Instability of trajectories of solid particles around vortex lines

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Abstract

Progress in implementing the Particle Image Velocimetry (PIV) visualisation technique in liquid helium has stimulated interest in understanding the dynamics of micron-size solid particles in a superfluid. We show that at sufficiently low temperatures, in the limit of a pure superfluid, the trajectories of small, neutrally buoyant, solid particles are unstable and deviate from the trajectories of superfluid particles, even in the simplest case of the motion around a single stationary straight vortex line. The result also applies to classical Euler flows. The implications of this result for the visualization of turbulent superflow at very low temperatures are discussed.

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The recent success in implementing the Particle Image Velocimetry (PIV) technique in liquid helium [1, 2] using micron-size, solid tracer particles has opened up the possibility of visualising flow patterns in helium II. This progress could help understanding the phenomenon of quantum turbulence. The interpretation of PIV data, however, is not trivial. Helium II consists of two co-penetrating fluid components [3], the (viscous) normal fluid and the (inviscid) superfluid. In a recent paper [4] we have shown that, under a number of reasonable assumptions, the equations of motion of a small tracer particle of radius a_p are the following:

$$\frac{d\mathbf{v}_p}{dt} = \frac{1}{\tau}(\mathbf{v}_n - \mathbf{v}_p) + \frac{\rho_n}{\rho}\left[\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n\right] + \frac{\rho_s}{\rho}\left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s\right], \tag{1}$$

$$\frac{d\mathbf{r}_p}{dt} = \mathbf{v}_p, \tag{2}$$

where $\mathbf{v}_p$ and $\mathbf{r}_p$ are the particle's velocity and position, $\tau = \rho a_p^2/(3\mu_n)$ is the relaxation time, μ_n the viscosity of helium II, $\mathbf{v}_n$ and $\mathbf{v}_s$ the normal fluid and superfluid velocities, ρ_n and ρ_s the normal fluid and superfluid densities, and $\rho = \rho_n + \rho_s$ the total density. In writing Eq. (1) we have assumed for the sake of simplicity that the particle is neutrally buoyant ($\rho_p = \rho$), but the Archimedes force can be easily taken into account in the model[4]. We have also assumed that the flow velocity does not vary significantly over the volume of the particle, which means that we have assumed the particle is not too close to, and has not been trapped by, any quantized vortex line. The three terms at the right hand side of Eq. (1) represent respectively the Stokes drag induced by the normal fluid and inertial effects associated with the normal fluid and the superfluid components.

In reference [4] we made a brief comment about the behaviour of a neutrally-buoyant particle in a turbulent superfluid at a very low temperature, when the normal fluid is effectively absent. We recall that each element of a vortex line moves with the local superfluid velocity (in the absence of normal fluid there is no mutual friction). Therefore we can expect that a particle trapped by a vortex line will follow the superfluid velocity (strictly speaking we have assumed here that the mass of the particle is not too large, as we explain in a later paper). However, we were not able to calculate the trapping probability. Nevertheless, we suggested that this might not matter. The motion of an untrapped particle is governed by the equation

$$\frac{d\mathbf{v}_p}{dt} = \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s\,. \tag{3}$$

which follows from Eq. (1) in the limit $T \to 0$. We suggested that in this case also the particle might follow the superfluid. The purpose of the present short note is to examine the extent to which this last statement is correct, especially in the case when the superfluid contains vortex lines, as must be the case in superfluid turbulence.

The statement was based on the fact that

$$\mathbf{v}_p(t) = \mathbf{v}_s(\mathbf{r}_s(t), t)\,, \quad \mathbf{r}_p(t) = \mathbf{r}_s(t), \tag{4}$$

where $\mathbf{r}_s(t)$ represents a Lagrangian trajectory of a superfluid particle, is a formal solution of Eqs. (2) and (3) (see ref. [4]). Here a superfluid particle or point $\mathbf{r}_s(t)$ is defined as the solution of the equation

$$\frac{d\mathbf{r}_s}{dt} = \mathbf{v}_s(\mathbf{r}_s, t), \tag{5}$$

where $\mathbf{v}_s$ is imposed, so that it is determined by a differential equation of lower order than $\mathbf{r}_p(t)$. It appears therefore that, in a pure superfluid, a neutrally buoyant particle ought to move with the superfluid. The physical reason is as follows. Using the Euler equation of motion for the superfluid, we see that the right hand side of Eq. (3) is equal to $-(1/\rho_s)\nabla p$, where p is the pressure; hence the force acting on the particle is the same as that acting on the volume of superfluid displaced by it. However, this is an oversimplification for the following obvious reason. Consider the solution (4) for times $t \geq 0$. This solution is valid only if the initial condition

$$\mathbf{v}_p(0) = \mathbf{v}_s(\mathbf{r}_s(0), 0) \tag{6}$$

holds; i.e. the velocity of the particle must initially match that of the surrounding superfluid. To illustrate the consequences of a violation of Eq. (6), and their implications, we shall now consider the motion of a particle in the superfluid velocity field due to a single fixed rectilinear vortex.

Let the vortex be along the z-axis of a system of cylindrical coordinates, $(r,\ \theta,\ z)$, so that the velocity field of the vortex is given by $v_{sr} = v_{sz} = 0$, $v_{s\theta} = \Gamma/(2\pi r)$, where Γ is the quantum of circulation. Eq. (3) reduces to

$$\frac{d\mathbf{v}_p}{dt} = \frac{\Gamma^2}{8\pi^2}\nabla\left(\frac{1}{r^2}\right), \tag{7}$$

which is the equation of motion of a particle of unit mass in a two-dimensional central potential $-\Gamma^2/(8\pi^2 r^2)$. We assume that the particle has no velocity along the vortex, its position and velocity in the $(r,\ \theta)$ plane being denoted respectively by $\mathbf{r}_p = (r, \theta)$ and $\mathbf{v}_p = (v_r, v_\theta) = (dr/dt, r d\theta/dt)$. The particle's angular momentum $l = r v_\theta$ is a constant of motion. Conservation of energy requires that

$$v_\theta^2 + v_r^2 - \frac{\Gamma^2}{4\pi^2 r^2} = \text{const.} \tag{8}$$

Suppose, for example, that $v_r = 0$ when $r = r_0$. Then, expressing v_θ in terms of the constant l, we find from Eq. (8) that

$$\frac{dr}{dt} = \left[\frac{\Gamma^2}{4\pi^2} - l^2\right]^{1/2} \left[\frac{1}{r^2} - \frac{1}{r_0^2}\right]^{1/2}. \tag{9}$$

We distinguish three cases, depending on the relative values of l and $\Gamma/2\pi$. For the case $l > \Gamma/2\pi$, it is easy to show from Eq. (9) that

$$\int_1^x \frac{x dx}{(x^2-1)^{1/2}} = \left(l^2 - \frac{\Gamma^2}{4\pi^2}\right)^{1/2} \frac{t}{r_0^2}, \tag{10}$$

where $x = r/r_0$ and $x = 1$ at $t = 0$; it follows that

$$x^2 = 1 + \left(l^2 - \frac{\Gamma^2}{4\pi^2}\right) \frac{t^2}{r_0^4}. \tag{11}$$

Integration of the equation $l = r^2 d\theta/dt$ then yields

$$\theta = \left(1 - \frac{\Gamma^2}{4\pi^2 l^2}\right)^{-1/2} \tan^{-1}\left[\left(l^2 - \frac{\Gamma^2}{4\pi^2}\right)^{1/2} \frac{t}{r_0^2}\right]. \tag{12}$$

The corresponding equations for the case $l < \Gamma/2\pi$ are

$$x^2 = 1 - \left(\frac{\Gamma^2}{4\pi^2} - l^2\right) \frac{t^2}{r_0^4}, \tag{13}$$

and

$$\theta = \left(\frac{\Gamma^2}{4\pi^2 l^2} - 1\right)^{-1/2} \tanh^{-1}\left[\left(\frac{\Gamma^2}{4\pi^2} - l^2\right)^{1/2} \frac{t}{r_0^2}\right]. \tag{14}$$

Eqs. (11)-(14) determine the path of the particle in these two cases. In the first case the particle spirals outwards towards $r = \infty$; in the second it spirals inwards towards $r = 0$ and is trapped by the vortex. In both cases the departure $|(r - r_0)|$ of the path of the particle from that of the superfluid is proportional to t^2 in the limit of small t. Only in a third case, when $l = \Gamma/2\pi$ exactly, does the particle follow the superfluid for all times; this is the case when initially the velocity of the particle is exactly equal to the velocity of the superfluid.

We see from this example that unless its velocity initially matches exactly that of the superfluid the particle is likely to move in such a way that it deviates increasingly from that of the superfluid as time goes on. In practice it is difficult to see how this initial matching could be achieved. The best hope, albeit unpromising, might be to inject the particles into helium that is initially at rest and at a temperature where there is enough normal fluid to provide viscous damping of the particle motion; then to cool the helium; and finally to generate the turbulent flow.

However, even if initial exact matching were achieved, any infinitesimal perturbation of the motion of the particle, associated with, for example, Brownian motion, would lead to increasing deviation from the superfluid velocity[5]. In this sense, therefore, trajectories that initially follow the superfluid are unstable. There is no relaxation process that can counteract this instability.

A similar linear instability of the motion of a neutrally buoyant particle around vortex cells in a viscous fluid (rather than in the inviscid fluid considered here) has been obtained by Babiano *et al.* [6], although in the viscous case the instability is exponential in time rather than quadratic. Even in the case of a simple time-independent flow, Babiano *et al* found that solid particles that start near a vortex can leave this vortex for one of the neighboring vortices, a motion that is not allowed to fluid particles. For time-dependent flows the situation is more complex and there are regions of phase space dominated by chaotic trajectories.

If the orbit of a particle around a single stationary vortex is unstable, it is very likely that the motion in the presence of many (moving) vortices suffers the same problem. To illustrate this case we consider the relatively simple case of motion in two dimensions of three vortex points of the same polarity - see Fig. 1. The calculation was performed by solving the equations in dimensionless form with a_p as length scale and a_p^2/Γ as time scale using a fourth-order Adams-Bashforth method, time step $\Delta t = 10^{-4}$ and periodic boundary conditions. Initially the vortices are set at the corners $(0, 1)$, $(0.866, -0.5)$ and $(-0.866, -0.5)$ of an equilateral triangle. A solid particle and a superfluid particle are initially set at position A, and the velocity of the solid particle is initially set to be equal to the velocity of the superfluid at A. Fig. 1 shows the orbit of the three vortices (closed dotted line), the trajectory of the superfluid particle (solid line from A to B) and the trajectory of the solid particle (dashed line from A to C). The final positions correspond to $t = 60$ (at $t = 50$ each vortex point has moved approximately 1/3 of one orbit). It is apparent that the solid particle's trajectory, $\mathbf{r}_p(t)$, quickly deviates from the superfluid particle's trajectory, $\mathbf{r}_s(t)$.

If the number of vortex points is larger, the instability of solid particle trajectories is likely to be further reinforced by chaos of the superfluid trajectories. A vortex point moves with the local superfluid velocity, and it is known that an N-point vortex system has a threshold for chaotic behaviour, beyond which it becomes extremely sensitive to the initial conditions [7]; for an unbound fluid chaos occurs for $N > 3$. In three dimensions the trajectories are likely to be even more sensitive to perturbations because it is known that two vortex rings are sufficient to produce chaotic streamlines, provided that the rings' axes point in different directions [8].

Finally, note that if the particle is not neutrally buoyant then, for $T \to 0$, the right hand side of Eq. (3) has a prefactor $3/(1 + 2\rho_p/\rho_s)$ which is different from unity, and we would not expect the solid particle to trace to superfluid.

We conclude that, at least in the presence of the vortices that are an essential component of superfluid turbulence, small particles cannot realistically be used to visualize the full superfluid velocity field at very low temperatures. This does not mean that such particles cannot be used to visualize time-dependent, but spatially-independent (vortex-free) superflows at low temperatures. Furthermore, it does not mean that important information about low-temperature superfluid turbulence cannot be obtained from the observation of particles that are trapped on vortex lines. This means that calculations or simulations that tell us something about the probability of such trapping become crucially important. We plan to return to this question in a later paper.

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[5] Consider a small perturbation $r = R + r'$, $\omega = \Omega + \omega'$ (where $r'/R \ll 1$, $\omega'/\Omega \ll 1$) of the circular orbit of radius $r = R$ and angular velocity $d\theta/dt = \omega = \Omega = \Gamma/(2\pi R^2)$ around the vortex line. It is easy to show that, after linearization, the perturbation obeys $d^3 r'/dt^3 = 0$ and $d^3\omega'/dt^3 = 0$, hence it grows quadratically with time. For example the perturbation of the radial position is

$$r'(t) = \frac{\Gamma}{2\pi R}\left(\frac{\Gamma}{\pi R^2}\, r'(0) + \omega'(0)\right) t^2 + \dot{r}'(0)t + r'(0)\,, \tag{15}$$

where $r'(0)$, $\dot{r}'(0)$ and $\omega'(0)$ are the initial values of the perturbations of the radial position, radial velocity and angular velocity; a similar expression holds for the perturbation of the particle's angular velocity.

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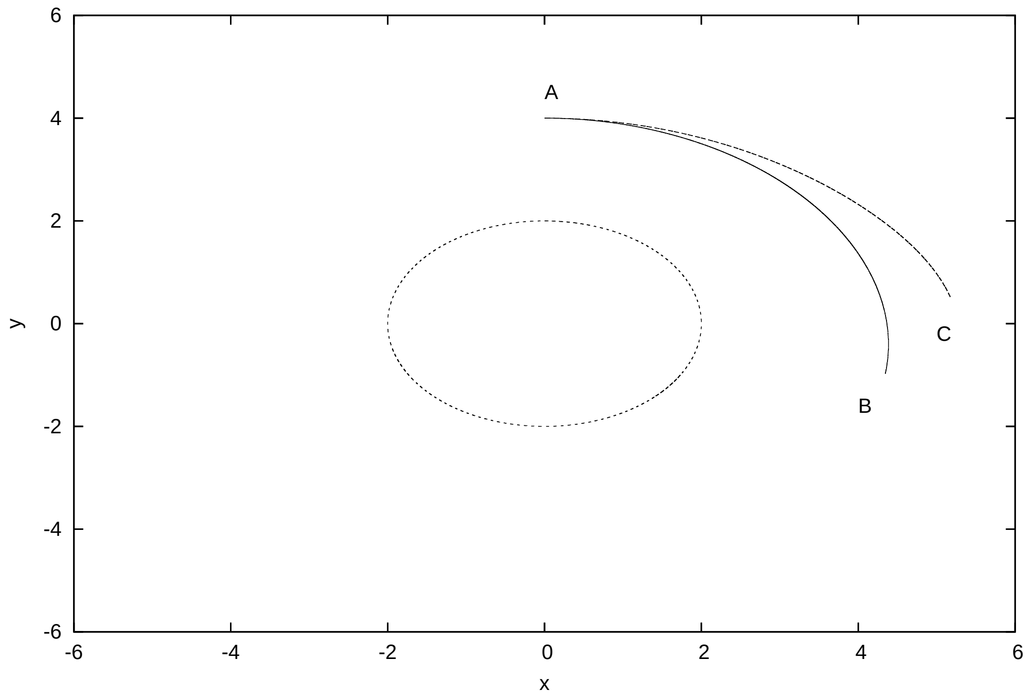

FIG. 1: Trajectories of solid particle (dashed line A to C), superfluid particle (solid line A to B) around three vortices (moving along the closed orbit shown).