Maximal subgroups of finite symplectic groups stabilizing spreads of lines

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Abstract

We prove that, for $q$ odd, the group $G = U_n(q^2) \cdot 2$ is maximal in the symplectic group $Sp_{2n}(q)$ except when $n = 2$ and $q = 3$. The group $G$ corresponds to the stabilizer of a spread of lines of $PG(2n-1, q)$ in which some lines are isotropic and some are non–isotropic.

Keywords: line–spread, finite unitary group, finite symplectic group, maximal subgroup

1 Introduction

In the study of classical groups over a finite field, Aschbacher’s theorem plays a major part, [1], [14]. Any subgroup of a classical group either lies inside a maximal subgroup belonging to one of eight classes or it is almost simple (with additional properties). The focus of this paper is the symplectic group $Sp_{2n}(q)$ and the Aschbacher class $C_3$, i.e. the class defined in terms of stabilizers of overfields of $GF(q)$. The subgroups in this class preserve a vector space structure given by an overfield; in projective terms they stabilize a spread of subspaces. Within this class Aschbacher lists two subclasses: normalizers of $Sp_{2m}(q^r)$ where $n = mr$ and $r$ is prime; and normalizers of $U_n(q^2)$. The first subclass was considered by R.H Dye in [6], [7], [8], [9], [11], where he proves the maximality in purely geometric terms. Our object is to do the same for the second subclass when $q$ is odd.

The unitary group $U_n(q^2)$ lies inside both $Sp_{2n}(q)$ and $O_{2n}^\varepsilon(q)$, $\varepsilon = (-1)^n$. For $q$ even, $O_{2n}^\varepsilon(q)$ is contained in $Sp_{2n}(q)$ and contains the normalizer in $Sp_{2n}(q)$ of $U_n(q^2)$, so that the normalizer will not usually be maximal in $Sp_{2n}(q)$. However this raises the question of the maximality of the normalizer of $U_n(q^2)$ in $O_{2n}^\varepsilon(q)$. Dye proves in [10] that maximality occurs for all $q$ when $n \geq 3$. When $q$ is odd, the normalizer of $U_n(q^2)$ in $Sp_{2n}(q)$ does not lie in $O_{2n}(q)$ although it has the same structure ($U_n(q^2) \cdot 2$) as the corresponding group in $O_{2n}^\varepsilon(q)$ (the two normalizers correspond to different subgroups of $\Gamma U_n(q^2)$). In the (projective) orthogonal case, $U_n(q^2) \cdot 2$ is the stabilizer of a spread of lines $\bar{K}_n \cup \bar{L}_n$ of $PG(2n-1, q)$ and at the same time is the stabilizer of a spread $\bar{K}_n$ of lines of a quadric. In the (projective) symplectic case, $U_n(q^2) \cdot 2$ is the stabilizer of the same spread $\bar{K}_n$ and of the partial spread $\bar{K}_n$ (still a spread of a quadric), but in terms of the symplectic form one can only say that $\bar{K}_n$ consists of isotropic lines and $\bar{L}_n$ of non–isotropic.
lines. Although our approach shares Dye’s philosophy, the techniques are different.

The maximality of \( U_2(q^2) \cdot 2 \) in \( Sp_4(q) \) (\( q \) odd and \( > 3 \)) was established by H.H. Mitchell many years ago. We include a different treatment, using the isomorphism between \( PSp_4(q) \) and \( \Omega_5(q) \), that gives a clearer view of the case \( q = 3 \). The maximality of \( U_n(q^2) \cdot 2 \) in \( Sp_{2n}(q) \) (for \( n \geq 6 \)) was demonstrated in [14] using the full weight of Aschbacher’s Theorem and the Classification of Finite Simple Groups. The approach in this paper is purely geometric, without reliance on the Classification, and is designed to complement Dye’s approach in [6], [7], [8], [9], [10], [11].

2 Spreads of lines and their stabilizers

Let \( L = GF(q^2) \) and \( K = GF(q) \), \( q \) odd. Let \( \omega \) be an element of \( L \) such that \( \omega^q = -\omega \). Then 1 and \( \omega \) form a basis for \( L \) over \( K \), and if \( \theta \in L \), then \( \theta = \alpha + \beta \omega \), with \( \alpha, \beta \in GF(q) \). Let \( \{e_1, \ldots, e_n\} \) be a basis of \( L^n \) as a vector space over \( L \).

Define a bijective map \( \Phi \) from \( L^n \) to \( K^{2n} \) by the rule
\[
(\theta_1, \ldots, \theta_n) \mapsto (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n),
\]
where \( \theta_i = \alpha_i + \beta_i \omega \), for each \( i = 1, \ldots, n \). We denote a vector of \( K^{2n} \) by \( z \) with the corresponding vector in \( L^n \) represented by \( \bar{z} \). The vectors of the 1–subspace \( \langle \bar{z} \rangle \) of \( L^n \) are \( K \)–linear combinations of the vectors \( \bar{z} \) and \( \omega \bar{z} \) which correspond in \( K^{2n} \) to the vectors of a 2–dimensional subspace we call \( k_z \). Since \( \Phi \) is a bijection, each non–zero vector in \( K^{2n} \) lies in exactly one \( k_z \).

Passing to the projective space \( PG(2n−1, q) \) whose underlying vector space is \( K^{2n} \), the subspace \( k_z \) gives a line \( s_z \) in \( PG(2n−1, q) \), and the set of all such lines gives a spread of lines (regular spread [12]) of \( PG(2n−1, q) \).

Let \( H \) be a non–degenerate Hermitian form on \( L^n \) with isometry group \( U_n(q^2) \). We can take \( \{e_1, \ldots, e_n\} \) to be an orthogonal basis for \( L^n \) with respect to \( H \). Starting from \( H \) we can define a non–degenerate alternating form \( A \) on \( K^{2n} \) by
\[
A(x, y) = Tr(\omega H(x, y)) = \omega H(x, y) + \omega^q H(x, y)^q,
\]
for any \( x, y \in K^{2n} \). In this setting isotropic 1–dimensional subspaces of \( L^n \) correspond to totally isotropic 2–dimensional subspaces of \( K^{2n} \), and
non–isotropic 1–dimensional subspaces of $L^n$ correspond to non–isotropic 2–dimensional subspaces of $K^{2n}$. Any linear map on $L^n$ preserving $H$ gives rise to a linear map on $K^{2n}$ preserving $A$. For other properties of the map $\Phi$ see [10, Lemma 1].

We obtain an embedding

$$\iota : U_n(q^2) \to \text{Sp}_{2n}(q).$$

Let

$$\mathcal{K}_n = \{ k_z : z \neq 0, H(z) = 0 \}; \quad \mathcal{L}_n = \{ k_z : H(z) \neq 0 \};$$

$$\tilde{\mathcal{K}}_n = \{ s_z : z \neq 0, H(z) = 0 \}; \quad \tilde{\mathcal{L}}_n = \{ s_z : H(z) \neq 0 \}.$$  

We have $k = |\mathcal{K}_n| = (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q^2 - 1) \ [16]$ and so $l = |\mathcal{L}_n| = (q^{2n} - 1)/(q^2 - 1) - k$. Of course $l > k$.

From our previous discussion, it follows that $\tilde{\mathcal{K}}_n \cup \tilde{\mathcal{L}}_n$ is a line–spread $S$ of $PG(2n - 1, q)$.

Let $\sigma : L \to L$ be the Frobenius automorphism of $L$: $\theta \mapsto \theta^q$, for each $\theta$ in $L$. Then $\sigma$ gives rise to a semi–linear map: $\theta e_i \mapsto \theta^q e_i$ on $L^n$ which corresponds to a linear map on $K^{2n}$. It turns out that $A(\sigma(x), \sigma(y)) = -A(x, y)$ and so $\sigma$ multiplies $A$ by $-1$. Hence $\sigma$ is an element of $GSp_{2n}(q)$.

If $\tau \in GU_n(q^2)$ is such that $\tau(e_i) = \lambda e_i$, $i = 1, \ldots n$, where $\lambda \in L$ and $\lambda^{q+1} = -1$, then it is easy to see that $\tau$ multiplies $H$ by $-1$ and corresponds to an element of $GSp_{2n}(q)$ again multiplying $A$ by $-1$. Thus $\tau \sigma \in Sp_{2n}(q)$; it has order 4 since its square is $-I_{2n}$, where $I$ denotes the identity matrix.

We denote by $G$ the group $\iota(U_n(q^2), \tau \sigma)$ and often write $G = U_n(q^2)$ · 2.

From our previous discussion it follows that $G$ is contained in the stabilizer in $Sp_{2n}(q)$ of $\mathcal{K}_n \cup \mathcal{L}_n$. Since the subspaces in $\mathcal{K}_n$ are isotropic while those in $\mathcal{L}_n$ are non–isotropic it follows that $G$ stabilizes each of $\mathcal{K}_n$ and $\mathcal{L}_n$. We shall prove that $G$ is maximal in $Sp_{2n}(q)$ from which it follows that $G$ is the stabilizer of $\mathcal{K}_n \cup \mathcal{L}_n$, and indeed the stabilizer of $\mathcal{K}_n$. Moreover $G$ contains the centre of $Sp_{2n}(q)$ so an immediate consequence is the maximality of the image $\bar{G}$ of $G$ in $PSp_{2n}(q)$.

We observe that $U_n(q^2)$ acts transitively on the 1–dimensional non–isotropic subspaces of $L^n$ and transitively on the non–zero singular vectors of $L^n$ [4], [5].
Hence $G$ acts transitively on $\mathcal{L}_n$ and transitively on the non–zero vectors lying in members of $\mathcal{K}_n$. The stabilizer in $U_n(q^2)$ of a non–isotropic 1–dimensional subspace $\langle \varphi \rangle$ of $L^n$ is isomorphic to $U_1(q^2) \times U_{n-1}(q^2)$ acting on $\langle \varphi \rangle \oplus \langle \varphi \rangle^\perp$. Thus the stabilizer in $G$ of $k_x$ is isomorphic to $(U_1(q^2) \times U_{n-1}(q^2)) \cdot 2$ and fixes the set $\mathcal{K}_{n-1} \cup \mathcal{L}_{n-1}$ where $\mathcal{K}_{n-1}$ (respectively $\mathcal{L}_{n-1}$) corresponds to the set of elements of $\mathcal{K}_n$ (respectively $\mathcal{L}_n$) contained in $k_x^\perp$.

In this paper we prove the following theorem.

**Theorem 2.1.** Assume $n \geq 3$ and $q$ odd. Then the group $G = U_n(q^2) \cdot 2$ is a maximal subgroup of $Sp_{2n}(q)$. If $n = 2$ and $q$ is odd then $U_2(q^2) \cdot 2$ is a maximal subgroup of $Sp_4(q)$ except for $q = 3$. In the excepted case there is a single group $H \cong 2 \cdot 2^4 : A_5$, such that $G < H < Sp_4(q)$.

The group $Sp_{2n}(q)$ is transitive on the set of all isotropic 2–dimensional subspaces of $K^{2n}$ so cannot stabilize $\mathcal{K}_n \cup \mathcal{L}_n$ or $\mathcal{K}_n$. It will be clear that in the excepted case, $H$ does not stabilize $\mathcal{K}_2 \cup \mathcal{L}_2$ or $\mathcal{K}_2$ either. Thus we have the following theorem.

**Theorem 2.2.** The stabilizer of $\mathcal{K}_n \cup \mathcal{L}_n$ in $Sp_{2n}(q)$ is the stabilizer of $\mathcal{K}_n$, is isomorphic to $U_n(q^2) \cdot 2$ and is a maximal subgroup of $Sp_{2n}(q)$ except when $n = 2$ and $q = 3$.

As we have already observed, $G$ contains the centre of $Sp_{2n}(q)$. Thus we have the further theorem.

**Theorem 2.3.** The stabilizer of the line spread $\tilde{\mathcal{K}}_n \cup \tilde{\mathcal{L}}_n$ of $PG(2n - 1, q)$ in $PSp_{2n}(q)$ is the stabilizer of the partial spread $\mathcal{K}_n$ and is a maximal subgroup of $PSp_{2n}(q)$ except when $n = 2$ and $q = 3$.

The following lemma will be useful in Section 4.

**Lemma 2.4.** Suppose $U$ is a 2–dimensional subspace of $K^{2n}$, not lying in $\mathcal{K}_n \cup \mathcal{L}_n$. Then $U$ lies in a uniquely defined 4–dimensional subspace of $K^{2n}$ corresponding to a 2–dimensional subspace of $L^n$. The 4–dimensional subspace may be written as $k_a \oplus k_b$, for any linearly independent vectors $a, b \in U$.

**Proof.** Let $a, b$ be linearly independent vectors in $U$ with $a, b$ the corresponding vectors of $L^n$. Then $a, b$ are linearly independent over $L$ (for otherwise $U = k_a$). Hence $\langle a, b \rangle$ corresponds to the 4–dimensional subspace $k_a \oplus k_b$ of $K^{2n}$. If $c, d$ are linearly independent in $U$, corresponding to $c, d \in L^n$, then each of $c, d$ is a $K$–linear combination of $a, b$ so $k_c, k_d \subseteq k_a \oplus k_b$. Therefore $k_c \oplus k_d = k_a \oplus k_b$. \qed
3 The case $\text{n} = 2$

In this section we establish the maximality of the group $U_2(q^2) \cdot 2$ inside $Sp_4(q)$. This result is originally due to H.H. Mitchell [15] who also approached the problem geometrically. Our approach uses the well–known isomorphism between $PSp_4(q)$ and $\Omega_5(q)$ (for odd $q$, $PO_5(q)$ and $\Omega_5(q)$ are isomorphic). It enables us to determine properties of the intermediate subgroup in the case $q = 3$ which in turn facilitate the proof of the maximality of $U_3(9) \cdot 2$ in $Sp_6(3)$.

As we shall see, the $q + 1$ members of $\mathcal{K}_2$ correspond to the points of a non–degenerate conic $\mathcal{C}$ inside a non–degenerate quadric $\mathcal{P}$ of $PG(4, q)$ (with $\mathcal{P}$ inside the Klein quadric) and having the property that $\mathcal{C}$ is orthogonal to a line of $PG(4, q)$ that is external to $\mathcal{P}$. In vector space terms we have a non–isotropic 3–dimensional subspace of $K^5$ stabilized along with its (anisotropic) complement. This motivates the following development. Let $V$ be a 5–dimensional vector space over $GF(q)$ with $q$ odd. Let $B$ be a non–degenerate symmetric bilinear form on $V$ with associated quadratic form $Q$ given by $Q(v) = B(v, v)/2$. For more details see [2], [4], [5]. In [13] the stabilizers of non–isotropic subspaces are studied. It is proved that the stabilizers in $O_5(q)$ and $SO_5(q)$ of a 2–dimensional non–isotropic subspace $W$ are maximal except when $q = 3$ and $W$ is anisotropic (i.e., if $w \in W$ with $Q(w) = 0$, then $w = 0$). We require the corresponding result for $\Omega_5(q)$. The proof follows similar lines to [13] and so we omit details where the argument is essentially identical.

If $w$ is a non–singular vector in $V$ (i.e. $Q(w) \neq 0$) then the symmetry $s_w$ centred on $w$ is given by:

$$s_w : v \mapsto v - [B(w, v)/Q(w)]w.$$  

The symmetry $s_w$ has determinant $-1$, and stabilizes a subspace $Z$ of $V$ if and only if $w \in Z \cup Z^\perp$ (where $Z^\perp$ is the orthogonal complement of $Z$). If $x$ is a non–zero singular vector in $V$ (i.e. $Q(x) = 0$) and if $w \in x^\perp$, then the semi–transvection $\rho_{x,w}$ centred on $x$ is given by

$$\rho_{x,w} : v \mapsto v + [B(w, v) - Q(w)B(x, v)]x - B(x, v)w.$$  

Each such semi–transvection lies in $\Omega_5(q)$ [17]. If $x$ lies in a subspace $Z$ of $V$ then $\rho_{x,w}$ stabilizes $Z$ if and only if $w \in Z$.  

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Proposition 3.1. Assume that $q \geq 5$ and let $W$ be an anisotropic 2–dimensional subspace of $V$. Then the stabilizer $\tilde{G}$ of $W$ in $\Omega_5(q)$ is a maximal subgroup of $\Omega_5(q)$.

Proof. Suppose $\tilde{G} < \tilde{F} \leq \Omega_5(q)$. We divide the proof into several steps. Steps 1, 2 and 2a establish that there is some $f_3 \in \tilde{F} \setminus \tilde{G}$ such that $f_3(x) = x$ for some non–zero singular vector $x \in W^\perp$. Step 3 deduces that $\rho_{x,u} \in \tilde{F}$ for all $u \in x^\perp$ and then Step 4 concludes that $\tilde{F}$ contains every semi–transvection in $\Omega_5(q)$ from which it follows that $\tilde{F} = \Omega_5(q)$.

Step 1. If $f_1 \in \tilde{F} \setminus \tilde{G}$, then $f_1(x) \not\in W^\perp$, for some non–zero singular vector $x \in W^\perp$. Write $f_1(x) = x_1 + x_2$, with $x_1 \in W$ and $x_2 \in W^\perp$. Then $x_1$ and $x_2$ are both non–isotropic. Further, we may assume that $f_1(W) \not\subset W^\perp$, as in [13, Propositions 4.3 and 4.4].

Step 2. There are two conjugacy classes of symmetries in $\Omega_5(q)$, each corresponding to a class of non–isotropic 1–dimensional subspaces. One of these classes has the property that for a symmetry $s_v$ in that class, $-s_v \in \Omega_5(q)$. Moreover, if $v \in W \cup W^\perp$, then $-s_v \in \tilde{G}$. The subspace $x_2^\perp \cap W^\perp = f_1(x)^\perp \cap W^\perp$ is non–isotropic of dimension 2 whereas $f_1(x)^\perp \cap f_1W^\perp$ is 2–dimensional and isotropic. Thus $x_2^\perp \cap W^\perp \not\subset f_1W^\perp$. Moreover $x_2^\perp \cap W^\perp$ is spanned by vectors of each class, and if $q \geq 7$, then there are at least three 1–dimensional subspaces of each class in $x_2^\perp \cap W^\perp$. Thus for $q \geq 7$ we can find a vector $v \in x_2^\perp \cap W^\perp$ such that $-s_v \notin \tilde{G}$ but $-s_v$ fixes neither $f_1W$ nor $f_1W^\perp$. Hence $f_2 = f_1^{-1}(-s_v)f_1 \in \tilde{F} \setminus \tilde{G}$ and $f_2(x) = -x$. Let $g_2 \in \tilde{G}$ such that $g_2(x) = -x$. Then $f_3 = g_2f_2 \in \tilde{F} \setminus \tilde{G}$ with $f_3(x) = x$.

Step 2a. If $q = 5$, then $x_1$ and $x_2$ are in the same class, so $s_1s_2 \in \tilde{G}$, where $s_i$ is the symmetry centered on $x_i$, and $f_2 = f_1^{-1}s_1s_2f_1 \in \tilde{F} \setminus \tilde{G}$, unless $x_1, x_2 \in f_1W^\perp$. If $x_1, x_2 \in f_1W^\perp$, then $f_2(x) = -x$ and the argument of Step 2 applies unless $x_2^\perp \cap W^\perp = Y_1 + Y_2$, where $Y_1 \subseteq f_1W$, $Y_2 \subseteq f_1W^\perp$, with $Y_1, Y_2$ in the same class. But in this case $x_1^\perp \cap W \subseteq f_1W$ and is in the opposite class to both $Y_1$ and $\langle x_1 \rangle$, so $Y_1, \langle x_1 \rangle, \langle x_2 \rangle$ and $Y_2$ all lie in the same class and $\langle x_2 \rangle + Y_2$ is a hyperbolic subspace of $W^\perp \cap f_1W^\perp$. It follows that $f_1(y) \in W^\perp$ for some non–zero singular vector $y \in W^\perp$ and hence there exists $g, g' \in \tilde{G}$ such that $f_3 = g'f_1g$ fixes $x$.

Step 3. As in [13] it follows now that $\rho_{x,z} \in \tilde{F}$ for some $0 \neq z \in W$. Let $y$
be singular in $W^\perp$ such that $B(x, y) = 1$. Then $\hat{G}$ contains elements fixing $z$ and taking $x$ to $\lambda^2 x$, $y$ to $\lambda^{-2} y$, for each $\lambda \in GF(q) \setminus \{0\}$. Hence $\hat{F}$ contains $\rho_{\lambda^2 x, z} = \rho_{x, \lambda^2 z}$, for each $\lambda$, see [17]. Any $\alpha \in GF(q)$ may be written as $\lambda^2 - \mu^2$, for some $\lambda, \mu \in GF(q)$, so $\rho_{x, \alpha z} = \rho_{x, \lambda^2 z} \cdot (\rho_{x, \mu^2 z}^{-1}) \in \hat{F}$. Moreover if $w \in W \setminus \langle z \rangle$ such that $Q(w) = Q(z)$ then $\hat{G}$ contains an element taking $z$ to $w$ and $x$ to $\beta x$, for some $\beta \in GF(q) \setminus \{0\}$. Thus $\hat{F}$ contains $\rho_{\beta x, w} = \rho_{x, \beta w}$. As we have just shown, this means that $\hat{F}$ contains $\rho_{x, \alpha w}$, for all $\alpha \in GF(q)$.

Step 4. There are three orbits of non–zero singular vectors under $\hat{G}$. One orbit consists of those vectors lying in $W^\perp$. The others correspond to representatives $w_1 + w_2$, with $w_1 \in W$ and $w_2 \in W^\perp$, with an orbit corresponding to each class of $w_1$ (i.e. $Q(w_1)$ square or non–square), see [13, Proposition 4.2]. Notice that $\text{Stab}_{\Omega_5(q)}(W)$ contains symmetries from each class fixing $w_1$ and $w_2$. Now observe that $\rho_{x, w_1}(y) = y - Q(w_1)x - w_1$ (as in Step 3), so $\hat{F}$ contains elements joining the orbit consisting of non–zero singular vectors in $W^\perp$ to each of the other orbits. Hence $\hat{F}$ is transitive on non–zero singular vectors of $V$.

In conclusion $\hat{F}$ contains every semi–transvection in $\Omega_5(q)$. Since $\Omega_5(q)$ is generated by its semi–transvections, see [17], it follows that $\hat{F} = \Omega_5(q)$ and $\hat{G}$ is a maximal subgroup of $\Omega_5(q).$ \qed

**Proposition 3.2.** If $q = 3$ and $\hat{G} < \hat{F} \leq \Omega_5(3)$, then either $\hat{F} = \Omega_5(3)$, or $\hat{F} \cong 2^4 \cdot A_5$ permuting five pairwise orthogonal non–isotropic 1–dimensional subspaces.

**Proof.** If $\hat{F} \setminus \hat{G}$ contains an element $f$ such that $W^\perp \cap fW^\perp$ contains a non–zero singular vector then the arguments of Steps 3 and 4 of the previous Proposition may be applied with the conclusion that $\hat{F} = \Omega_5(3)$. Thus we may assume that $W^\perp \cap fW^\perp$ is anisotropic for all $f \in \hat{F} \setminus \hat{G}$.

We observe that $W$ has two 1–dimensional non–isotropic subspaces belonging to each class, while $W^\perp$ has four singular 1–dimensional subspaces, six non–isotropic 1–dimensional subspaces of one class and three of the other class; moreover the three are pairwise orthogonal. Thus for one class of non–isotropic 1–dimensional subspaces there is a set $\Delta$ of five of these (two from $W$ and three from $W^\perp$) preserved by $\hat{G}$. We show that the stabilizer in $\Omega_5(3)$ of $\Delta$ has structure $2^4 \cdot A_5$ and that $\hat{F}$ is precisely this stabilizer. We denote
by “+” the class of non–isotropic 1–dimensional subspaces corresponding to
Δ, and by “−” the other class. The subspaces in Δ are pairwise orthogonal
and so the stabilizer of Δ in O₅(3) acts as S₅ on Δ. Symmetries centred on
subspaces in Δ all lie in one conjugacy class of O₅(3) and fix each element
of Δ. An element of O₅(3) corresponding to a transposition of Δ must arise
as the product of one symmetry from the class − multiplied by any num-
er of symmetries centered on subspaces in Δ. Thus a transposition cannot
correspond to an element of Ω₅(3). Hence the stabilizer of Δ in Ω₅(3) acts
as A₅ on Δ with kernel consisting of products of symmetries centered on
subspaces in Δ. As Ω₅(3) contains no such symmetry by itself, but contains
the product of any pair, we conclude that the stabilizer of Δ in Ω₅(3) has
structure 2⁴ · A₅.

Now consider \( f \in \tilde{F} \setminus \tilde{G} \) and let \( x \) be a non–zero singular vector of \( W \)
with \( f(x) = x_1 + x_2, (x_1 \in W, x_2 \in W^\perp) \). Recall from Proposition 3.1
that \( \Omega_5(q) \) contains \(-s_v\) for symmetries \( s_v \) belonging to one class. Here that
class is the + class. We can use the argument of Step 2 of Proposition 3.1
(and hence conclude that \( \tilde{F} = \Omega_5(3) \)) unless \( v \in fW \) or \( fW^\perp \), for every
\( v \in x_2^\perp \cap W^\perp \) of class +.

We write \( X_1 = \langle x_1 \rangle, X_2 = \langle x_2 \rangle \) and \( Z = \langle z \rangle = X_1^\perp \cap W \). The subspace
\( X_2^\perp \cap W \) cannot lie in \( fW^\perp \), so we have three possibilities:

i) \( fW = X_2^\perp \cap W^\perp = Y_1 \oplus Y_2 \), with \( Y_1, Y_2 \) both of class +;

ii) \( X_2^\perp \cap W^\perp = Y_1 \oplus Y_2 \) is anisotropic with \( Y_1, Y_2 \) both of class +, \( Y_1 \subseteq fW \),
\( Y_2 \subseteq fW^\perp \);

iii) \( X_2^\perp \cap W^\perp = Y_1 \oplus Y_2 \) is hyperbolic with \( Y_1 \) of class − and \( Y_2 \) of class +.

In case (i) the subspace \( fW^\perp \) is \( X_2 \oplus X_1 \oplus Z \) and has just three 1–
dimensional subspaces of class +: \( X_2, \langle x_1 + z \rangle, \langle x_1 - z \rangle \), and we see that
\( f \) preserves Δ. In case (ii) \( fW \) has a subspace of class + contained in
\( X_1 \oplus X_2 \oplus Z \), but no such subspace exists that is also orthogonal to \( x_1 + x_2 \).
In case (iii), \( X_2 \) is of class − and \( X_1, Z \) of class +: if \( Z \not\subseteq fW \cup fW^\perp \) then
\( f_3 = f^{-1}(-s_z)f \in \tilde{F} \setminus \tilde{G} \) with \( f_3(x) = -x \), so we may further assume that
\( Z \subseteq fW \) or \( Z \subseteq fW^\perp \). It is not possible for \( Z \) and \( Y_2 \) both lie in \( fW^\perp \) so
we have three subcases:

a) \( Z, Y_2 \subseteq fW \);

b) \( Y_2 \subseteq fW, Z \subseteq fW^\perp \);
c) \( Y_2 \subseteq fW^\perp, \ Z \subseteq fW \).

In (a), \( fW^\perp = Y_1 \oplus X_1 \oplus X_2 \) with \( X_1 \) as one of the subspaces of class + and the other two in \( Y_1 \oplus X_2 \), and we see that \( f \) preserves \( \Delta \). In (b) and (c), \( fW \) has a subspace of class + contained in \( X_1 \oplus X_2 \oplus Y_1 \), but no such subspace exists that is also orthogonal to \( x_1 + x_2 \). Hence \( f \) stabilizes \( \Delta \) in all cases and so \( \tilde{F} \leq \text{Stab}_{\Omega_5(q)}(\Delta) \). Finally, \( \tilde{G} \) contains the subgroup \( 2^4 \) generated by pairs of symmetries centered on subspaces in \( \Delta \). Further \( \tilde{G} \) acts on \( \Delta \) as the maximal subgroup \( S_3 \) of \( A_5 \). Hence \( \tilde{F} \) is maximal in \( \text{Stab}_{\Omega_5(q)}(\Delta) \).

Remark 3.3. We observe that the subspace \( W^\perp \) in 3.2 has four singular 1–dimensional subspaces and these span \( W^\perp \). Therefore \( \tilde{F} \) does not stabilize this set of four 1–dimensional subspaces.

Theorem 3.4. The group \( U_2(q^2) \cdot 2 \) is a maximal subgroup of \( Sp_4(q) \) when \( q \) is odd and \( q \neq 3 \).

Proof. Let \( G = U_2(q^2) \cdot 2 \) and let \( \tilde{G} \) be the image of \( G \) in \( PSp_4(q) \). Then \( \tilde{G} \) preserves the spread \( \bar{K}_2 \cup \bar{L}_2 \) of \( PG(3, q) \) and, since \( G \) contains the centre of \( Sp(4, q) \), \(|\tilde{G}| = |G|/2 = (q+1)q(q^2-1)\). We use the well known isomorphism between \( PSp_4(q) \) and \( \Omega_5(q) \) to establish the maximality of \( \tilde{G} \) and hence \( G \), see [3], [18].

Recall that under the Plücker correspondence, lines of \( PG(3, q) \) are represented as points of the Klein quadric \( Q \) in \( PG(5, q) \), and given a non–degenerate symplectic polarity on \( PG(3, q) \), the isotropic lines correspond to points of a parabolic quadric \( P \) of \( PG(4, q) \) lying inside \( Q \). The \( q+1 \) isotropic lines of \( \bar{K}_2 \) form a regulus of isotropic lines of \( PG(3, q) \) so correspond to the points of a non–degenerate conic on \( P \), [12].

Thus \( \tilde{G} \) is isomorphic to a subgroup of \( PO_5(q) \) fixing a non–degenerate conic, i.e. (given that \( PO_5(q) \) and \( \Omega_5(q) \) are isomorphic) isomorphic to a subgroup of \( \Omega_5(q) \) fixing a non–isotropic subspace \( W^\perp \) of \( K^5 \) and its orthogonal complement \( W \).

The stabilizer of \( W \) and \( W^\perp \) in \( \Omega_5(q) \) has order \((q-1)q(q^2-1)\) when \( W \) is hyperbolic and \((q+1)q(q^2-1)\) when \( W \) is anisotropic. We thus see that \( W \) must be anisotropic and \( G \) is isomorphic to the stabilizer \( \tilde{G} \) of \( W \) in \( \Omega_5(q) \). By Proposition 3.1, \( \tilde{G} \) is maximal in \( \Omega_5(q) \) so \( \tilde{G} \) is maximal in \( PSp_4(q) \) and \( G \) is maximal in \( Sp_4(q) \).

Theorem 3.5. If \( q = 3 \), then \( U_2(q^2) \cdot 2 \) is not a maximal subgroup of \( Sp_4(q) \). There is a single intermediate subgroup \( H \) with structure \( 2 \cdot 2^4 \cdot A_5 \). Moreover
if $k_x$ is a non–isotropic subspace in $L_2$, then the projection of $\text{Stab}_H(k_x)$, acting on each of $k_x$ and $k_x^\perp$ is $\text{Sp}_{2}(3)$.

Proof. If we write $G$ and $\bar{G}$ as in the proof of Theorem 3.4, then the correspondence described in the theorem applies equally to $q = 3$: $\bar{G}$ is isomorphic to $\bar{G}$, the stabilizer in $\Omega_5(3)$ of an anisotropic 2–dimensional subspace of $K^5$. By Proposition 3.2, $\bar{G}$ is not maximal, there being a single intermediate subgroup $\bar{H}$ of $\Omega_5(3)$ with structure $2^4 \cdot A_5$. The corresponding subgroup $\bar{H}$ of $\text{PSp}_{4}(3)$ has preimage $\bar{H} \cong 2 \cdot 2^4 \cdot A_5$ in $\text{Sp}_{4}(3)$.

Under the Plücker correspondence, a non–isotropic line of $PG(3, q)$ corresponds to a point of the Klein quadric $Q$ that does not lie on $P$ (we use the notation of Theorem 3.4).

Let us write $K^6$ as the orthogonal sum $K \oplus K^5$, where $K^5$ corresponds to the $PG(4, q)$ containing $P$. Then a non–zero singular vector of $K^6$ not in $K^5$ can be written in the form $a + b$, where $a$ is in class $-$ and $b$ in class $+$. The 2–dimensional subspace $\langle a, b \rangle$ has two singular 1–dimensional subspaces: given the actions of $\Omega_5(3)$ and $\text{PSp}_{4}(3)$, on points of $PG(4, 3)$ and lines of $PG(3, 3)$ respectively, we conclude that a subspace of class $+$ of $K^5$ corresponds to a pair of lines of $PG(3, 3)$, namely a non–isotropic line and its orthogonal complement.

There are six non–isotropic lines in $\bar{L}_2$ and because $\bar{K}_2 \cup \bar{L}_2$ is a spread, the lines of $\bar{L}_2$ correspond to subspaces of class $+$ of $K^5$ that are not orthogonal to any singular 1–dimensional subspace of $W^\perp$. This can only mean that the lines of $\bar{L}_2$ (three pairs) correspond to the three 1–dimensional subspaces of class $+$ in $W^\perp$. Hence $\bar{H}$ permutes a set of five pairs of non–isotropic lines of $PG(3, q)$, acting as $A_5$ on this set. If we take a particular subspace $k_x$ in $L_2$ with image $\bar{k}_x$ in $\bar{L}_2$, then the stabilizer in $\bar{H}$ of $\{ \bar{k}_x, \bar{k}_x^\perp \}$ acts as $A_4$ on the remaining four pairs. Thus $H$ has an element acting as a 3–cycle, and a suitable power of this element has order 3. In fact, a suitable power of a preimage in $H$ has order 3. Let us write $h$ for such an element with $h_1$ and $h_2$ its projections acting on $k_x$ and $k_x^\perp$, respectively. One or both of $h_1$, $h_2$ has order 3. As $G$ contains elements switching $k_x$ and $k_x^\perp$, we may assume that $h_1$ has order 3. Now $G$ contains $[U_1(q^2) \times U_1(q^2)] \cdot 2$ which stabilizes each of $k_x$ and $k_x^\perp$, and the projection acting on $k_x$ has order 8. Considering also $h_1$, we see that the projection of $\text{Stab}_H(k_x)$ acting on $k_x$ has order divisible by 24, i.e. it is the whole of $\text{Sp}_{2}(3)$. The remark above on switching $k_x$ and $k_x^\perp$ now ensures that $\text{Stab}_H(k_x)$ also acts on $k_x^\perp$ as the whole of $\text{Sp}_{2}(3)$. □
4 The case $n \geq 3$: The Reduction Argument

In this Section we assume that $n \geq 3$. We start with the following lemma.

**Lemma 4.1.** Suppose that $G = U_n(q^2) \cdot 2 \leq F < Sp_{2n}(q)$. Then there is

a non–isotropic 2–dimensional subspace $k_x$ of $L_n$ such that if $F_1$ and $F_2$ are

the projections of $Stab_F(k_x)$ acting on $k_x$ and $k_x^\perp$ respectively, then either

$U_1(q^2) \cdot 2 < F_1$ or $U_{n-1}(q^2) \cdot 2 < F_2$ (or both).

Proof. Let $T_n$ be the set of 1–dimensional subspaces of $K^{2n}$ lying in

members of $L_n$ and let $S_n$ be the set of 1–dimensional subspaces of $K^{2n}$ lying

in members of $K_n$. Then $PG(2n-1,q) = T_n \cup S_n$, $|T_n| = (q+1)|L_n|$ and

$|S_n| = (q+1)|K_n|$. As observed in Section 2, $|L_n| > |K_n|$ so if $f \in F \setminus G$, then

the intersection $fT_n \cap T_n$ is non–empty, i.e. $f(k_a) \cap k_b \neq \{0\}$, for some

$k_a, k_b \in L_n$. There exists $g \in G$ such that $g(k_b) = k_a$ so that $gf(k_a) \cap k_a \neq \{0\}$

with $gf \in F \setminus G$. Thus we may assume that $f(k_a) \cap k_a \neq \{0\}$.

Suppose that $f(k_a) = k_a$. Then $f$ can be written as $(f_1, f_2)$, with $f_1$ acting on

$k_a$ and $f_2$ on $k_a^\perp$. If either $f_1 \not\in U_1(q^2) \cdot 2$ or $f_2 \not\in U_{n-1}(q^2) \cdot 2$, then we may take $k_x = k_a$. If $f_1 \in U_1(q^2) \cdot 2$ and $f_2 \in U_{n-1}(q^2) \cdot 2$, then since $G$ contains $[U_1(q^2) \times U_{n-1}(q^2)] \cdot 2 = \langle U_1(q^2) \times U_{n-1}(q^2), t \rangle$, with $t = \iota(	au a) = (t_1, t_2)$, we conclude that $F$ contains both $(t_1, 1)$ and $(1, t_2)$. In this case let $k_x$ be a member of the $L_{n-1}$ lying inside $k_a^\perp$ and write $k_a^\perp = W \oplus k_x$, where $W = k_x^\perp \cap k_a^\perp$. Then $K^{2n} = k_a \oplus W \oplus k_x$ and with respect to this sum $F$

contains $(t_1, 1)$. In other words, the projection of $Stab_F(k_x)$ acting on $k_x^\perp$

contains $(t_1, 1) \not\in U_{n-1}(q^2) \cdot 2$ i.e., $U_{n-1}(q^2) \cdot 2 < F_2$.

Suppose that $f(k_a) \neq k_a$. Then $f(k_a) \not\in L_n$ and we can write $f(k_a) =

\langle b, c + d \rangle$, where $0 \neq b, c \in k_a$ and $0 \neq d \in k_a^\perp$. Then by Lemma 2.4

$f(k_a) \subseteq k_b \oplus k_{c+d} = k_b \oplus k_d$ (since $k_b = k_c = k_a$). The subspace $\langle a, d \rangle$ of $L^n$

is not totally isotropic, so if $n = 3$ and $\langle a, d \rangle$ is non–isotropic or if $n \geq 4$, there

exists a non–isotropic vector $x \in \langle a, d \rangle^\perp$. In these cases $k_a \oplus k_d \subseteq k_x^\perp$. Let $s$

be a quasi–symmetry of order $q + 1$ centered on $a$ i.e., $s$ fixes every element of $a^\perp$ and takes $a$ to $\mu a$ for some $\mu \in L$ with order $q + 1$ in $L \setminus \{0\}$. We may think of $s$ as an element of $U_n(q^2) \leq G$; it fixes every vector in $k_a^\perp$ and fixes no 1–dimensional subspace of $k_a$. Thus $fsf^{-1}$ is an element of $F$ that fixes no 1–dimensional subspace of $f(k_a)$ but fixes every vector in $f(k_a)^\perp$ and, in particular, fixes every vector in $k_x$. Since $f(k_a)^\perp \neq k_x^\perp$ and since both contain $k_x$, there is a non–isotropic $y \in \langle x, a \rangle^\perp$ such that $k_y \not\subseteq f(k_a)^\perp$. Now $k_y$ and $f(k_a)^\perp$ are both subspaces of $b^\perp$ so $k_y \cap f(k_a)^\perp$ has dimension 1. Thus we can write $k_y = \langle u, v+w \rangle$, where $0 \neq u, v \in f(k_a)^\perp$ and $0 \neq w \in f(k_a)$. Returning
to $fsf^{-1}$: it fixes $u$ and $v$ but moves $w$ to a different 1-dimensional subspace of $f(k_a)$. Hence $fsf^{-1}(k_y) \not\subseteq \mathcal{L}_n$ and furthermore $fsf^{-1}$ does not stabilize $\mathcal{L}_{n-1}$ in $k^+_n$. Therefore with this choice of $k_x$ we have $U_{n-1}(q^2) \cdot 2 < F_2$.

We now have to consider the possibility that $n = 3$ and $(a, d)$ is isotropic. We show however that there is a choice of $f \in F \setminus G$ and of $k_a$ such that $(a, d)$ is non–isotropic. We can simplify $f$ in two ways in order to minimize the algebra that follows.

Choose a basis $e_1, e_2$ for $k_a$ such that $f(e_1) \in k_a$ and write $f(e_2) = z + u$, with $z \in k_a$ and $0 \neq u \in k^+_a$. Moreover, the vector $u$ corresponds to $d$ above, so $k_u \in \mathcal{K}_3$.

Choose $g \in 1 \times U_2(q^2) \leq G$ (acting on $k_a \oplus k^+_a$) such that $g(u) = 2u$. Then $f^{-1}gf(e_1) = e_1$ and $f^{-1}gf(e_2) = e_2 + f^{-1}(u) \not\subseteq k_a$ (because $f^{-1}(u) \in e^+_1$ and $f(e_1) \not\subseteq k^+_a$) so $f^{-1}gf \in F \setminus G$. Thus we may assume that $f(e_1) = e_1$. Now $f^{-1}(u) = \mu e_1 + w$ for some $\mu \in K$ and some $w \in k^+_a$. Either $k_w$ is non–isotropic, in which case we can apply the previous paragraph to $f^{-1}gf$, or $k_w$ is isotropic, in which case there exists $h \in 1 \times U_2(q^2) \leq G$ (acting on $k_a \oplus k^+_a$) such that $h(w) = u$. We only need to pursue the latter case: $hf^{-1}(u) = \mu e_1 + u$ so $fh^{-1}(u) = u - \mu e_1$. Replacing $f$ by $fh^{-1}$ we may assume that $f(e_1) = e_1$, and $f(e_2) = z + u$, with $f(u) = u - \mu e_1$.

We have seen that $k_a$ is isotropic so there is a $k_v \subseteq k^+_a$ such that $k^+_a = k_u \oplus k_v$ and $A(u, v) \neq 0$. We show that there is a vector $y \in k_v$ such that $f^{-1}(y - u) = y_1 + y_2$, with $y_1 \in k_a$, $y_2 \in k^+_a$ and $k_{y_2}$ non–isotropic, and use this to construct an element $F \setminus G$ of the required form.

In $L^3$, $u, v$ are isotropic with $H(u, v) \neq 0$. We may assume that $H(u, v) = \epsilon^{-1}$, where $\epsilon = 2\omega^2 \in K$ (recall that $\omega \in K$ such that $\omega^q = -\omega$). We write $u_1 = u$, $u_2 \in k_a$ such that $u_2 = -\omega u_1$, $v_2 = v$ and $v_1 \in k_v$ such that $v_1 = \omega v_2$. Then $u_1, u_2, v_1, v_2$ form a symplectic basis for $k^+_a$, i.e. $A(u_1, v_1) = A(u_2, v_2) = 1$, $A(u_1, u_2) = A(v_1, v_2) = A(u_1, v_2) = A(u_2, v_1) = 0$. We can provide a test to decide whether a vector $w = \alpha u_1 + \beta u_2 + \gamma v_1 + \delta v_2$ belongs to a member of $\mathcal{K}_3$ or $\mathcal{L}_3$: it is simply a question of whether $w = (\alpha - \beta \omega) u + (\gamma \omega + \delta) v$ is isotropic or non–isotropic in $L^3$. We calculate that $w$ is isotropic precisely when $\alpha \beta - \beta \gamma \omega w^2 = 0$.

We can write $f^{-1}(v_1) = \alpha_1 u_1 + \alpha_2 u_2 + v_1 + \alpha_3 v_2 + \alpha_4 e_1$ and $f^{-1}(v_2) = \beta_1 u_1 + \beta_2 u_2 + \beta_3 v_2 + \beta_4 e_1$ for some $\alpha_i, \beta_i \in K$, with $\beta_2, \beta_3$ not both zero. If $y = \theta v_1 + \varphi v_2$, with $\theta, \varphi \in K$, then the condition for $f^{-1}(y - u) = y_1 + y_2$ with $k_{y_2}$ isotropic is:

$$(\theta \alpha_1 + \varphi \beta_1 - 1)(\theta \alpha_3 + \varphi \beta_3) + (\theta \alpha_2 + \varphi \beta_2)\theta w^2 = 0,$$
\[ \theta^2(\alpha_1 \alpha_3 + \alpha_2 w^2) + \varphi^2 \beta_1 \beta_3 + \theta \varphi (\alpha_1 \beta_3 - \alpha_3 \beta_1 + \beta_2 w^2) - \alpha_3 \theta - \beta_3 \varphi = 0. \]

This equation holds for all values of \( \theta \) and \( \varphi \) if and only if all the coefficients are zero, i.e. if and only if \( \alpha_3 = \beta_3 = 0 \) and (hence) \( \alpha_2 = \beta_2 = 0 \). However \( \beta_2 \) and \( \beta_3 \) are not both zero, so for some choice of \( \theta, \varphi \), we have \( k_{y_2} \) non–isotropic.

The subspaces \( k_u \) and \( k_v \) of \( k_a^\perp \) are both isotropic and \( 1 \times U_2(q^2) \leq G \) (acting on \( k_a \oplus k_a^\perp \)) is transitive on non–zero vectors of \( k_a^\perp \) lying inside members of \( K_3 \). Thus there exists \( g \in 1 \times U_2(q^2) \) such that \( g(u) = y \) (with \( y \) chosen as above). Then \( f^{-1}gf(e_1) = e_1 \) and \( f^{-1}gf(e_2) = e_2 + f^{-1}(y - u) = e_2 + y_1 + y_2 \), with \( f^{-1}gf \in F \backslash G \) and \( f^{-1}gf(k_a) \leq k_a \oplus k_{y_2} \), this last subspace being non–isotropic. In conclusion, we can say that there is a choice of \( f \in F \backslash G \) and a choice of \( k_a \) such that \( f(k_a) \cap k_a \neq \{0\} \) and (using earlier notation) \( f(k_a) \subseteq k_a \oplus k_d \) with \( \langle g, f \rangle \) non–isotropic. This completes the proof of the Lemma.

\[ \square \]

5 The case \( n \geq 3, q \neq 3 \): Conclusions

In this Section we assume \( n \geq 3 \) and \( q \neq 3 \).

In Theorem 3.4 we have an initial case in an induction hypothesis: Theorem 2.1 holds when \( n = 2 \). Assume as an inductive argument that \( U_{n-1}(q^2) \cdot 2 \) is a maximal subgroup of \( Sp_{2n-2}(q) \). Note also that \( U_1(q^2) \cdot 2 \) is a maximal subgroup of \( Sp_2(q) \). By Lemma 4.1, if \( G < F \leq Sp_{2n}(q) \) then there is a non–isotropic 2–dimensional subspace \( k_x \in L_n \) such that if \( F_1 \) and \( F_2 \) are the projections of \( Stab_F(k_x) \) acting on \( k_x \) and \( k_x^\perp \), respectively, then either \( U_1(q^2) \cdot 2 < F_1 \) or \( U_{n-1}(q^2) \cdot 2 < F_2 \) (or both). It follows that either \( F_1 = Sp_2(q) \) or \( F_2 = Sp_{2n-2}(q) \).

Suppose that \( F_2 = U_{n-1}(q^2) \cdot 2 \). Then \( F_1 = Sp_2(q) \) and the subgroup \( \{ f_1 \in F_1 : (f_1, f_2) \in F \text{ for some } f_2 \in U_{n-1}(q^2) \} \) forms a subgroup of \( F_1 \) of index at most two, but \( Sp_2(q) \) has no subgroup of index two. Furthermore \( 1 \times U_{n-1}(q^2) \leq G \) and so \( Sp_2(q) \times 1 \leq F \). There exists \( g \in G \) such that \( g(k_x) = k_u \subseteq k_x^\perp \). Expressing \( V \) as \( k_x \oplus (k_x^\perp \cap k_u^\perp) \oplus k_u \), we see that \( F \) contains \( Sp_2(q) \times 1 \times 1 \) and \( g(Sp_2(q) \times 1 \times 1)g^{-1} = (1 \times 1 \times Sp_2(q)) \). The last subgroup is contained in \( Stab_F(k_x) \) but not in \( Sp_2(q) \times (U_{n-1}(q^2) \cdot 2) \). We conclude that \( F_2 \) cannot be just \( U_{n-1}(q^2) \cdot 2 \) and therefore \( F_2 = Sp_{2n-2}(q) \).
The subgroup \( \{f_2 \in F_2 : (1, f_2) \in F\} \) of \( F_2 \) is a normal subgroup of index at most \(|Sp_2(q)|\), but \( PSp_{2n-2}(q) \) is simple and the centre of \( Sp_{2n-2}(q) \) has order 2 so \( 1 \times F_2 \leq F \). Utilizing \( u \) and \( g \) as above, \( F \) contains \( g(1 \times Sp_{2n-2}(q))^{-1} = Sp_{2n-2}(q) \times 1 \) (where the first expression is acting on \( k_x \oplus k_z^\perp \) and the second on \( k_u \oplus k_a \)). In particular \( F \) contains \( Sp_2(q) \times 1 \times 1 \so contains \( Sp_2(q) \times Sp_{2n-2}(q) \), the stabilizer of \( k_x \) in \( Sp_{2n}(q) \). This stabilizer is maximal in \( Sp_{2n}(q) \), [13, Section 3] but does not contain \( Sp_{2n-2}(q) \times 1 \), so \( F = Sp_{2n}(q) \). Hence \( G \) is maximal in \( Sp_{2n}(q) \). We have proved the Theorem 2.1 except in the case \( q = 3 \).}

\[ \square \]

6 The case \( n \geq 3, q = 3 \): Conclusions

To begin with let us note that the only shortcoming in the previous Section when applied to the case \( q = 3 \) is the non-maximality of \( U_2(q^2) \cdot 2 \) in \( Sp_4(q) \). In this Section we show that \( U_3(9) \cdot 2 \) is maximal in \( Sp_6(3) \). It will then follow that \( U_n(9) \cdot 2 \) is maximal in \( Sp_{2n}(3) \) for all \( n \geq 3 \).

Suppose that \( n = 3 \) and \( q = 3 \). There are just three possibilities for \( F_2 \): \( F_2 = U_2(9) \cdot 2 \) with \(|F_2| = 2^4 \cdot 6\); \( F_2 \cong 2 \cdot 2^4 \cdot A_5 \); and \( F_2 = Sp_4(3) \).

If \( F_2 = U_2(9) \cdot 2 \), then \( F_1 = Sp_2(3) \) and, as in the general case, \( Sp_2(3) \times 1 \leq F \) and we can construct \( 1 \times 1 \times Sp_2(3) \triangleleft F \) to conclude that \( U_2(9) \cdot 2 \) is maximal in \( Sp_{2n}(3) \) for all \( n \geq 3 \).

If \( F_2 = Sp_4(3) \) then the arguments of the previous Section apply without modification: \( 1 \times Sp_4(3) \leq F \) and \( Sp_4(3) \times 1 \leq F \), leading to \( F = Sp_6(3) \).

We concentrate on the remaining possibility (and demonstrate that it cannot occur): \( F_2 \cong 2 \cdot 2^4 \cdot A_5 \). The order of \( Sp_3(3) \) is 24, so the subgroup \( R = \{f_2 \in F_2 : (1, f_2) \in F\} \) is a normal subgroup of \( F_2 \) of index at most 24 containing \( U_2(9) \). If \( N \) is the normal subgroup \( 2 \cdot 2^4 \) of \( F_2 \), then \( NR/N \) is isomorphic to a normal subgroup of \( A_5 \) of index at most 24, i.e. \( NR \cong F_2 \).

Given that \( R \) contains \( U_2(9) \) with structure \( 2 \cdot 2^3 \cdot S_3 \), the only possibilities for \( R \) are \( R = F_2 \) and \( R \cap N = 2 \cdot 2^3 \), with \( R/(2 \cdot 2^3) \cong A_5 \). However any element of \( F_2 \) of order 5 corresponds to an element of \( PSp_4(3) \) acting on \( 2^4 \) by conjugation, with three orbits of length five, so the image of \( F_2 \) in \( PSp_4(3) \) has no subgroup with structure \( 2 \cdot 2^3 \cdot A_5 \). Hence \( R = F_2 \) and \( F \) contains \( 1 \times F_2 \). In particular if \( k_y \) is a member of \( L_3 \) contained in \( k_z \) and we write \( k_z^\perp = k_z \), then by Theorem 3.5, \( Stab_{F_2}(k_y) \) has projections acting on each of \( k_y \) and \( k_z \) as \( Sp_2(3) \). Let \( g \in G \) such that \( g(k_x) = k_z \). Then \( gF_2g^{-1} \leq F \) fixes each vector in \( k_z \). Moreover \( Stab_{gF_2g^{-1}}(k_x) \) has projections acting on each of \( k_z \) and \( k_y \) as \( Sp_2(3) \). But \( Stab_{gF_2g^{-1}}(k_x) \leq F_1 \times F_2 \). It follows that
$F_1 = \text{Sp}_2(3)$, that $F_2$ contains $\text{Sp}_2(3) \times 1$ (written with respect to $k_y \oplus k_z$) and hence that $F_2$ contains also $1 \times \text{Sp}_2(3)$. Therefore $F_2$ contains a subgroup $\text{Sp}_2(3) \times \text{Sp}_2(3)$, which is impossible given the order of $F_2$. In conclusion $F_2$ cannot be isomorphic to $2 \cdot 2^4 \cdot A_5$. Thus $F_2 = \text{Sp}_4(3)$ and $F = \text{Sp}_6(3)$.

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**References**


