Maximal orthogonal subgroups of finite unitary groups

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Abstract. Certain orthogonal subgroups of finite unitary groups belonging to the fifth Aschbacher class $C_5$ are studied and their maximality is proved using the geometry of permutable polarities.

1 Introduction

Aschbacher’s Theorem, proved in [1], states that a maximal subgroup of a finite classical group either belongs to one of eight ‘geometric’ classes $C_1, \ldots, C_8$, or has a non-abelian simple group as its generalized Fitting subgroup. At least seven of these eight classes can be described as stabilizers of geometric configurations. In consequence, one might prefer a direct geometric approach to the classification of maximal subgroups over one dependent on the classification of finite simple groups. This is the approach adopted by R. H. Dye and O. H. King in a number of papers (e.g. [7], [12] and [13]) and is also the approach taken by the authors in [3] and [4].

As in [3], we are interested in Aschbacher’s class $C_5$. For a classical group $G$ acting on an $n$-dimensional vector space $V$ over a field $K$, the class $C_5$ is the collection of normalizers of the classical groups acting on the $n$-dimensional vector spaces $V_0$ over maximal subfields $K_0$ of $K$ such that $V = K \otimes_{K_0} V_0$. Apart from the work of Kleidman and Liebeck [14], very little has been done for subgroups belonging to this class. As far as we know there are just three other papers, by Li [15], [16] and Li and Zha [17], devoted to this class.

This paper can be considered a continuation of [3], where the maximality of certain symplectic groups of the unitary groups was proved. Here we prove the maximality of certain orthogonal subgroups of the finite unitary group $\text{PSU}_n(q^2)$ for $n \geq 3$. Our main result is expressed in terms of, and our approach to the proof depends on, the geometry of permutable polarities in projective spaces, described in [19]. We prove the following result.

Theorem. Suppose that $n \geq 3$ and that $q$ is odd. If $\mathcal{U}$ a non-degenerate unitary polarity on $\text{PG}(n - 1, q^2)$ and if $\mathcal{B}$ is an orthogonal polarity commuting with $\mathcal{U}$, then the set of absolute points of $\mathcal{U}$ fixed by the non-linear collineation $\mathcal{V} = \mathcal{U}\mathcal{B}$ forms a non-
degenerate quadric in a subgeometry PG(n − 1, q). The stabilizer of the quadric in 
PSU_n(q^2) is maximal except when n = 3 and q = 3 or 5 and when n = 4, q = 3 and the 
quadric is hyperbolic.

Our approach is essentially an induction argument in which a reduction to lower 
dimension is achieved via 'hyperbolic rotations'. These are elements of order q − 1 
that 'rotate' most of the points on a hyperbolic line and fix two points of the line and 
the points on its orthogonal complement. To a certain extent, our result has also been 
obtained by Li and Zha in [16], using suitable subgroups of unitary transvections.
Throughout the paper, q will be a power of an odd prime p and n ≥ 3. We note that 
over fields of even order the orthogonal groups have symplectic groups as overgroups 
and so cannot be maximal.

2 The geometric setting: orthogonal polarities commuting with a unitary polarity

In PG(n − 1, q^2) a non-singular Hermitian variety is defined to be the set of all abso-
lute points of a non-degenerate unitary polarity \( U \), and it is denoted by \( \mathcal{U}(n − 1, q^2) \).
The number of points on \( \mathcal{U}(n − 1, q^2) \) is shown in [19] to be

\[ [q^n + (-1)^{n−1}][q^{n−1} − (-1)^{n−1}]/(q^2 − 1). \]

We write \( \mathcal{B} \) for PG(n − 1, q^2) and \( \mathcal{H} \) for \( \mathcal{H}(n − 1, q^2) \) when the context is clear.

Let \( \mathcal{B} \) be an orthogonal polarity commuting with the unitary polarity \( \mathcal{U} \) associated 
with \( \mathcal{H} \). Set \( \mathcal{V} = \mathcal{B} \mathcal{U} = \mathcal{U} \mathcal{B} \). Then \( \mathcal{V} \) is a non-linear collineation and from [19], the 
fixed points of \( \mathcal{V} \) on \( \mathcal{H} \) form a non-degenerate quadric \( \mathcal{Q} \). Moreover, the complete 
set of points of \( \Sigma \) fixed by \( \mathcal{V} \) forms a subgeometry \( \Sigma_0 \) isomorphic to PG(n − 1, q) 
such that \( \mathcal{Q} = \Sigma_0 \cap \mathcal{H} \). Notice that the points of \( \Sigma \) fixed under \( \mathcal{V} \) are those admitting 
the same tangent or polar space with respect to both the unitary polarity and the 
orthogonal polarity. If \( \Pi \) is a subspace of \( \Sigma \) such that \( \mathcal{V}(\Pi) = \Pi \), then \( \Pi \) is a subs-
pace of \( \Sigma \) generated by a subspace \( \Pi_0 \) of \( \Sigma_0 \); we also say that \( \Pi \) is an extension of \( \Pi_0 \) 
to a subspace of \( \Sigma \). If \( P \) is a point of \( \Sigma \\setminus \Sigma_0 \), then \( P + \mathcal{V}(P) \) is a line of \( \Sigma \) extending a 
line of \( \Sigma_0 \). Moreover if \( L \) is a line of \( \Sigma \) extending a line \( L_0 \) of \( \Sigma_0 \) and if \( P \in L \setminus L_0 \), then 
\( P + \mathcal{V}(P) = L \). We use the terms secant line and external line to refer to lines of \( \Sigma_0 \) 
that are secant and external with respect to \( \mathcal{Q} \).

In terms of forms, let us assume that \( (V, H) \) is an n-dimensional unitary space over 
\( K = GF(q^2) \); \( H \) is a non-degenerate hermitian form on \( V \) with \( H(\alpha u, \beta v) = \alpha \beta^* H(u, v) \) 
for all \( \alpha, \beta \in K \) and all \( u, v \in V \). Let \( K_0 \) be the subfield \( GF(q) \) of \( K \). Choose a basis 
v_1, \ldots, v_n of \( V \) such that \( H(v_i, v_j) \in K_0 \) for all \( i, j \) and let \( W \) denote the \( K_0 \)-span of 
these vectors. It turns out that the restriction \( H_0 \) of \( H \) to \( W \) is a non-degenerate sym-
metric bilinear form. If the basis is an orthonormal basis, then the discriminant of \( H_0 \) 
is a square. Replacing \( v_1 \) by \( \xi v_1 \), where \( \xi \) is a generator of the multiplicative group 
\( GF(q^2)^* \) of \( GF(q^2) \), the discriminant of \( H_0 \) is a non-square. Therefore when \( n \) is even 
we obtain embeddings \( O_n(q) \subset U_n(q^2) \) for both \( \varepsilon = + \) and \( \varepsilon = − \).

As above, let \( \xi \) be a generator for \( GF(q^2)^* \). Then \( \lambda = \xi q^\varepsilon + 1 \) is a generator for the 
multiplicative group \( GF(q^2)^* \) of \( GF(q) \) and is necessarily a non-square in \( GF(q) \). Let
$$G = \langle \text{GO}_n(q), \xi I_n \rangle$$ (the centre of GU_n(q^2)), let $$\mathcal{G}_1 = G \cap U_n(q^2)$$, let $$\mathcal{G}_2 = G \cap SU_n(q^2)$$ and let $$\mathcal{G}_3 = \langle \text{SO}_n(q), \xi I_n \rangle \cap SU_n(q^2)$$. We write G and S for the stabilizers of 2 in SU(n, q^2) and U(n, q^2) and $$\bar{G}$$ for the stabilizer of 2 in PSU(n, q^2). It is of some interest to know the structure of G and related groups.

**Proposition 2.1.** $$S = \mathcal{G}_1$$ and $$G = \mathcal{G}_2$$ with $$\bar{G}$$ the image of G in PSU(n, q^2).

**Proof.** Suppose that $$h \in S$$. Observe that if L is a line of Σ extending a secant line $$L_0$$ of $$\Sigma_0$$ and if $$\{X, Y\} = 2 \cap L$$, then $$hX, hY$$ are non-orthogonal points in 2 and so $$hL$$ also extends a secant line of $$\Sigma_0$$. Moreover if there is a third point R of $$L_0$$ such that $$hR \in \Sigma_0$$, then $$hP \in \Sigma_0$$ for all $$P \in L_0$$. Suppose that $$n \geq 4$$. For any point Z of $$\Sigma_0 \backslash 2$$, the polar space $$Z^\perp$$ is spanned by points of 2 and the same is true of $$h(Z^\perp)$$. Then $$h(Z^\perp)$$ extends a subspace of $$\Sigma_0$$ to one of Σ and therefore the same is true of $$hZ$$, i.e. $$hZ \in \Sigma_0$$. If $$n = 3$$, then the same argument works for one class of points of $$\Sigma_0 \backslash 2$$.

If Z lies in the second class, then $$Z^\perp$$ is spanned by points of the first class and the same argument can be applied again. Hence h fixes $$\Sigma_0$$ and h must be the product of an element of GL_n(q) and a central element of U(n, q^2). It is well known that the stabilizer of 2 in GL_n(q) is GO_n(q); see [13]. Therefore $$h \in \mathcal{G}_1$$. The elements of $$\mathcal{G}_1$$ stabilize 2, so that $$S = \mathcal{G}_1$$ and the stabilizer G of 2 in SU_n(q^2) is $$\mathcal{G}_2$$. The stabilizer $$\bar{G}$$ of 2 in PSU_n(q^2) is then the image of G in PSU_n(q^2).

**Proposition 2.2.** The stabilizer $$\bar{G}$$ of 2 in PSU_n(q^2) is isomorphic to $$\text{PSO}_n(q) \cdot 2$$ when n is even and to $$\text{PSO}_n(q)$$ when n is odd.

**Proof.** We identify the elements of $$\mathcal{G}$$ and GO_n(q) that lie in U_n(q^2) and can then consider $$\bar{G}$$ as $$\text{PGO}_n(q) \cap \text{PSU}_n(q^2)$$. Recall that λ is a generator for GF(q)^*.

Assume that n is even with $$n = 2m$$. First we construct an element $$g_\lambda \in \text{GO}_n(q)$$ such that $$\text{GO}_n(q) = \langle \text{O}_n(q), g_\lambda \rangle$$. The structure of $$g_\lambda$$ depends on the Witt index of $$H_0$$ and on the value of q. We take a basis $$x_1, x_2, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}, x_m, y_m$$ for V such that

$$H_0(x_i, x_j) = H_0(y_i, y_j) = H_0(x_i, x_m) = H_0(x_i, y_m) = H_0(y_i, x_m) = H_0(y_i, y_m) = 0$$

and $$H_0(x_i, y_j) = \delta_{ij}$$ for $$i, j \leq m - 1$$. If the Witt index of $$H_0$$ is $$m$$ (i.e., in the hyperbolic case), we take

$$H_0(x_m, x_m) = H_0(y_m, y_m) = 0 \quad \text{and} \quad H_0(x_m, y_m) = 1.$$

If the Witt index is $$m - 1$$ (i.e., in the elliptic case), we take

$$H_0(x_m, x_m) = 1, \quad H_0(y_m, y_m) = \lambda, \quad H_0(x_m, y_m) = 0$$

when $$q \equiv 1 \pmod{4}$$ and

$$H_0(x_m, x_m) = 1, \quad H_0(y_m, y_m) = 1, \quad H_0(x_m, y_m) = 0$$
when \( q \equiv 3 \pmod{4} \). Let \( g_q \) be the block-diagonal matrix \((\lambda I_{m-1}, I_{m-1}, X)\), where \( X \) is the \( 2 \times 2 \) matrix given in the three cases above by

\[
\begin{pmatrix}
\lambda & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
-\lambda & 0
\end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}
\]

(where \( a^2 + b^2 = \lambda \)) respectively. The element \( g_q \) so constructed has multiplier \( \lambda \) (i.e., \( H_0(g_q(x), g_q(y)) = \lambda H_0(x, y) \)). Any element of \( G_0(n, q) \) may be written \( h g_q^i \) with \( 0 \leq i \leq q-2 \) and with \( h \in O_n(q) \) (as \( g_q^{q-1} \in O_n(q) \)) and any element in \( G \) can be written \( h g_q^i z^t I_n \). Note that \( z^t I_n \) has multiplier \( \lambda \), so that \( k \lambda = g_q^i z^t I_n \) has multiplier 1, i.e., \( k \lambda \in U_n(q^2) \). Examining the elements in \( S \) we see that they may be written \( k \lambda^j z^t I_n = h g_q^i z^t I_n \) with \( z^{(q+1)(r+t)} = 1 \), so that \( z^{r+t} = z^{(q-1)s} \) for some \( 0 \leq s \leq q \). In other words elements in \( S \) may be written \( h k \lambda^j z^t I_n \) where \( l = q^{q-1} I_n \) and \( 0 \leq s \leq q \). In fact \( g_q^2 \) has the same multiplier as \( \lambda I_n \) and so \( g_q^2 = h_1 \lambda I_n \) (for some \( h_1 \in O_n(q) \)) and therefore

\[
k \lambda^2 = g_q^2 z^{-2} I_n = h_1 \lambda I_n z^{-2} I_n = h_1 I,
\]

so that elements in \( S \) may be written \( h k \lambda^j z^t I_n \) with \( r = 0 \) or 1. We can characterize the elements of \( G \) as lying in exactly one of the following classes:

1. \( G_3^* \);
2. \( \{ h l^s : h \in O_n(q); \det h = -1; \det l^s = -1 \} \);
3. \( \{ h k \lambda^j I_n : h \in SO_n(q); \det k \lambda^j I_n = 1 \} \);
4. \( \{ h k \lambda^j I_n : h \in O_n(q); \det h = -1; \det k \lambda^j I_n = 1 \} \).

We can characterize the classes (II), (III) and (IV) as follows:

Class II: \( z^{(q-1)} n \equiv -1 \pmod{q} \), i.e. \( (q + 1) \) divides \( n - \frac{1}{2} (q + 1) \).

Class III: \( z^{(q-1)} n \equiv 1 \pmod{q} \), i.e. \( (q + 1) \) divides \( n + \frac{1}{2} n \).

Class IV: \( z^{(q-1)} n \equiv 0 \pmod{q} \), i.e. \( (q + 1) \) divides \( n \) and \( n - (q + 1) \).

Notice that no two of II, III, IV can occur simultaneously. Let \( d = \text{HCF}(n, q + 1) \).

There are three possibilities. If \( (q + 1)/d \) is even but \( n/d \) is odd, let \( s = (q + 1)/(2d) \).

Then \( sn - \frac{1}{2} (q + 1) = \frac{1}{2} (q + 1) [(n/d) - 1] \) and so Case II occurs. If \( (q + 1)/d \) is odd but \( n/d \) is even, let \( s = \frac{1}{2} [((q + 1)/d) - 1] \). Then \( sn + \frac{1}{2} n = (q + 1)[n/(2d)] \) and so Case III occurs. Finally if \( (q + 1)/d \) and \( n/d \) are both odd, let \( s = \frac{1}{4} [((q + 1)/d) - 1] \). Then \( sn + \frac{1}{2} n = \frac{1}{2} (q + 1) [(n/d) - 1] \) and so Case IV occurs.

Hence exactly one of Cases II, III, IV occurs, depending on the values of \( n \) and \( q \). Moreover two elements of a given class differ by an element of \( G_3 \), so that \( |G : G_3| = 2 \). Factoring out by scalars we conclude that \( G = \text{PSO}_n(q), 2 \); in Case II we have \( G = \text{PO}_n(q) \) but not in the others.

Now assume that \( n \) is odd with \( n = 2m + 1 \). In this case

\[
G_0(n, q) = \langle O_n(q), \lambda I_n \rangle = \langle \text{SO}_n(q), \lambda I_n \rangle,
\]
since \(-I_n\) has determinant \(-1\) and is a power of \(\lambda I_n\). Similarly, \(G = \langle SO_n(q), \xi I_n \rangle\) and we now see that \(G = G_3\), from which it follows that \(G = \text{PSO}_n(q)\).

**Proposition 2.3.** \(G\) is the normalizer of \(SO_n(q)\) in \(SU_n(q^2)\) and \(\tilde{G}\) is the normalizer of \(\text{PSO}_n(q)\) in \(\text{PSU}_n(q^2)\).

**Proof.** Certainly the normalizer of \(SO_n(q)\) in \(SU_n(q^2)\) contains \(G\). One orbit of \(SO_n(q)\) on \(\Sigma\) is \(\mathcal{A}\) and so if \(h \in SU_n(q^2)\) normalizes \(G\) then \(h\mathcal{A}\) is also an orbit of \(SO_n(q)\). We show that \(h\mathcal{A}\) must be \(\mathcal{A}\) so that \(h \in G\).

There are four types of line (three in small cases) in \(\Sigma_0\): external, secant, tangent to \(\mathcal{A}\), on \(\mathcal{A}\). Suppose that \(L_0\) is a line of \(\Sigma_0\) and \(L\) is its extension to a line of \(\Sigma\). If \(L_0\) is an external line, then \(L\) is non-isotropic with \(q+1\) points on \(\mathcal{H} \setminus \mathcal{A}\) and these lie in a single orbit under \(O_\Sigma^-(q)\) (the action on \(L\) of the stabilizer in \(SO_n(q)\) of \(L_0\)). If \(L_0\) is a secant line, then \(L\) is non-isotropic with two points on \(\mathcal{A}\), and \(q-1\) singular points on \(\mathcal{H} \setminus \mathcal{A}\) in a single orbit under \(O_\Sigma^+(q)\) (again, the action on \(L\) of the stabilizer in \(\text{PSO}_n(q)\) of \(L_0\)). If \(L_0\) is a tangent line, then \(L\) contains exactly one point of \(\mathcal{H}\) and this point on \(\mathcal{A}\). If \(L_0\) is a line on \(\mathcal{A}\), then \(L\) contains \(q+1\) points of \(\mathcal{A}\) and \(q^2 - q\) of \(\mathcal{H}\) that lie off \(\mathcal{A}\) but lie in a single orbit under the stabilizer in \(SO_n(q)\) of \(L_0\) acting as \(\text{PGL}_2(q)\) on \(L\).

If \(n \geq 5\), then \(SO_n(q)\) has a single orbit of each of the three types of line: external, secant, and on \(\mathcal{A}\). If \(n = 4\) and \(\mathcal{A}\) is an elliptic quadric or if \(n = 3\), then there are no lines on \(\mathcal{A}\) but \(SO_n(q)\) has a single orbit of external lines and a single orbit of secant lines. If \(n = 4\) and \(\mathcal{A}\) is a hyperbolic quadric, then \(SO_n(q)\) has a single orbit of external lines and a single orbit of secant lines, but has two orbits (of equal size) of lines on \(\mathcal{A}\). Moreover \(SO_n(q)\) is transitive on the points of \(\mathcal{A}\) for all \(n \geq 3\). Thus \(SO_n(q)\) has 3, 4 or 5 orbits of points on \(\mathcal{H}\), depending on \(n\) and the nature of \(\mathcal{A}\).

The numbers of points and lines in \(\Sigma_0\) of various types are given in [11, pp. 23, 25]. In each case we list the relevant number for (i) the elliptic quadric \((n = 2m)\), for (ii) the hyperbolic quadric \((n = 2m)\), and for (iii) the parabolic quadric \((n = 2m + 1)\).

The number \(N_0\) of points on \(\mathcal{A}\) is

1. \([(q^m + 1)(q^{m-1} - 1)]/[q - 1],
2. \([(q^m - 1)(q^{m-1} + 1)]/[q - 1],
3. \([(q^m - 1)(q^m + 1)]/[q - 1].

The number \(N_1\) of lines on \(\mathcal{A}\) is

1. \([(q^m + 1)(q^{m-1} + 1)(q^{m-1} - 1)(q^{m-2} - 1)]/[\{(q - 1)(q^2 - 1)\}],
2. \([(q^m - 1)(q^{m-1} - 1)(q^{m-1} + 1)(q^{m-2} + 1)]/[\{(q - 1)(q^2 - 1)\}],
3. \([(q^m - 1)(q^{m-1} - 1)(q^m + 1)(q^{m-1} + 1)]/[\{(q - 1)(q^2 - 1)\}].

The number \(N_2\) of secant lines to \(\mathcal{A}\) is

1. \([(q^m + 1)(q^{m-1} - 1)q^{2m-2}]/[2(q - 1)],
2. \([(q^m - 1 + 1)(q^m - 1)q^{2m-2}]/[2(q - 1)],
3. \([(q^m + 1)(q^m - 1)q^{2m-1}]/[2(q - 1)].

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The number $\mathcal{N}_3$ of external lines to $\mathcal{O}$ is

1. $[(q^m + 1)(q^{m-1} + 1)q^{2m-2}]/[2(q + 1)]$.
2. $[(q^m - 1)(q^{m-1} - 1)q^{2m-2}]/[2(q + 1)]$.
3. $[q^{2m-1}(q^m - 1)]/[2(q + 1)]$.

The number of points of $\mathcal{H}$ arising from lines on $\mathcal{O}$, secant lines to $\mathcal{O}$ and external lines to $\mathcal{O}$ is $(q^2 - q)\cdot \mathcal{N}_1$, $(q - 1)\cdot \mathcal{N}_2$ and $(q + 1)\cdot \mathcal{N}_3$ respectively, these numbers being greater than $2\cdot \mathcal{N}_0$. $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively (excluding the cases where $\mathcal{N}_1 = 0$). Thus $\mathcal{O}$ represents the smallest orbit of $G$ and since $h\mathcal{O}$ is an orbit we conclude that $h\mathcal{O} = \mathcal{O}$, i.e. $h \in G$. Hence $G$ is the normalizer of $\text{PSO}_n(q)$ in $\text{PSU}_n(q^2)$.

3 The maximality

In this section we prove that $G$ is maximal in $\text{PSU}_n(q^2)$ (with a small number of exceptions). What we prove is a little stronger: if $F$ is a subgroup such that $\text{SO}_n(q) \leq F \leq \text{SU}_n(q^2)$ and $F \not\leq G$, then $F = \text{SU}_n(q^2)$. We perform a dimension reduction argument using elements of $\text{SO}_n(q)$ known as hyperbolic rotations. These are discussed in [6, p. 26]: a hyperbolic rotation is an element of $\text{SO}_n(q)$ that fixes both points of $\mathcal{O}$ on a secant line $L_0$ of $\Sigma_0$ and acts as the identity on $L_0^\perp$; in matrix terms it has the form $\text{diag}(x, x^{-1}, 1, \ldots, 1)$ with respect to an appropriate basis, where $0 \neq x \in \text{GF}(q)$. It is shown in [6] that $\text{SO}_n(q)$ is generated by such elements. More generally there are such elements in $\text{SU}_n(q^2)$ fixing non-orthogonal points $X, Y$ of $\mathcal{H}$ and acting as the identity on $(X + Y)^\perp$: we continue to take $z \in \text{GF}(q)$, and when $z$ is equal to $\lambda$, a generator of $\text{GF}(q)^\ast$ we denote the element by $h_{XY}$. If $x, y$ are vectors representing $X, Y$, chosen such that $H(x, y) = 1$, then

$$h_{XY}(v) = v + (\lambda - 1)H(v, y)x + (\lambda^{-1} - 1)H(v, x)y.$$ 

If $g \in \text{SU}_n(q^2)$ then $gh_{XY}g^{-1} = h_{X'Y'}$, where $X' = gx$ and $Y' = gy$. The subgroup $H_{XY} = H_{YX}$ generated by $h_{XY}$ has order $q - 1$, fixes $X$, $Y$ and each point in $(X + Y)^\perp$, has orbits of length $\frac{1}{q-1}$ on the remaining points of $X + Y$ and orbits of length $q - 1$ on points of $\Sigma$ not in $X + Y$ or $(X + Y)^\perp$. In our reduction argument we show that usually $F$ contains a hyperbolic rotation $h_{XY}$ not in $G$ such that $(X + Y)^\perp$ contains a point $Z \in \Sigma_0 \setminus \mathcal{O}$. Let $F_{n-1}$ be the subgroup of $\text{SU}_{n-1}(q^2)$ such that $1 \times F_{n-1} = F \cap (1 \times \text{SU}_{n-1}(q^2))$ (written with respect to the decomposition $Z \oplus Z^\perp$). We may regard $h_{XY}$ as an element of $F_{n-1}$ that does not lie in $\text{SO}_{n-1}(q)$ and we can apply induction to conclude that $F_{n-1} = \text{SU}_{n-1}(q^2)$. The initial case when $n = 3$ was handled by Mitchell [18], although exceptions when $q = 3$ or 5 necessitate careful argument.

The first result largely characterizes the hyperbolic rotations that lie in $G$. If $X, Y \in \mathcal{O}$ (with $Y \notin X^\perp$), then $h_{XY} \in \text{SO}_n(q)$. Otherwise $h_{XY}$ does not usually lie in $G$. 

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Proposition 3.1. Suppose that \( n \geq 4 \).

(i) If \( X \in \mathcal{A} \) and \( Y \in \mathcal{H} \setminus \mathcal{A} \) (with \( Y \neq X^\perp \)), then \( h_{XY} \notin G \) except when \( q = 3 \) and \( X \in Y + \mathcal{V}(Y) \).

(ii) If \( X, Y \in \mathcal{H} \setminus \mathcal{A} \) (with \( Y \neq X^\perp \)) and \( X + \mathcal{V}(X) = Y + \mathcal{V}(Y) \), then \( h_{XY} \notin G \) except when \( q = 3 \).

(iii) If \( X, Y \in \mathcal{H} \setminus \mathcal{A} \) (with \( Y \neq X^\perp \)) and \( L = X + \mathcal{V}(X) \neq N = Y + \mathcal{V}(Y) \), then \( h_{XY} \notin G \) except (conceivably) when \( n = 4 \), \( L^\perp = L \) and \( N^\perp = N \).

Proof. (i) Let \( L = Y + \mathcal{V}(Y) \). Then \( L \) is a line of \( \Sigma \) fixed by \( \mathcal{V} \) and is thus the extension to \( \Sigma \) of a line \( L_0 \) of \( \Sigma_0 \). If \( X \in L \), then \( L_0 \) is a secant line of \( \Sigma_0 \) and \( H_{XY} \) fixes two points of \( L_0 \) with all other points lying in orbits of length \( \frac{1}{2}(q - 1) \). Thus, if \( q \neq 3 \), then the second point of \( \mathcal{A} \) lying on \( L \) is mapped by \( h_{XY} \) to a point off \( \mathcal{A} \). Now assume that \( X \) does not lie on \( L \) and write \( \Pi_0 \) for the plane \( X + L_0 \) in \( \Sigma_0 \) and \( \Pi \) for its extension to a plane of \( \Sigma \) fixed by \( H_{XY} \). Then \( \Pi_0 \cap X^\perp \) is a line \( N_0 \) of \( \Pi_0 \) whose extension to \( \Sigma \) is a line \( N \) fixed by \( H_{XY} \). The remaining \( q^2 \) points of \( \Pi_0 \) lie neither in \( X + Y \) nor \( (X + Y)^\perp \) and hence lie in orbits of length \( q - 1 \) under \( H_{XY} \). Therefore for some point \( P \in \Pi_0 \setminus N_0 \), we have \( h_{XY}(P) \notin \Sigma_0 \). Thus \( h_{XY} \notin G \).

(ii) This time \( H_{XY} \) fixes two points of \( L = Y + \mathcal{V}(Y) \) (and the two points do not lie on \( \mathcal{A} \)) with all other points lying in orbits of length \( \frac{1}{2}(q - 1) \). Assume that \( q \neq 3 \). Then \( H_{XY} \) has exactly two fixed points on \( L \), namely \( X \) and \( Y \). Suppose that \( L_0 \) is the line of \( \Sigma_0 \) underlying \( L \). If \( L_0 \) is a secant line of \( \Sigma_0 \), then the subgroup of \( G \) that fixes \( L \) and acts as the identity on \( L^\perp \) must act on \( L \) as \( \text{SO}_3^+ \) \( (q) \), having order \( q - 1 \) and fixing exactly two points of \( L \), both in \( L_0 \); hence \( h_{XY} \notin G \). If \( L_0 \) is an external line, then the subgroup of \( G \) that fixes \( L \) and acts as the identity on \( L^\perp \) must act on \( L \) as \( \text{SO}_3^+ \) \( (q) \), having order \( q + 1 \) and therefore not containing \( H_{XY} \).

(iii) Given that \( L \neq N \) it follows that \( L^\perp \neq N^\perp \) and therefore \( L^\perp \) is spanned by points \( P \) of \( \Sigma_0 \) such that \( P \notin N^\perp \). Suppose that \( h = h_{XY} \) fixes \( \mathcal{A} \) and that \( P \in L^\perp \cap \Sigma_0 \) with \( P \notin N^\perp \). Then \( hP \neq P \) and \( hP \in P + X \), so that \( X \in P + hP \). Thus \( L = P + hP \) for all such \( P \). This can only happen if \( L^\perp = L \) in which case \( n = 4 \). The same argument applies to \( N \).

The following two propositions establish conditions sufficient for the reduction to lower dimension when \( n \geq 5 \).

Proposition 3.2. Suppose that \( X, Y \in \mathcal{H} \) with \( Y \neq X^\perp \). If either \( n \geq 5 \) and \( X \in \mathcal{A} \) or \( n \geq 7 \), then there is a point \( Z \) of \( \Sigma_0 \setminus \mathcal{A} \) lying in \( (X + Y)^\perp \). Moreover, if \( h_{XY} \in F \setminus G \), then \( h_{XY} \) does not fix \( Z \setminus \mathcal{A} \).

Proof. We write \( M \) for \( X + \mathcal{V}(X) + Y + \mathcal{V}(Y) \). Then \( M \) is fixed by \( \mathcal{V} \) so extends a subspace \( M_0 \) of \( \Sigma_0 \) and the projective dimension of \( M_0 \) can be no greater than 3.

If \( n \geq 8 \), then \( M_0 \) cannot contain \( M_0^\perp \cap \Sigma_0 \), so that \( M_0^\perp \cap \Sigma_0 \) cannot lie on \( \mathcal{A} \) (if \( n = 8 \), then the fact that \( Y \neq X^\perp \) is pertinent). Hence there is a point of \( \Sigma_0 \setminus \mathcal{A} \) lying in \( M^\perp \).

If \( n = 7 \), then there is the possibility that \( M_0^\perp \cap \Sigma_0 \) lies on \( \mathcal{A} \). However this can only happen if \( M^\perp \) is a plane and \( M \) a solid, and in this case \( M^\perp \) contains all points of \( M \) lying on \( \mathcal{H} \), including \( X \) and \( Y \), contrary to \( Y \neq X^\perp \).
Now suppose that \( n \geq 5 \) with \( X \in \mathcal{Q} \). Then the projective dimension of \( M_0 \) can be no greater than 2 and \( M_0 \cap \Sigma_0 \) cannot lie on \( \mathcal{Q} \) unless \( n = 5 \), \( M \) is a plane and \( M^{\perp} \) is a line containing all the points of \( M \) lying on \( \mathcal{H} \), including \( X \) and \( Y \), contrary to \( Y \not\in X^{\perp} \).

In all cases here there is a point \( Z \) of \( \Sigma_0 \setminus \mathcal{Q} \) lying in \( M^{\perp} \), and such a point necessarily lies in \((X + Y)^{\perp}\). Finally, \( h_{XY} \) acts as an element of \( 1 \times \text{SU}_{n-1} (q^2) \) with respect to the decomposition \( Z \oplus Z^{\perp} \) and it follows from Proposition 3.1 that \( h_{XY} \) cannot fix \( Z^{\perp} \cap \mathcal{Q} \).

**Proposition 3.3.** Suppose that \( n = 5 \) or 6 and that \( F \) contains an element \( h_{XY} \notin G \) with \( X, Y \notin \mathcal{Q} \). Then either there is a point \( Z \) of \( \Sigma_0 \setminus \mathcal{Q} \) lying in \((X + Y)^{\perp} \) such that \( h_{XY} \) does not fix \( Z^{\perp} \cap \mathcal{Q} \) or there are \( X' \in \mathcal{Q} \), \( Y' \in \mathcal{H} \) such that \( h_{XY'} \in F \setminus G \).

**Proof.** Unless \( M \) is a solid and either \( n = 5 \) with \( M^{\perp} \) a point of \( \mathcal{Q} \) or \( n = 6 \) with \( M^{\perp} \) a line extending a line on \( \mathcal{Q} \), we can argue as in Proposition 3.2 that there is a point \( Z \) of \( \Sigma_0 \setminus \mathcal{Q} \) lying in \((X + Y)^{\perp} \) and that \( h_{XY} \) acts as an element of \( 1 \times \text{SU}_{n-1} (q^2) \) with respect to the decomposition \( Z \oplus Z^{\perp} \). If \( n = 6 \), then it follows from Proposition 3.1 that \( h_{XY} \) cannot fix \( Z^{\perp} \cap \mathcal{Q} \). Suppose that \( n = 5 \) and that \( h_{XY} \) fixes \( Z^{\perp} \cap \mathcal{Q} \). Then \( h = h_{XY} \) cannot lie in \( 1 \times \text{SO}_4 (q) \). Let \( B \) be a point of \( \mathcal{Q} \) not fixed by \( h \). Then \( B \) does not lie in \( Z^{\perp} \) or \( A^{\perp} \). Let \( A \) be a point \( Z^{\perp} \cap \mathcal{Q} \) not in \( B^{\perp} \). Then \( hA \in \mathcal{Q} \) but \( hB \notin \mathcal{Q} \). Moreover there is a point \( C \) of \( \Sigma_0 \setminus \mathcal{Q} \) such that \( B \) lies on the line \( Z + C \) and \( h(Z + C) \) extends a line of \( \Sigma_0 \). Hence \( hB + \varphi^{-1} (hB) = h(Z + C) \) and this line cannot contain \( hA \). Writing \( X' = hA \), \( Y' = hB \), we conclude from Proposition 3.1 that \( h_{X'Y'} \in F \setminus G \).

Suppose now that \( M \) is a solid and that either \( n = 5 \) with \( M^{\perp} \) a point of \( \mathcal{Q} \) or \( n = 6 \) with \( M^{\perp} \) a line extending a line on \( \mathcal{Q} \), then \( X, \varphi (X), Y, \varphi (Y) \) form a simplex for \( M \). We choose a point \( X' \in \Sigma_0 \) if \( n = 5 \) we choose \( X' = M^{\perp} \) and if \( n = 6 \) we choose \( X' \) to be any point of \( M^{\perp} \) on \( \mathcal{Q} \). Note that if \( B \in \mathcal{Q} \), then \( B \in X^{\perp} \) if and only if \( B \in \varphi^{-1} (X)^{\perp} \) with the corresponding statement for \( Y \) also true. Thus if \( B \in \mathcal{Q} \setminus X'^{\perp} \), then \( B \notin M \) and so \( B \notin X^{\perp} \cap Y^{\perp} \). In consequence \( Y' = h_{XY} (B) \notin \mathcal{Q} \). Now \( h_{X'Y'} \notin G \) and \( h_{X'Y'} = h_{XY} h_{X'B} h_{X'Y}^{-1} \in F \setminus G \) by Proposition 3.1, except possibly when \( q = 3 \).

Suppose that \( q = 3 \). If \( n = 5 \), then \( X' = M^{\perp} \) (as chosen above) cannot lie in both \( X + \varphi (X) \) and \( Y + \varphi (Y) \) (as the two lines have no points in common): without loss of generality we may assume that \( X' \notin L = Y + \varphi (Y) \). We write \( L_0 \) for the line of \( \Sigma_0 \) that is extended to give \( L \); it must be a secant or external line of \( \Sigma_0 \) and \( L_0 \cap \Sigma_0 \) is a non-isotropic plane containing \( X' \). Thus we may choose \( B \in (\mathcal{Q} \cap L^{\perp}) \setminus X'^{\perp} \) so that \( Y' = h_{XY} (B) \) lies on \((B + Y) \setminus \mathcal{Q} \), and then \( Y' + \varphi^{-1} (Y') \) lies in \( B + L \). Thus \( Y' + \varphi^{-1} (Y') \) cannot contain \( X' \): by Proposition 3.1, \( h_{X'Y'} \notin G \). If \( n = 6 \), then one possibility is that \( L = Y + \varphi^{-1} (Y) \) meets \( M \) in a point \( M^{\perp} \setminus \mathcal{Q} \), when we can choose \( X' \) to be any other point of \( M^{\perp} \) on \( \mathcal{Q} \) and choose \( B \in (\mathcal{Q} \cap L^{\perp}) \setminus X'^{\perp} \) (so necessarily not in \( M \)) with \( X' \notin B + L \). Otherwise \( L \) does not meet \( M^{\perp} \), and then any choice of \( X' \in M^{\perp} \) and \( B \in L^{\perp} \) such that \( B \notin \mathcal{Q} \) leads to \( X' \notin B + L \): again by Proposition 3.1 we have \( h_{X'Y'} \notin G \). In both cases \((n = 5 \text{ and } n = 6)\), the hyperbolic rotation \( h_{X'B} \) lies in \( G \) and so \( F \) contains \( h_{X'Y'} = h_{XY} h_{X'B} h_{X'Y}^{-1} \).

We need the next four propositions for the reduction argument when \( n = 4 \).
Suppose that \( n = 4 \). Then either there exists \( k \in F \setminus G \) such that \( \mathcal{A} \cap k \mathcal{A} \) is non-empty or there exist non-orthogonal points \( X, Y \in \mathcal{A} \) and \( k \in F \) such that \( k(X + Y) = X + Y \) but \( k(X), k(Y) \notin \mathcal{A} \).

**Proof.** Suppose that for all \( k \in F \setminus G \) and for all \( X \notin \mathcal{A} \) we have \( kX \notin \mathcal{A} \).

Suppose that for some \( X \in \mathcal{A} \), the line \( N = kX + \mathcal{A}(kX) \) extends a line \( N_0 \) of \( \Sigma_0 \) that lies on \( \mathcal{A} \). Then \( \mathcal{A} \) is hyperbolic and, as there are only 2\((q + 1)\) lines on \( \mathcal{A} \), there is a choice of \( X \) such that \( N \) contains a second point \( kA \) of \( \mathcal{A} \). It follows that all \( q + 1 \) points of \( X + A \in \Sigma_0 \) are mapped by \( k \) into \( N \) and none of the images lies on \( \mathcal{A} \). The stabilizer of \( N_0 \) in \( \text{SO}_+^4(q) \) acts as \( \text{GL}_2(q) \) on \( N \) with order \( (q^2 - 1)(q^2 - q) \), there being two orbits on \( N \); the \( q + 1 \) points of \( N_0 \) and the remaining \( q^2 - q \) points.

The stabilizer of one of the points of \( N \setminus N_0 \) has two fixed points of \( N \setminus N_0 \) and has orbits of length \( q + 1 \) on the remainder. It follows that we can find \( P \in k \mathcal{A} \cap N \) and \( q \in \text{Stab}_{\text{SO}_+^4(q)}(N_0) \) such that \( gkX = kX \) but \( gP \notin k \mathcal{A} \). Hence \( k^{-1}gk \in F \setminus G \) with \( k^{-1}gk \mathcal{A} \cap \mathcal{A} \) non-empty.

Suppose that \( \mathcal{A} \) is hyperbolic and that, for some \( k \in F \setminus G \) and some \( X \in \mathcal{A} \), the line \( N = kX + \mathcal{A}(kX) \) extends an external line \( N_0 \) of \( \Sigma_0 \). Then \( G \) has a subgroup \( E \) acting as \( \text{SO}_+^2(q) \) on \( N \) and as the identity on \( N \); the order of \( E \) is \( q + 1 \). The orbits of \( E \) acting on the points of \( \mathcal{A} \) not in \( N \) or \( N \setminus N_0 \) have length \( q + 1 \). The number of points of \( k \mathcal{A} \) not in \( N \) or \( N \setminus N_0 \) lies between \( q^2 + 2q - 3 \) and \( q^2 + 2q \), so that these points do not comprise a union of orbits under \( E \). Thus for some \( g \in E \) we have \( k^{-1}gk \in F \setminus G \) with \( k^{-1}gk \mathcal{A} \cap \mathcal{A} \) non-empty.

Suppose that \( \mathcal{A} \) is elliptic. Then there are \( q^2 + 1 \) points on \( \mathcal{A} \) and \( \frac{1}{2}(q^2 + 1)q^2(q + 1) \) points of \( \mathcal{A} \) lying on extensions of external lines of \( \Sigma_0 \). However \( \text{SU}_4(q^2) \) has order \( q^6(q^2 - 1)(q^2 + 1)(q^4 - 1) \), which is not divisible by \( q^2 + 1 + \frac{1}{2}(q^2 + 1)q^2(q + 1) \), so that these points cannot form an orbit under \( F \). Hence we must be some \( k \in F \setminus G \) and some \( X \in \mathcal{A} \) such that the line \( N = kX + \mathcal{A}(kX) \) extends a secant line \( N_0 \) of \( \Sigma_0 \).

Finally suppose that for some \( k \in F \setminus G \) and some \( X \in \mathcal{A} \), the line \( N = kX + \mathcal{A}(kX) \) extends a secant line \( N_0 \) of \( \Sigma_0 \). This time \( G \) has a subgroup \( E \) acting as \( \text{SO}_+^2(q) \) (respectively \( \text{SO}_-^2(q) \)) on \( N \) if \( \mathcal{A} \) is hyperbolic (respectively elliptic) and as the identity on \( N \); the order of \( E \) is \( q - 1 \) (respectively \( q + 1 \)). The orbits of \( E \) acting on the points of \( \mathcal{A} \) not in \( N \) or \( N \setminus N_0 \) have length \( q - 1 \) (respectively \( q + 1 \)). The number of points of \( k \mathcal{A} \) not in \( N \) or \( N \setminus N_0 \) lies between \( q^2 + 2q - 3 \) and \( q^2 + 2q \) in the hyperbolic case and between \( q^2 - 1 \) and \( q^2 \) in the elliptic case. Either these points do not comprise a union of orbits under \( E \), in which case for some \( g \in E \) we have \( k^{-1}gk \in F \setminus G \) with \( k^{-1}gk \mathcal{A} \cap \mathcal{A} \) non-empty (as above), or \( N \) contains two points of \( k \mathcal{A} \). In this latter case we can find \( Y \notin \mathcal{A} \) and \( g \in G \) such that \( N = k(X + Y) \) and \( gk(X + Y) = X + Y \), with neither \( U = gkX \) nor \( W = gkY \) in \( \mathcal{A} \).

**Proposition 3.5.** Suppose that \( n = 4 \) and \( q \neq 3 \). If there exists \( k \in F \setminus G \) with \( \mathcal{A} \cap k \mathcal{A} \) non-empty, then there exists \( h_{X,Y} \in F \setminus G \) with \( X' \in \mathcal{A} \). If no such \( k \) exists, then there exists \( h_{X',Y'} \in F \setminus G \) with \( X', Y' \notin \mathcal{A} \) but \( X' + Y' \) extending a secant line of \( \Sigma_0 \).

**Proof.** Suppose that there exists \( k \in F \setminus G \) with \( \mathcal{A} \cap k \mathcal{A} \) non-empty. Then (as \( G \) is transitive on \( \mathcal{A} \)) we may assume that \( kA = A \) for some \( A \in \mathcal{A} \). If for some
\( B \in \mathcal{P} \setminus A^\perp \) we have \( kB \not\in \mathcal{P} \), then \( h_{AB} \in G \) and we can take \( X' = A \), \( Y' = kB \) so that \( h_{X',Y'} = kh_{AB}k^{-1} \) has the required properties. Otherwise for some \( C \in \mathcal{P} \cap A^\perp \) we have \( kC \not\in \mathcal{P} \). There must exist \( B \in \mathcal{P} \) such that \( B \not\in A^\perp \) and \( B \not\in C^\perp \); now \( h_{BC} \in G \) and we can take \( X' = kB \), \( Y' = kC \) with \( h_{X',Y'} = kh_{BC}k^{-1} \) having the required properties.

Otherwise, by Proposition 3.4, there exist non-orthogonal points \( X, Y \in \mathcal{P} \) and \( k \in F \) such that \( k(X + Y) = X + Y \) but \( k(X), k(Y) \not\in \mathcal{P} \). Let \( X' = kX \), \( Y' = kY \). Then \( h_{X',Y'} = kh_{XY}k^{-1} \in F \setminus G \) by Proposition 3.1, and so \( h_{X',Y'} \) has the required properties.

**Proposition 3.6.** Suppose that \( n = 4 \), \( q \neq 3 \) and that \( \mathcal{P} \) is elliptic. Suppose also that \( F \) contains \( h_{XY} \not\in G \) with \( X \in \mathcal{P} \) but \( Y \not\in \mathcal{P} \) (and \( Y \not\in X^\perp \)). Then there is a point \( Z \) of \( \Sigma_0 \setminus \mathcal{P} \) lying in \((X + Y)^\perp \) and moreover \( h_{XY} \) does not fix \( Z^\perp \cap \mathcal{P} \).

**Proof.** The argument is similar to that for \( n = 5 \) and makes use of Proposition 3.1. Either \( M = X + Y + \mathcal{P}(Y) \) extends a secant line of \( \Sigma_0 \) and any point \( Z \) of \( \Sigma_0 \setminus \mathcal{P} \) lying on \( M^\perp \) has the required properties; or \( M \) is a plane with \( Z = M^\perp \) a point of \( \Sigma_0 \), necessarily not on \( \mathcal{P} \) since \( \mathcal{P} \) is elliptic, and \( Z \) has the required properties.

**Proposition 3.7.** Suppose that \( n = 4 \) and that \( \mathcal{P} \) is hyperbolic. Suppose also that \( F \) contains an element \( h_{XY} \not\in G \) with \( X \in \mathcal{P} \). If \( q \neq 3 \), then there is a point \( Z \in \Sigma_0 \setminus \mathcal{P} \) such that \( F \) contains an element \( h_{XY} \not\in G \) with \( X' \), \( Y' \in Z^\perp \) and \( X' \in \mathcal{P} \) and \( X' \in \mathcal{P} \) such that \( h_{X',Y'} \) does not fix \( Z^\perp \cap \mathcal{P} \). If \( q = 3 \), then either there exists an element \( h_{X',Y'} \), with the same properties or there exist a line \( L \) of \( \Sigma \) extending a secant line of \( \Sigma_0 \) and an element of \( F \) fixing \( L \) and fixing exactly one point of \( L \cap \mathcal{P} \).

**Proof.** Let \( M = X + Y + \mathcal{P}(Y) \). If \( M^\perp \) is not a point of \( \mathcal{P} \) (i.e., \( M^\perp \) either is a line or is a point of \( \Sigma_0 \setminus \mathcal{P} \)), then \( M^\perp \) contains a point \( Z \in \Sigma_0 \setminus \mathcal{P} \) and we may take \( X' = X \), \( Y' = Y \).

Suppose that \( M^\perp \) is a point \( U \) of \( \mathcal{P} \). Let \( T \) be a point of \( \mathcal{P} \setminus \mathcal{P}_{\perp} \) such that \( T \not\in U^\perp \) and let \( W \) be the second point of \( \mathcal{P} \) on \((U + T)^\perp \). Let \( u, t, x, w \) be vectors representing \( U, T, X, W \), chosen so that points of \( \Sigma_0 \) are represented by \( GF(q) \)-linear combinations of \( u, t, x, w \) and so that \( H(u, t) = H(x, w) = 1 \). Then \( h_{XY} \) takes \( u \) to \( u \), \( x \) to \( \lambda x \) and \( t \) to \( t + xw \) for some \( x \in GF(q^2) \).

Suppose that \( x \in GF(q) \). Then \( SO_4(q) \) contains an element \( g \) such that \( g(u) = u, g(x) = \lambda^{-1}x \), and \( g(t + xw) = t \). Thus \( f = gh_{XY} \in F \setminus G \) and fixes each of \( u, x, t \). If \( q = 3 \), then \( f \) has the required property with \( L = (U + T)^\perp \). If \( q \neq 3 \), then \( f(w) = w + \beta x \) for some \( \beta \in GF(q) \) and \( fW \not\in \mathcal{P} \). We may take \( X' = X, Y' = fW \), since \( h_{XY} \in G \) and \( h_{X',Y'} = fh_{XY}f^{-1} \in F \setminus G \) with \( Z \) any point of \((U + T) \cap \Sigma_0 \) but not in \( \mathcal{P} \).

Suppose that \( x \not\in GF(q) \). Let \( r = t + xw \) and let \( R \) be the corresponding point of \( \Sigma \). Then \( G \) contains \( h_{UR} \) and \( F \) contains \( h_{UR} = h_{XY}h_{UT}h_{XY}^{-1} \). Let \( k = h_{UT}^{-1}h_{UR} \). Then

\[
k(u) = u, \quad k(x) = x, \quad k(t) = t + (\lambda^{-1} - 1)xw
\]

and

\[
k(w) = w + (\lambda^{-1} - 1)xw,
\]
so that
\[ k(t + w) = t + w + (\lambda^{-1} - 1)(zx - \bar{z}u) \]
(here \( \bar{z} \) denotes \( z^q \)). Let us write \( A, B \) and \( C \) for the points of \( \Sigma \) represented by \( t + w \), \( zx - \bar{z}u \) and \( x - u \). Observe that \( kA \in A + B \), that \( kA + C \) contains exactly one point of \( \Sigma_0 \) (namely \( C \)), and that \( \varphi(kA) \in A + \varphi(B) \). Therefore \( kA + \varphi(kA) \) is a line of \( \Sigma \) extending a line of \( \Sigma_0 \) (so that it cannot contain \( C \)) and it meets \( X + U \) at a point \( X' \neq C \). Note that \( C = (X + U) \cap A^\perp \), so that \( X' \notin A^\perp \). Let \( Y' = kA \). Then \( h_{X', A} \in G \) and \( kh_{X', A}k^{-1} = h_{X', Y'} \in F \) with \( Y' \notin \mathcal{2} \) but \( X' \in Y' + \varphi(Y') \). If \( q \neq 3 \), then \( h_{X', Y'} \notin G \) and any point \( Z \) of \( (X' + Y')^\perp \) lying in \( \Sigma_0 \) but not in \( \mathcal{2} \) has the required properties. If \( q = 3 \), then \( h_{X', Y'} \in G \). However \( X' + A \) and \( k(X' + A) \) are both lines of \( \Sigma \) extending secant lines of \( \Sigma_0 \), so that for some \( g \in G \) we have \( gk(X' + A) = X' + A \) and \( gkX' = X' \). Writing \( L = X' + A \), we have \( gkA = gY' \notin \mathcal{2} \), so that \( f = gk \) is in \( F \) and fixes exactly one point of \( L \cap \mathcal{2} \).

We now turn to the main theorem. The following results represent initial cases in an induction argument. They verify the theorem for \( n = 3 \) and also in the elliptic case when \( n = 4 \) and \( q = 3 \).

**Result 3.8** (see [2]). For \( q = 3 \), any subgroup of \( \text{PSU}_4(q^2) \) containing \( \text{PSO}_4^+(q) \) either lies in \( \text{PSO}_4^-(q) \), or is isomorphic to \( \text{PSL}_3(4) \); in particular \( \text{PSO}_4^+(q) \) is maximal in \( \text{PSU}_4(q^2) \).

In [10, Theorem 19.3.18], the group \( \text{PSL}_3(4) \) is identified as the subgroup of \( \text{PGU}_4(9) \) fixing a hemisystem whose dual structure in \( \text{PG}(5, 3) \) is the Hill’s cap, studied in [5] and [9]. In [5] the stabilizer in \( \Omega^+_6(3) \) of Hill’s cap is shown to be maximal.

**Result 3.9** (see [18]). For odd \( q \), the subgroup \( \text{PSO}_3(q) \) is maximal in \( \text{PSU}_3(q^2) \) except when \( q = 3, 5 \). When \( q = 3 \), any proper overgroup of \( \text{PSO}_3(q) \) properly contained in \( \text{PSU}_3(q^2) \) is isomorphic to \( \text{PSL}_2(7) \). When \( q = 5 \), any proper overgroup of \( \text{PSO}_3(q) \) properly contained in \( \text{PSU}_3(q^2) \) is isomorphic to \( A_7 \). In all cases, \( \text{PSO}_3(q) \) is self-normalizing in \( \text{PSU}_3(q^2) \).

We noted earlier that some care is necessary when \( q = 3 \) and when \( q = 5 \). When \( n = 5 \) and \( q = 3 \) we have to ensure that our reduction leads to an elliptic quadric and that the overgroup of \( \text{SO}_3^-(q) \) corresponding to \( \text{PSL}_3(4) \) is avoided. When \( n = 4 \) and \( q = 5 \) we have to ensure that our reduction to \( n = 3 \) avoids the overgroup of \( \text{SO}_3(q) \) corresponding to \( A_7 \).

**Corollary 3.10.** Suppose that \( q = 3 \) or \( 5 \), that \( \text{PSO}_3(q) < \bar{F} \leq \text{PSU}_3(q^2) \) and that \( \text{Stab}_P \bar{F} \neq \text{Stab}_{\text{PSO}_3(q)} \bar{F} \) for some \( P \in \mathcal{2} \). Then \( \bar{F} = \text{PSU}_3(q^2) \).

**Proof.** Suppose first that \( q = 3 \). There are 28 points of \( \mathcal{H} \) and 4 of \( \mathcal{2} \). The remaining
points of \( \mathcal{H} \) fall into two orbits under \( PSO_3(q) \), each of size 12. The order of \( PSL_2(7) \) is 168, not divisible by 16, and so such a subgroup of \( PSU_3(q^2) \) is transitive on \( \mathcal{H} \) and the stabilizer in such a group of a point of \( \mathfrak{A} \) is the same as the stabilizer in \( PSO_3(q) \). Hence an overgroup \( \bar{F} \), as given, cannot be isomorphic to \( PSL_2(7) \) and must therefore be \( PSU_3(q^2) \). Now suppose that \( q = 5 \). Essentially the same arguments work, but now there are 126 points of \( \mathcal{H} \) and 6 of \( \mathfrak{A} \), with the remaining points of \( \mathcal{H} \) falling into two orbits of size 60 under \( PSO_3(q) \), and the order of \( A_7 \) is 2520.

**Corollary 3.11.** Suppose that \( SO_3(q) \leq F \leq SU_3(q^2) \) and \( F \not\cong G \). If \( q = 3 \) or 5 suppose further that \( Stab_F P \neq Stab_G P \) for any \( P \in \mathfrak{A} \). Then \( F = SU_3(q^2) \).

**Proof.** Let \( \bar{F} \) denote the image of \( F \) in \( PSU_3(q^2) \). Then by Result 3.9 and Corollary 3.10 above, \( \bar{F} = PSU_3(q^2) \). Hence \( F.C = SU_3(q^2) \) where \( C \) is the centre of \( SU_3(q^2) \). Let \( \sigma \) be any transvection in \( SU_3(q^2) \). Then \( \sigma = I \) for some \( v \in GF(q^2)^* \), with the orders of \( \sigma \) and \( v \) coprime so that \( \sigma \in F \). Now \( SU_3(q^2) \) is generated by its transvections and so \( F = SU_3(q^2) \).

**Proposition 3.12.** Suppose that \( n = 5 \) and \( q = 3 \) and that \( F \) contains some element \( h_{XY} \notin G \) with \( Y \notin \mathfrak{A} \). Then there is a point \( Z \in \Sigma_0 \setminus \mathfrak{A} \) such that \( F \) contains \( 1 \times SU_4(9) \) (written with respect to the decomposition \( Z \oplus Z^\perp \)).

**Proof.** By Proposition 3.1 we can find \( R \in \Sigma_0 \setminus \mathfrak{A} \) and \( h_{XY}, \; X' \in R^\perp \) and \( h_{X'Y'}, \; Y' \in R^\perp \) does not fix \( R^\perp \setminus \mathfrak{A} \). Let \( F_4 \) be the subgroup of \( SU_4(9) \) determined by \( 1 \times F_4 = (1 \times SU_4(9)) \cap F \) (written with respect to the decomposition \( R \oplus R^\perp \)). Then \( h_{X'Y'}, \; Y' \in 1 \times F_4 \) and \( SO_4(3) \leq F_4 \) but \( F_4 \not\cong SO_4(3).2 \). We show that there is a plane \( \Pi \) extending a plane \( \Pi_0 \) of \( \Sigma_0 \) such that \( \Pi_0 \) meets \( \mathfrak{A} \) in a conic, and such that the stabilizer in \( F \) of \( \Pi \) and a point of \( \Pi \cap \mathfrak{A} \) does not fix \( \Pi \cap \mathfrak{A} \). We consider separately the two cases when \( R^\perp \setminus \mathfrak{A} \) is hyperbolic or elliptic.

Suppose first that \( R^\perp \setminus \mathfrak{A} \) is elliptic. Then \( SO_4^- (3) \leq F_4 \) but \( F_4 \not\cong SO_4^- (3).2 \), and so, by Result 3.8, the image of \( F_4 \) in \( PSU_4(9) \) is either \( PSU_4(9) \) or \( PSL_3(4) \); in each case the image of \( F_4 \) contains \( PSL_3(4) \). The centre of \( SU_4(9) \) is cyclic of order 4 and the centre of \( SO_4^- (3) \) has order 2, so that \( F_4 \) contains a subgroup with structure \( 2.PSL_3(4) \) or \( 4.PSL_3(4) \) and the order of \( F_4 \) is divisible by either 40320 or 80640. Let \( T \in R^\perp \setminus \mathfrak{A} \). The number of points in \( R^\perp \cap \mathcal{H} \) is 280, so that the stabilizer of \( T \) in \( 1 \times F_4 \) has order at least 144 if \( F_1 = 2.PSL_3(4) \) and at least 288 if \( F_1 = 4.PSL_3(4) \). On the other hand there are 10 points in \( R^\perp \cap \mathfrak{A} \) and \( SO_4^- (3) \) has structure \( 2.A_6 \) and hence the stabilizer of \( T \) in \( 1 \times SO_4^- (3) \) has order 72. It follows that \( 1 \times F_4 \) contains an element \( f = (1, f') \) with \( f' \) not in the centre of \( F_4 \) such that \( f' \notin SO_4^- (3).2 \) and \( f'.T = T \). As \( f' \) cannot fix \( R^\perp \setminus \mathfrak{A} \), there must be a point \( U \in R^\perp \setminus \mathfrak{A} \) such that \( f'.U \notin \mathfrak{A} \). If \( T \notin f'.U + \mathfrak{A} \), then \( \Pi = T + f'.U + \mathfrak{A} \) is a plane of \( \Sigma \) extending a plane of \( \Sigma_0 \), the point \( f' \cap R^\perp \) is not on \( \mathfrak{A} \) (so that \( \Pi \) meets \( \mathfrak{A} \) in a conic) and \( f'.T.U // f'.U = T \) fixes \( \Pi \). If \( T \in f'.U + \mathfrak{A} \), then \( L = T + f'.U + \mathfrak{A} \) is a secant line of \( \Sigma \) extending a secant line of \( \Sigma_0 \) and so we can find \( g \in 1 \times SO_4^- (3) \) such that \( g.T = T, \; g.(T + U) = T + U \) and \( g.U \neq U \). This time let \( \Pi = R + L \); then \( \Pi \) has the required properties.
Suppose now that $R^\perp \cap \mathcal{L}$ is hyperbolic. By Proposition 3.7, either there is a point $W \in (R^\perp \cap \Sigma_0) \setminus \mathcal{L}$ such that $1 \times F_1$ contains an element $h_{X',Y'} \notin G$ with $X', Y' \in (R + W)^\perp$ and $X' \notin \mathcal{L}$ and such that $h_{X',Y'}$ does not fix $(R + W)^\perp \cap \mathcal{L}$, or there exist a line $L$ of $R^\perp$ extending a secant line of $\Sigma_0$ and an element of $1 \times F_4$ fixing $L$ and fixing exactly one point of $L \cap \mathcal{L}$. In the first case we take $\Pi = (R + W)^\perp$ and in the second we take $\Pi = R + L$.

Consider the orthogonal decomposition $\Pi \oplus \Pi^\perp$ of $\Sigma$. The projection $\tilde{F}_3$ of $\text{Stab}_F(\Pi)$ onto $\Pi$ is a subgroup of $U_3(9)$ containing $SO_3(3)$ in which the stabilizer of a point of $\Pi \cap \mathcal{L}$ does not fix $\Pi \cap \mathcal{L}$. Since $PSU_3(9) = PU_3(9)$, it follows from Corollary 3.10 that $\tilde{F}_3$ maps onto $PU_3(9)$. The subgroup $F_3$ of $SU_3(9)$ determined by $1 \times F_3 = (1 \times SU_3(9)) \cap F$ is normal in $\tilde{F}_3$ and maps onto a normal subgroup of $PSU_3(9)$, and so it must map onto $PSU_3(9)$. By Corollary 3.11 we have $F_3 = SU_3(9)$.

Choose a point $Z$ of $\pi^\perp \cap \Sigma_0$ such that $Z \notin \mathcal{L}$ and $Z^\perp \cap \mathcal{L}$ is elliptic. With respect to the decomposition $Z \oplus Z^\perp$, $F$ contains subgroups $1 \times SO_4(3)$ and $1 \times (1 \times SU_3(9))$.

The latter group has order 6048 which does not divide 80640. Hence

$$F \cap (1 \times SU_4(9)) = 1 \times SU_4(9)$$

and $F$ contains $1 \times SU_4(9)$ as required.

**Proposition 3.13.** Suppose that $n = 4$ and $q = 5$. Then there exists a point $Z$ of $\Sigma_0 \setminus \mathcal{L}$ such that $F$ contains $1 \times SU_3(q^2)$, written with respect to the decomposition $Z \oplus Z^\perp$.

**Proof.** We first show that there exist a point $Z$ of $\Sigma_0 \setminus \mathcal{L}$, a point $X \in Z^\perp \cap \mathcal{L}$ and an element of $F$ acting as an element of $1 \times SU_3(q^2)$ with respect to the decomposition $Z \oplus Z^\perp$, fixing $X$ but not fixing $Z^\perp \cap \mathcal{L}$.

By Proposition 3.5, either there exists $h_{X',Y'} \in F \setminus G$ with $X' \in \mathcal{L}$ or there exists an element $h_{X',Y'} \in F \setminus G$ with $X', Y' \notin \mathcal{L}$ but $X' + Y'$ extending a secant line of $\Sigma_0$. For the former case, the existence of an element in $F$ with the required properties was established in Propositions 3.6 and 3.7. It remains to consider the latter possibility. Let $L$ be the line $X' + Y'$ and let $X, Y$ be the points of $\mathcal{L}$ on $L$. The elements of $F$ stabilizing $L$ act as elements of $U_2(q^2) \times U_2(q^2)$ with respect to the decomposition $\Sigma = L \oplus L^\perp$. Denote by $F_a$ the projection of $F$ onto the first copy of $U_2(q^2)$ and by $F_b$ the kernel of the projection of $F$ onto the second copy of $U_2(q^2)$. Thus $F_b \times 1 \leq \text{Stab}_F L \leq F_a \times U_2(q^2)$ and $F_b$ is normal in $F_a$. Let $G_a$ and $G_b$ denote the corresponding groups for $G$ and let $F_a, F_b, G_a, G_b$ denote the images of $F_a, F_b, G_a, G_b$ in the group $PU_2(q^2)$, which is isomorphic to $S_5$. Then $F_b, G_b \leq PSU_2(q^2)$, but $G_a \neq PSU_2(q^2)$. We observe that $G$ contains $SO_2^+(q) \times 1$, the element $g_b$ described in Proposition 2.2 can be regarded as lying in $G$ and further $G$ contains an element $g$ of $O_2(q) \times O_2(q)$ that has determinant $-1$ in each component. (Such an element is a product of symmetries and switches $X$ and $Y$.) All elements of $G_a$ and $G_b$ are identified in this way: $|G_a| = 16$, $|G_b| = 4$, $|\bar{G}_a| = 8$, $|\bar{G}_b| = 2$. Now $F$ contains the element $h_{X',Y'}$, which is in $F_b \times 1$. The possibilities for $(F_a, F_b)$, in terms of subgroups of $S_5$, are $(S_4, A_5), (S_4, A_4), (S_4, V_4)$ and $(D_8, V_4)$. In each case $F_b$ contains the four-group $V_4$ that is the normalizer of $\bar{G}_b$ in $A_5$. Let $f \in F_b \setminus G_b$ such that $\langle f, G_b \rangle$ corresponds

\[ F \cap (1 \times SU_4(9)) = 1 \times SU_4(9) \]
to \( V_4 \). Then \( f \) must switch \( X \) and \( Y \) and so \( gf \) fixes each of \( X \) and \( Y \). Moreover there are orthogonal points \( Z, W \) of \((L^\perp \cap \Sigma_0) \setminus \emptyset\) such that \( g \) fixes each of \( Z \) and \( W \) and \( g \) lies in \( 1 \times SU_3(q^2) \) with respect to the decomposition \( \Sigma = Z \oplus Z^\perp \). Finally \( gf \in 1 \times SU_3(q^2) \), \( gfX = X \) but \( gf \) cannot fix \( Z^\perp \cap \emptyset \).

Now let \( F_3 \) be the subgroup of \( SU_3(q^2) \) determined by \( 1 \times F_3 = (1 \times SU_4(9)) \cap F \). The argument above shows that \( SO_3(q) \leq F_3 \) with the stabilizer in \( F_3 \) of a point of \( Z^\perp \cap \emptyset \) not fixing \( Z^\perp \cap \emptyset \). Then, by Corollary 3.11, we have \( F_3 = SU_3(q^2) \).

**Theorem 3.14.** Suppose that \( n \geq 3 \) and that \( F \) is a subgroup such that \( SO_n(q) \leq F \leq SU_n(q^2) \) and \( F \not\leq G \). Then \( F = SU_n(q^2) \) except when \( n = 3 \) and \( q = 3 \) or 5 and when \( n = 4 \) and \( q = 3 \).

*Proof.* The case \( n = 3 \) (and \( q \neq 3, 5 \)) is handled in Corollary 3.11. We assume now that \( n \geq 4 \) if \( q \neq 3 \) and \( n \geq 5 \) if \( q = 3 \). The main part of the proof is concerned with identifying a point \( Z \in \Sigma_0 \setminus \emptyset \) such that, with respect to the decomposition \( Z \oplus Z^\perp \) of \( \Sigma \), \( F \) contains \( 1 \times SU_{n-1}(q^2) \). This is already established in Propositions 3.12 and 3.13 in the cases when \( n = 4, q = 5 \) and \( n = 5, q = 3 \).

Suppose that \( n \geq 4 \) when \( q > 5 \), that \( n \geq 5 \) when \( q = 5 \), that \( n \geq 6 \) when \( q = 3 \) and that the theorem holds for subgroups of \( SU_{n-1}(q^2) \) containing \( SO_{n-1}(q) \).

Suppose first that \( n \geq 5 \). We have noted that \( SO_n(q) \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]

Then \( X = fA \) and \( Y = fB \) cannot both be in \( \emptyset \) and by Proposition 3.1 we have \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus \emptyset \) such that \( F \) contains an element \( h_{X', Y'} \) with \( X', Y' \in Z^\perp \) and \( F \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]

Then \( X = fA \) and \( Y = fB \) cannot both be in \( \emptyset \) and by Proposition 3.1 we have \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus \emptyset \) such that \( F \) contains an element \( h_{X', Y'} \) with \( X', Y' \in Z^\perp \) and \( F \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]

Then \( X = fA \) and \( Y = fB \) cannot both be in \( \emptyset \) and by Proposition 3.1 we have \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus \emptyset \) such that \( F \) contains an element \( h_{X', Y'} \) with \( X', Y' \in Z^\perp \) and \( F \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]

Then \( X = fA \) and \( Y = fB \) cannot both be in \( \emptyset \) and by Proposition 3.1 we have \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus \emptyset \) such that \( F \) contains an element \( h_{X', Y'} \) with \( X', Y' \in Z^\perp \) and \( F \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]

Then \( X = fA \) and \( Y = fB \) cannot both be in \( \emptyset \) and by Proposition 3.1 we have \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus \emptyset \) such that \( F \) contains an element \( h_{X', Y'} \) with \( X', Y' \in Z^\perp \) and \( F \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in \emptyset \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have

\[
fh_A h f^{-1} \in F \setminus SO_n(q).
\]
The following corollary follows immediately from Theorem 3.14 and Results 3.8 and 3.9, and it establishes our main theorem.

**Corollary 3.15.** Suppose that $n \geq 3$. The normalizer of $\text{SO}_n(q)$ in $\text{SU}_n(q^2)$ is a maximal subgroup of $\text{SU}_n(q^2)$. The stabilizer of $\mathcal{Q}$ in $\text{PSU}_n(q^2)$ is a maximal subgroup of $\text{PSU}_n(q^2)$. In both cases there are exceptions when $n = 3$ and $q = 3$ or 5 and when $n = 4$, $q = 3$ and $\mathcal{Q}$ is hyperbolic.

### 4 The exceptional case: $n = 4$ and $q = 3$

We consider here the case where $n = 4$ and $q = 3$ and where $\mathcal{Q}$ is a hyperbolic quadric. In this case $G$ is not maximal (the maximal subgroups are listed in [2], for example). We describe here, briefly, an overgroup of $G$ that fixes a geometric configuration.

There are eight lines on $\mathcal{Q}$, falling in two sets of four lines, each a regulus of $\Sigma_0$. There are 112 lines on $\mathcal{H}$. It can be shown that $G$ has orbits of lengths 8, 16, 16 and 72 on this set. The orbits of size 16 are each complete spans of $\mathcal{H}$. Given either of these spans, the stabilizer in $\text{PSU}_4(9)$ of the span is isomorphic to $2^4.A_6$ and contains $G$. From [2] it is apparent that the subgroup $2^4.A_6$ is maximal in $\text{PSU}_4(9)$. More information on spans of $\mathcal{H}(n-1,q^2)$ in general, and $\mathcal{H}(3,9)$ in particular, is given in [8].

### References


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