, as shown in Fig. (3).

Two forces arise from the presence of a vortex near (or attached to) the sphere. The first, caused by the non–uniform distribution of pressure around the sphere, is

$$\mathbf{F} = \int_S p\hat{\mathbf{n}} dS, \tag{6}$$

where p is the pressure, S is the surface of the sphere and $\hat{\mathbf{n}}$ is the radial unit vector normal to S and directed into the fluid. In another context, Schwarz[13] calculated this expression and found that

$$\mathbf{F} = \frac{\rho_s}{2} \int_S (\mathbf{v}_\ell + \mathbf{v}_b)^2 \hat{\mathbf{n}} dS, \tag{7}$$

Here $\mathbf{v}_b$ is a contribution arising from the boundary condition that the normal component of the total superfluid velocity at S vanishes and $\mathbf{v}_\ell$ is the velocity field around the vortex line. Given an arbitrary vortex configuration, the determination of $\mathbf{v}_b$ requires a numerical calculation; however, if the radius of curvature of the vortex is sufficiently larger than the radius of the sphere, then $\mathbf{v}_b$ is negligible, because the velocity field $\mathbf{v}_\ell$ at S is approximately tangential to S. In this approximation, the force becomes:

$$\mathbf{F} \approx (\frac{\rho_s}{2}) \int_S (\mathbf{v}_\ell)^2 \hat{\mathbf{n}} dS \approx (\frac{\rho_s}{2}) 2\pi \Big(\int_\xi^a \Big(\frac{\kappa}{2\pi r} \Big)^2 r dr \Big) \hat{\mathbf{n}}_0, \tag{8}$$

where we have used a local Cartesian approximation to evaluate the surface integral and introduced radial cutoffs a and $\xi \approx 10^{-8}$ cm (the vortex core radius). Note that $\hat{\mathbf{n}}_0$ is the normal unit vector along one vortex strand pointing out of the plane which represents S. Thus the force, resulting from the non–uniform pressure distribution which brings sphere and vortex together, is:

$$\mathbf{F} \approx \frac{\rho_s \kappa^2}{4\pi} \ln{(a/\xi)} \hat{\mathbf{n}}_0, \tag{9}$$

The generalisation to N vortices attached to the sphere is:

$$\mathbf{F} \approx \frac{\rho_s \kappa^2}{4\pi} \ln{(a/\xi)} \sum_{i=1}^{N} \hat{\mathbf{n}}_i. \tag{10}$$

A schematic example for two vortex strands is shown in Fig.4. Clearly, if the configuration of vortices is symmetric, the net force will be zero, as in Fig. 5(a) and (b). If one or more vortex loops are asymmetrically attached to the sphere, as in Fig. 5(c) and (d), the contributions from individual vortices will not cancel out, and the pressure distribution cause a net body force which attracts the sphere to the vortices, thus opposing the motion of the sphere.

There are two limiting cases to consider: $a \ll \delta$ (dilute case) and $a \gg \delta$ (dense case). In the first case, which is relevant to the experiment of Zhang and Van Sciver [2] as shown in Fig. 3, it is likely that the sphere, as it moves through the tangle, carries along one or more vortex lines or even separate loops as the result of previous close encounters with vortices. The simplest scenario is that the sphere, after connecting to a vortex line, keeps moving, dragging a vortex loop (two attachments) along for a fraction of the relative distance to the next vortex line of the network with respect to its own size. We thus expect that the force in the direction opposite to the motion of the sphere is a body force which has magnitude

$$F \approx \frac{\rho_s \kappa^2}{4\pi} \ln(a/\xi) \left(\frac{2\beta a}{\delta}\right), \tag{11}$$

where β is a geometrical factor of the order unity which depends on the number of pair of vortices attached to the sphere and the relative distance of travel where they remain attached. In the case of a single vortex, we can interpret $2\beta a$ as the length of this vortex.

The second force on the sphere arises from the drag of the attached vortex loop with the normal fluid. It is possible that, under the action of this force, the vortex loop slides around the sphere or changes its size, but it reasonable to expect that, on the average, some vortex length will always be attached to the sphere. The friction force has magnitude approximately equal to $\gamma_0 \ell (v_n - v_\ell)$ where γ_0 is a known temperature–dependence friction coefficient[14]; setting $v_\ell = v_p$ (as vortex and particle move together) and interpreting $\ell = 2\beta a$, the friction is thus $2\beta a \gamma_0 (v_n - v_p)$. Finally, it should be mentioned that if a vortex is attached to a particle at one end and to another particle (or a wall) at the other end, there would be a tension force. In the case of experimental interest (dilute system of particles, large experimental cell compared to particle size and intervortex separation) this effect is not important.

Adding the body force and the friction force to Eq. 4 we have

$$v_n - v_p = v'_{slip} + v_{add}\,, \tag{12}$$

where $v'_{slip} = v_{slip}/f$, $f = 1 + \beta\gamma_0/(3\pi\mu)$ and

$$\begin{aligned} v_{add} &= \frac{\beta \kappa^2 \rho_s}{12\pi^2 f \mu \delta} \ln(a/\xi) = \\ &= \frac{\beta \kappa^2 \gamma \rho}{12\pi^2 f \mu} \ln(a/\xi) v_n \end{aligned} \tag{13}$$

because $1/\delta \approx L^{1/2} = \gamma v_{ns} = \gamma \rho v_n / \rho_s$. Substituting into Eq. 12 we obtain

$$v_{pa} = \Big(\frac{f-1}{f}\Big) v_{slip} + \Big(1 - \frac{\beta \kappa^2 \gamma \rho \ln{(a/\xi)}}{12\pi^2 \mu f}\Big) v_n \, . \tag{14}$$

III. RESULTS

The first term at the right hand side of Eq. 14, $v_{slip}(f-1)/f$, is negligible, because in the (q,T) range of interest $0.3 < (f-1)/f < 0.4$ and v_{slip} (of the order of few mm/s) is much smaller than v_n. We conclude that v_{pa} is essentially proportional to v_n. Values [16] of v_{pa} versus v_n computed using Eq. 14 for $\beta = 4.5$ over the independent ranges of T and q used by Zhang and Van Sciver [2] are shown in Fig. 5, which must be compared with Fig. 1. The agreement is good. Even without adjusting the value of the undetermined geometrical parameter β we would have order–of–magnitude agreement with the observed result, which is remarkable, given the relative simplicity of our model. More importantly, our result has the same linear dependence of v_{pa} on v_n and the same temperature independence of the slope v_{pa}/v_n which was observed in the experiment[2]. The value of β which best fit the observed slope ($\beta = 4.5$) suggests that the loops which remain attached to the particles are not big.

From Eq. 13 we have also

$$\frac{v_{add}}{q} = \frac{\beta \kappa^2 \ln{(a/\xi)}}{12\pi^2} \Big(\frac{\gamma}{\mu f s T}\Big), \tag{15}$$

which shows that the temperature dependence of the ratio v_{add}/q is that of the combination $\gamma/(\mu f s T)$; this quantity decreases with increasing T, as indeed found in the experiments [2]. The agreement is only qualitative, however, because the temperature dependence of $\gamma/(\mu f s T)$ scales approximately as $T^{-4.5}$, whereas v_{add}/q in the experiments scales as $T^{-6.6}$.

The other case to consider is the limit of very dense vortex tangle, for which $a \gg \delta$. In this case we expect that reconnections between sphere and vortices happen all the time, that is the sphere is always attached to vortex lines. The force is likely to have the form

$$F \approx \frac{\rho_s \kappa^2}{4\pi} \ln{(a/\xi)} 2\beta \Big(\frac{a}{\delta}\Big)^2, \tag{16}$$

where β is again a geometrical factor, and the term $(a/\delta)^2$ represents the geometrical cross-section of the sphere with the network of vortices. Proceeding as before, we prdict a different dependence of v_{pa} on v_n, a and T:

$$v_{pa} = \left(\frac{f-1}{f}\right)v_{slip} + v_n\left(1 - \frac{\beta a\kappa^2\gamma^2\rho^2 \ln{(a/\xi)}}{24\pi^2\mu\rho_s}v_n\right), \tag{17}$$

IV. CONCLUSION

In conclusion, we have developed a simple model of the interaction of small particles with a turbulent counterflow in He II. The model takes into account the interaction of the particles with the quantized vortices. In the dilute limit $a \ll \delta$ which applies to experiment of Zhang and VanSciver[2], we assume that small vortex loops remain attached to the particles as they move through the tangle. We have found that the adjusted terminal velocity of the particles is only half of the value, v_n, which would arise from a balance of buoyancy and Stokes drag which ignore the quantized vortices. The discrepancy, v_{add}, is due to the interaction of the particles with the quantized vortices, and is proportional to v_n independently of temperature, in agreement with the observations of Zhang and VanSciver [2]. Indeed, Zhang and VanSciver realized that a body force, not a pure friction force, must be responsible for the observed effect; our work provides a microscopic justification to their insight. Finally, we also predict that in the dense limit that $a \gg \delta$ the scaling is quadratic rather than linear. Numerical simulations are in progress to investigate the details of the close interaction of particles and quantized vortices[17].

V. ACKNOWLEDGEMENTS

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[1] R. J. Donnelly, A. N. Karpetis, J. J. Niemela, K. R. Sreenivasan, and W. F. Vinen, J. Low Temp. Physics **126**,327 (2002); D. Celik and S. W. Van Sciver, Exp. Therm. Fluid Sci. **26**, 971 (2002); T. Zhang and S. W. VanSciver, in *Advances in Cryogenics Engineering*, edited by S. Breon *et al.*, AIP Conf. Proc. No. 613 (AIP, Melville, NY, 2002), p. 1372; T. Zhang, D. Celik, and S. W. VanSciver, J. Low Temp. Phys. **134**, 985 (2004); T. Zhang and S. W.

Van Sciver, Nature Physics **1**, 36 (2005); G. P. Bewley, D. P. Lathrop, and K. R. Sreenivasan, Nature **44**, 588 (2006).

[2] T. Zhang and S. W. Van Sciver, J. Low Temp. Phys. **138**, 865 (2005).

[3] R. J. Donnelly, *Quantized Vortices in Helium II*, Cambridge University Press, Cambridge, UK, (1991).

[4] W. F. Vinen, Phys. Rev. B **64**, 134520 (2001); M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **94**, 065302 (2005); W. F. Vinen, Phys. Rev. B **71**, 024513 (2005); S. Nemirowskii, Phys. Rev. B **57**, 5972 (1996); C.F. Barenghi and L. Skrbek, submitted (2006).

[5] A.P. Finne, T. Araki, R. Blaauwgeers, V.B. Eltsov, N.B. Kopnin, M. Krusius and G.E. Volovik, Nature **424**, 1022 (2003); H. A. Nichol, L. Skrbek, P. C. Hendry, and P. V. E. McClintock, Phys. Rev. E **70** 056307 (2004).

[6] A. P. Finne, V. B. Eltsov, G. Eska, R. Hanninen, J. Kopu, M. Krusius, E. V. Thuneberg, and M. Tsubota, Phys. Rev. Lett. **96**, 085301 (2006).

[7] D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M.Guenault, R. P. Haley, C. J. Matthews, G. R. Pickett, V. Tsepelin and K. Zaki, Phys. Rev. Lett. **96**, 035301 (2006).

[8] W. F. Vinen, Proc. Roy. Soc. (London) A **242**, 493 (1957).

[9] J. T. Tough, *Superfluid turbulence*, in Prog. in Low Temp. Phys., Vol.VIII, North-Holland Publ. Co., (1982), p. 133.

[10] K. W. Schwarz, Phys. Rev. B **38**, 2398 (1988).

[11] D. R. Poole, C. F. Barenghi, Y. A. Sergeev, and W.F. Vinen, Phys. Rev. B **71**, 064514 (2005).

[12] D. J. Melotte and C. F. Barenghi, Phys. Rev. Lett. **80**, 4181 1998.

[13] K. W. Schwarz, Phys. Rev. A **10**, 2306 (1974).

[14] C.F. Barenghi. R.J.Donnelly and W.F. Vinen, J. Low Temp. Physics **52**, 189 (1982).

[15] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).

[16] The values of ρ_s, ρ_n, ρ, μ and s are from tables in: R. J. Donnelly and C. F. Barenghi, J. Phys. Chem. Ref. Data **27**, 1217 (1998); the values of γ, calculated by K. W. Schwarz [10], are from a table in Ref. [9]: $\gamma = 166$, 193 (interpolated), 219, 262 (interpolated), and 304 s/cm^2 respectively at $T = 1.6$, 1.7, 1.8, 1.9 and 2.0 K. γ_0 is obtained from Ref. [14]. Finally the vortex core radius is $\xi = 10^{-8}$ cm.

[17] C.F. Barenghi, D. Kivotides and Y. Sergeev, in preparation.

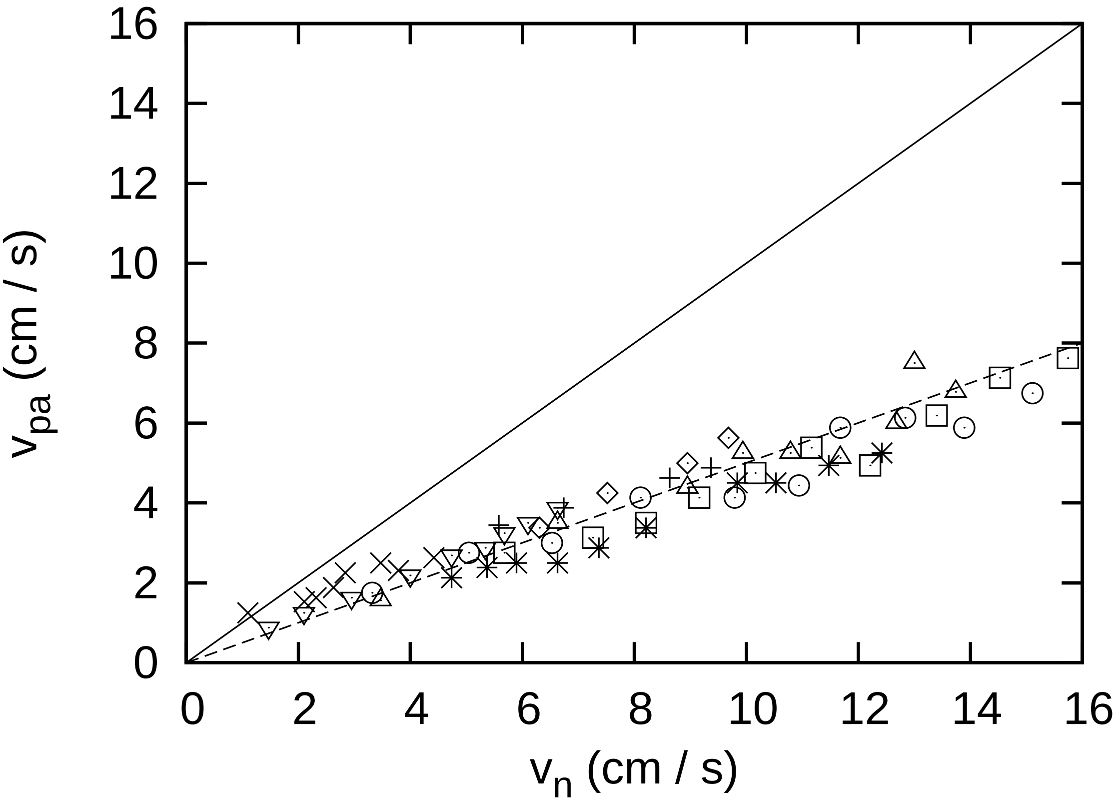

FIG. 1: Experimental data of Zhang and VanSciver [2] (reproduced to graphical accuracy): v_{pa} versus v_n at the following temperatures: $T = 1.62$ K (squares), $T = 1.65$ K (circles), $T = 1.68$ K (triangles pointing up), $T = 1.70$ K (stars), $T = 1.77$ K (diamonds), $T = 1.80$ K (crosses), $T = 1.90$ K (triangles pointing down) and $T = 2.0$ K (diagonal crosses). The solid line is $v_{pa} = v_n$ and the dashed line is $v_{pa} = 0.5v_n$. Note that v_{pa} is proportional to v_n and temperature independent.

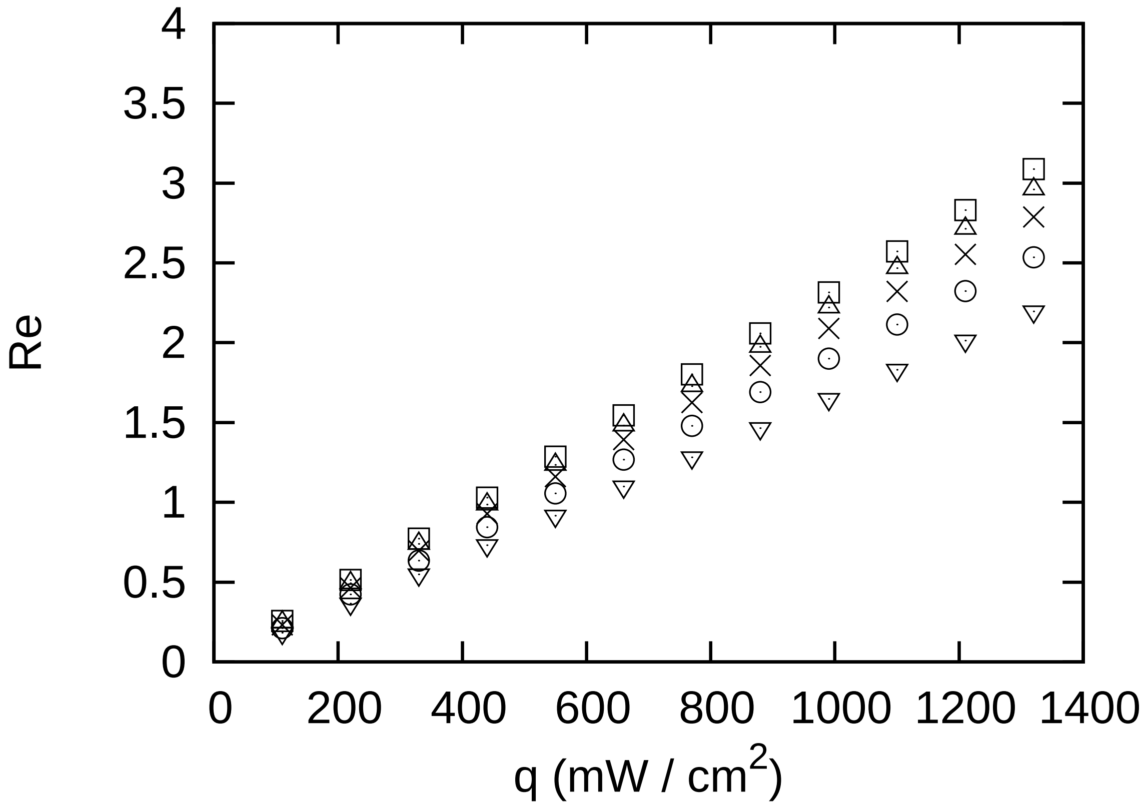

FIG. 2: Computed particle Reynolds numbers Re in the parameter range of the experiment of Zhang and VanSciver [2] versus heat flux q at the following temperatures: $T = 1.6$ K (squares), $T = 1.7$ K (triangles pointing up), $T = 1.8$ K (crosses), $T = 1.9$ K (circles), $T = 2.0$ K (triangles pointing down). Note that Re is of order unity.

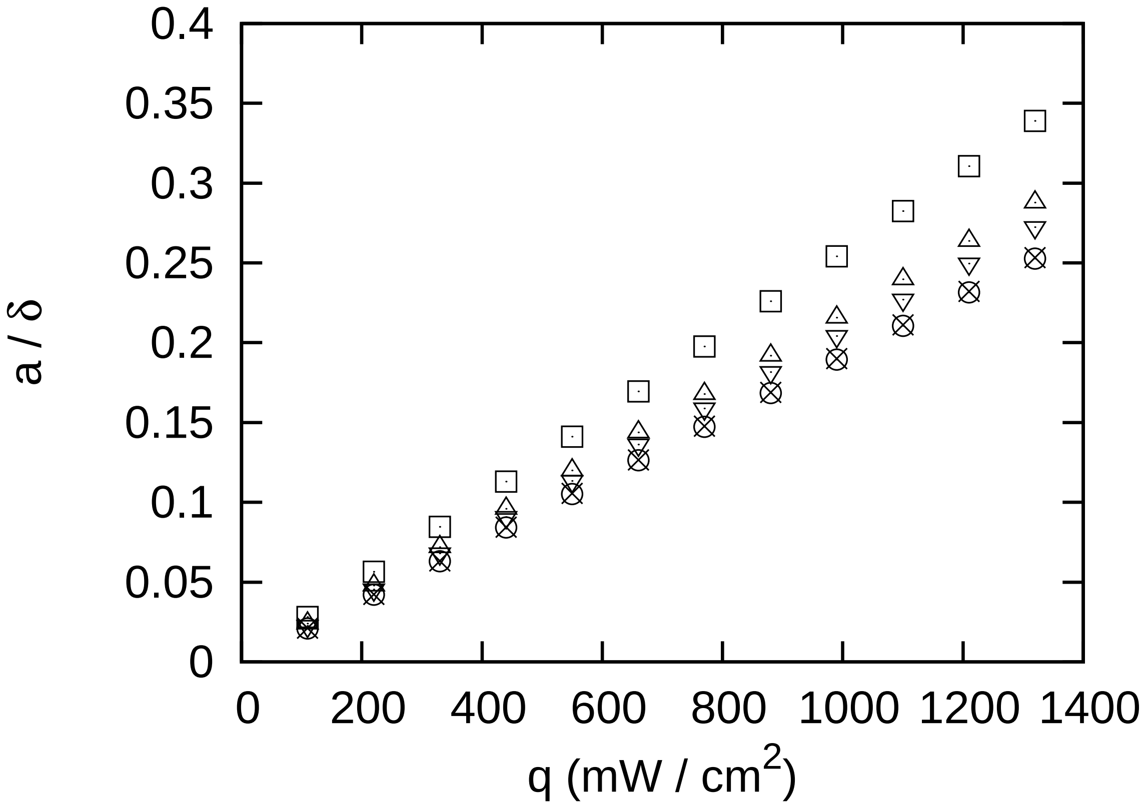

FIG. 3: Ratio a/δ of particle radius to vortex separation versus heat flux, q, in the experiment of Zhang and VanSciver [2] at different temperatures (labels as in Fig. 1). Note that $a/\delta < 1$

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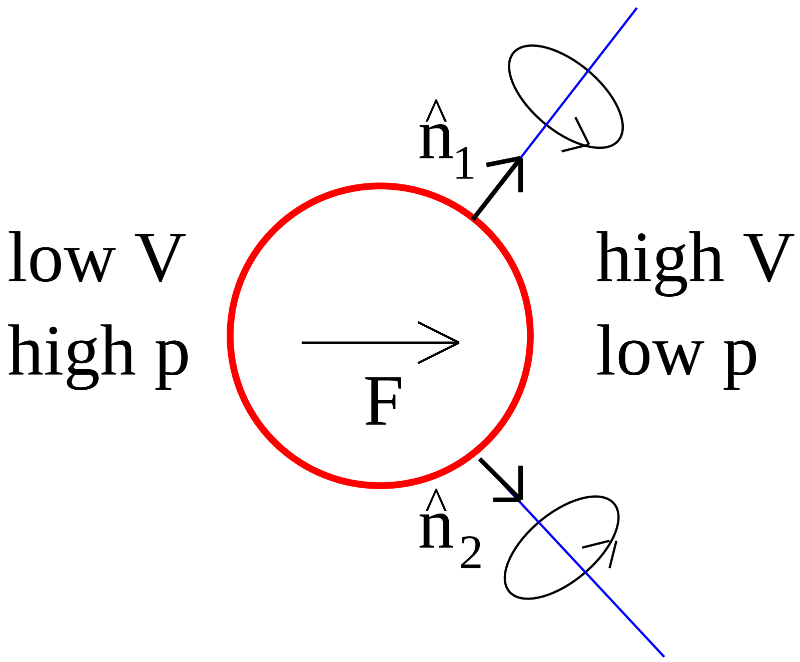

FIG. 4: (Colour online). Schematic of force arising from two attached vortex strands. The vortices have the same polarity because they originate from one vortex which has reconnected with the sphere.

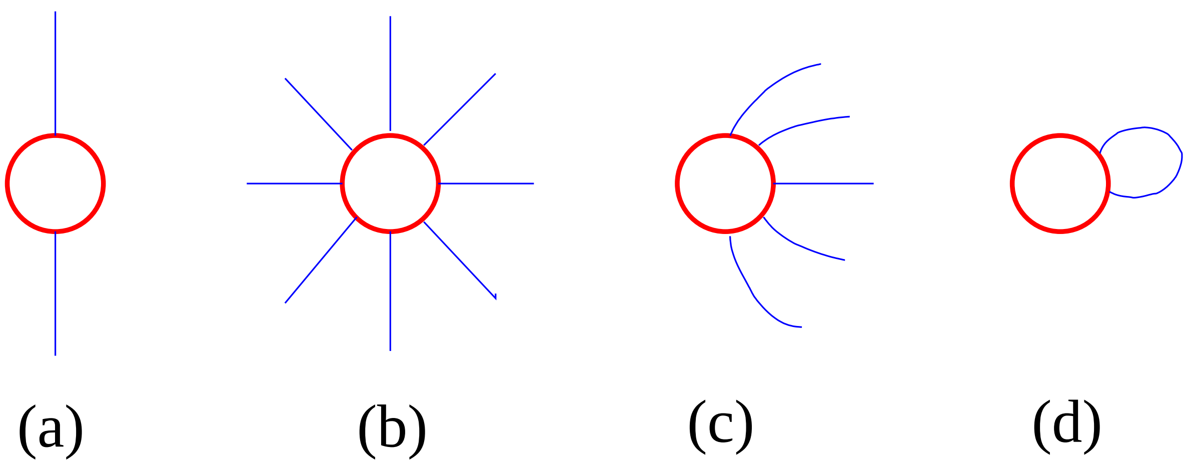

FIG. 5: (Colour online). Some possible sphere-vortex configurations.

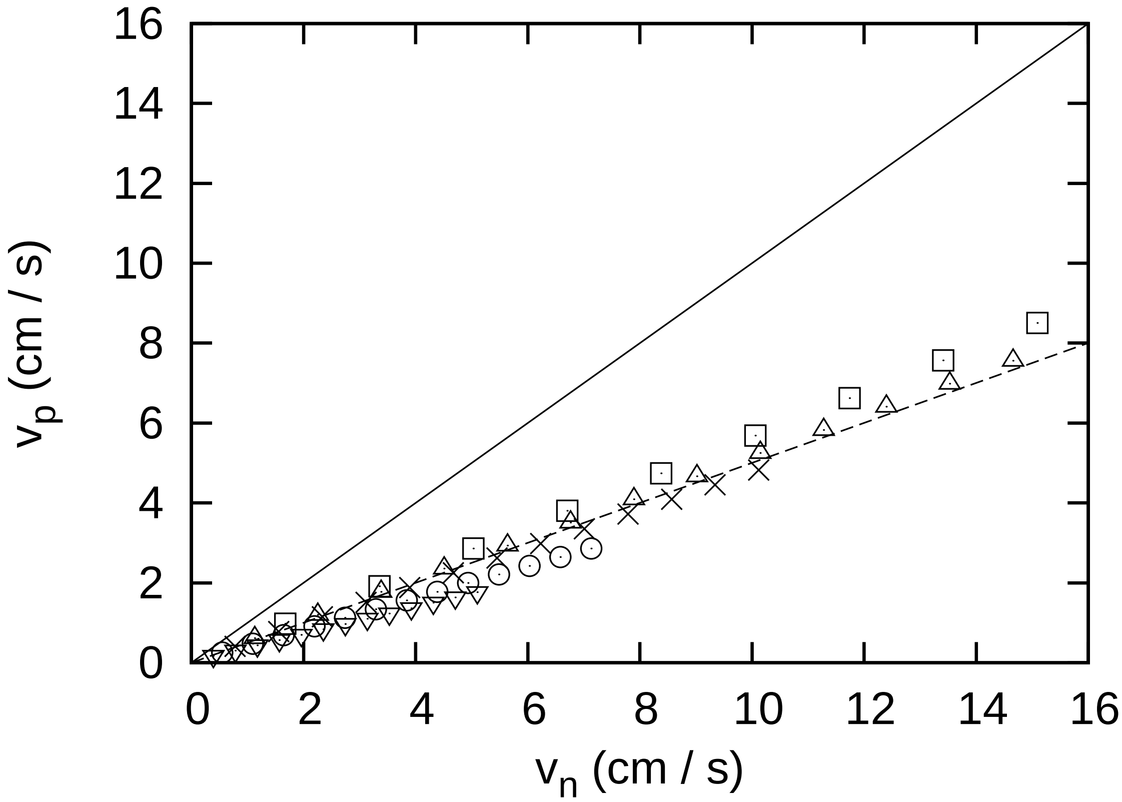

FIG. 6: Calculated v_{pa} versus v_n according to Eq. (14) with $\beta = 3$. The solid line is $v_{pa} = v_n$ and the dashed line is $v_{pa} = 0.5v_n$. The results compare well with the experimental data [2] shown in Fig. 1.