Group–theoretic characterizations of classical ovoids

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1 Introduction

An ovoid of $PG(3,q)$, $q > 2$, is a set of q^2+1 points of $PG(3,q)$, no three of which are collinear. The only known ovoids of $PG(3,q)$ are the elliptic quadrics, which exist for all q, and the Suzuki-Tits ovoids, which exist for $q = 2^e$, $e \geq 3$ odd, [10]. It is well known that for odd q, the only ovoids are the elliptic quadrics. For even q, the ovoids have been classified only for q up to and including 32. Elliptic ovoids and Suzuki-Tits ovoids are usually called "classical ovoids".

There are several results characterizing ovoids in $PG(3,q)$, some involving geometry, some involving group-theory.

Since a plane of $PG(3,q)$ meets an ovoid either in a single point or in an oval, a successful technique in studying ovoids has involved examining their plane sections.

The plane sections of an elliptic ovoid are all conics, while those of the Suzuki-Tits ovoid are all translation ovals, namely, ovals invariant under a group of elations of order q having a common axis, which are not conics. Conversely, it has been shown recently [14], that an ovoid admitting a pencil of translation ovals must be either an elliptic quadric or a Suzuki-Tits ovoid (by a pencil of an ovoid with carrier L is meant the set of ovals occurring as secant plane sections for the planes on a fixed tangent line L). The result by O'Keefe and Penttila is a refinement of a previous result by D.G. Glynn [9]. For other results in this direction see also [15], [19] and [13] for a survey on ovoids in $PG(3,q)$.

On the other hand, in 1966 Lüneburg [17], proved that if an ovoid $\mathcal{O}$ in $PG(3,q)$, q even, admits an automorphism group containing a subgroup of even order which is transitive on the points of the ovoid, then $\mathcal{O}$ is a classical ovoid.

Using linear codes, in 1987 Bagchi and Sastry [2] extended Luneburg's result proving that if an ovoid of $PG(3,q)$, q even, admits a point-transitive automorphism group, the the ovoid must be classical.

In this paper, we give another proof of the result by Bagchi and Sastry, using only group-theoretic tools.
In particular, we use the fundamental and celebrated theorem by Aschbacher and the description of certain maximal subgroups of the 4-dimensional symplectic group $PSp(4, q)$, q even, as given in a paper by D. Flesner.

2 Singer cyclic groups in symplectic and Suzuki groups

In this section we give some information about Singer cyclic groups in symplectic, orthogonal and Suzuki groups.
Let q be a power of a prime p and let $V = V(4, q)$ be a vector space of dimension 4 over the Galois field $GF(q)$.

Definition 1 *A cyclic subgroup of $GL(V)$ acting transitively on the non–zero vectors in V will be called a* **Singer cyclic group** *in $GL(V)$. A generator of a Singer cyclic group will be called a* **Singer cycle** *in $GL(V)$.*

From [11], we recall the following results

Proposition 2 *Let $Sp(V)$ be a symplectic group in $GL(V)$. Then there exists a Singer cyclic group S of $GL(V)$ such that $|S \cap Sp(V)| = q^2 + 1$.*

We call $S \cap Sp(V)$ a Singer cyclic group of $Sp(V)$.

Proposition 3 *Let $O^-(V)$ be a non–maximal orthogonal group in $GL(V)$. Then there exists a Singer cyclic group S of $GL(V)$ such that $|S \cap O^-(V)| = q^2 + 1$.*

We call $S \cap O^-(V)$ a Singer cyclic group of $O^-(V)$.
Assume q even. In this case, the center of $Sp(V)$ is trivial and so $PSp(V) = Sp(V)$. Moreover $O^-(V) \leq Sp(V)$, and a Singer cyclic group of $O^-(V)$ always is a Singer cyclic group of $Sp(V)$.
It turns out that $S \cap O^-(V) = S \cap Sp(V)$ preserves an elliptic quadric $\mathcal{E}_3$ of $PG(3, q)$ acting transitively on its points. We recall that an elliptic quadric of $PG(3, q)$, (any q), is an ovoid, namely a set of $q^2 + 1$ points such that no three of them are collinear [10]. In fact $S \cap O^-(V)$ has $q + 1$ orbits, each of which is an elliptic quadric. At the same time $S \cap O^-(V)$ has $q + 1$ orbits of totally isotropic lines of length $q^2 + 1$ and each is a regular spread (elliptic congruence) [10, 17.1]. A partial converse also holds.

Theorem 4 *[2] Let H be a subgroup of $Sp(4, q)$, such that the order of H is divisible by $q^2 + 1$. Suppose $q > 2$ and H is not transitive on the points of $PG(3, q)$. Then H fixes either an elliptic quadric or a Suzuki–Tits ovoid.*

On the other hand, given an ovoid $\mathcal{O}$ of $PG(3,q)$, q even, at each point of $\mathcal{O}$, there is a tangent plane containing $q+1$ tangent lines. These $(q+1)(q^2+1)$ lines form a general linear complex, Λ, say ([20]).
This way, to the ovoid $\mathcal{O}$ we can associate a symplectic polarity, say π, interchanging a point of $\mathcal{O}$ with the tangent plane at that point, and a non–tangent plane of $\mathcal{O}$ with the nucleus of the $q+1$–arc, which is the intersection between $\mathcal{O}$ and the plane. Also, the polarity π maps chords of $\mathcal{O}$ to external lines and vice versa. The collineation group G fixing Λ is, of course, an isomorphic copy of $Sp(4,q)$.
Assume that $q = 2^h$, $h \geq 3$ odd. Set $r = 2^{(h+1)/2}$.
Denote by $Sz(q)$ a Suzuki group of $PGL(4,q)$. Associated with $Sz(q)$ is an ovoid $\mathcal{O}$, the Suzuki–Tits ovoid, in the sense that $\mathcal{O}$ is invariant under $Sz(q)$. Using Segre's construction, the group $Sz(q)$ can always be represented as a subgroup of a 4–dimensional symplectic group $Sp(4,q)$.
We have that $|Sz(q)| = q^2(q-1)(q^2+1)$ and for $q > 2$, $Sz(q)$ admits subgroups of order q^2, $q-1$, $q+1+r$ and $q+1-r$ [12]. In particular, $Sz(q)$ has no element of order q^2+1.
The subgroups of order $q+1+r$ and $q+1-r$ are cyclic and they are irreducible subgroups of distinct Singer cyclic groups of $PGL(4,q)$ [12].
In [8], Glaubermann gave a nice descriptions of these Singer subgroups in terms of the field multiplication and the trace linear map of $GF(q^4)$ see [8, Theorem 4.2] allowing him to define $Sz(q)$ containing a Singer cyclic group of order $q+1+\epsilon r$ and a "mild perturbation" of it of order $q+1-\epsilon r$, with $\epsilon = \pm 1$.

3 Aschbacher's Theorem

In this section we identify subgroups of $Sp(4,q)$ whose order is divisible by q^2+1. Note that $Sp(4,q)$ has order $q^4(q^2-1)^2(q^2+1)$.

Aschbacher's famous theorem ([1]) essentiailly states that a subgroup of a classical group either lies in a member of one of eight naturally defined classes, $C_1 - C_8$, or is almost simple.
The following are the classes of maximal subgroups of $Sp(4,q)$ with q even. This list comes, in the main, from ([16]).
Class C_1: Reducible sugroups, the stabilizers of either totally isotropic or non-isotropic subspaces. A non-isotropic subspace would have dimension 2 and so be isometric to its conjugate; stabilizers of such subspaces lie inside subgroups in class C_2. This leaves totally isotropic subspaces of dimensions $1,2$ (with order $q^4(q-1)(q^2-1)$ in each case).
Class C_2: Imprimitive subgroups. Here we have the stabilizer of a pair of

2-dimensional orthogonal non-isotropic subspaces (with order $2q^2(q^2-1)^2$).
Class C_3: Stabilizers of field extensions: $Sp(2,q^2).2$ (with order $2q^2(q^2-1)(q^2+1)$).
Class C_5: Stabilizers of a subfield structure: $Sp(4,q')$ where q is a power of q' (with order $q'^4(q'^2-1)^2(q'^2+1)$).
Class C_8: Stabilizers of forms. The only possibilities here are orthogonal groups, one is the group of a hyperbolic quadratic form (with order $2q^2(q^2-1)^2$) and the other is the group of an elliptic quadratic form (with order $2q^2(q^2-1)(q^2+1)$).
$\mathcal{S}$: Almost simple subgroups (satisfying a number of further restrictions).

The classes C_4, C_6, C_7 don't occur here (C_4 and C_7 would be stabilizers of tensor product structures and C_6 would be symplectic-type groups).

It is useful to have the following lemma. The proof is elementary and thus omitted.

Lemma 1 *Suppose that a,b are positive integers with $c = hcf(a,b)$. Then*

$$(2^a+1, 2^b+1) = 2^c+1 \text{ if } a/s, b/s \text{ are both odd, and } 1 \text{ otherwise;}$$
$$(2^a+1, 2^b-1) = 2^c+1 \text{ if } a/s \text{ is odd and } b/s \text{ is even, and } 1 \text{ otherwise;}$$
$$(2^a-1, 2^b-1) = 2^c-1.$$

Theorem 2 *If G is a subgroup of $Sp(4,q)$ with $q \geq 8$ even such that q^2+1 divides the order of G, then either G stabilizes a regular spread of lines in $PG(3,q)$ or an elliptic ovoid in $PG(3,q)$, or G is almost simple.*

Proof. The orders of the groups in classes C_1, C_2 and C_5, and the order of $O^+(4,q)$ are not divisible by q^2+1, so only C_3, one group in C_8, and $\mathcal{S}$ remain.

□

We now turn attention to papers by David Flesner ([5],[6],[7]). Here he addresses the question of maximal subgroups of $PSp(4,2^a)$, in the main concentrating on subgroups containing central elations or non-centred skew elations. The first thing to note is that $PSp(4,2^a)$ is isomorphic to $Sp(4,2^a)$ so the theorems give us information about $Sp(4,2^a)$; the second thing is that central elations are just the images of transvections in $PSp(4,2^a)$ while non-centred skew elations are dual to elations, i.e., they are the images of elations under the outer automorphism of $PSp(4,2^a)$. We need to note the first theorem, but for us it is the second which is more significant. Both theorems appear in the third paper but refer to ideas developed in the earlier papers.

Theorem 3 *(Flesner)*
The conjugacy classes of those maximal subgroups of $PSp(4,2^a)$ which contain

central elations or non-centred skew elations are as follows:
(a) *stabilizer of a point;*
(a^*) *stabilizer of a totally isotropic line;*
(b) *maximal index orthogonal group;*
(b^*) *stabilizer of a pair of hyperbolic lines;*
(c) *nonmaximal index orthogonal group;*
(c^*) *dual of non-maximal index orthogonal group;*
(d_r) *(for each prime r dividing a) stabilizer of subgeometry over the maximal subfield $GF(2^{a/r})$.*

Theorem 4 *(Flesner)*
If M is a maximal subgroup of $PSp(4, 2^a)$ which contains no central elations or non-centred skew elations, then either $q = 2$ and M is isomorphic to A_6, or M contains normal subgroups M_1 and M_2 such that $M \geq M_1 \geq M_2 \geq 1$, where M/M_1 and M_2 are of odd order, and M_1/M_2 is isomorphic to either $PSL(2, q')$ or $Sz(q')$ for some power q' of 2.

Flesner's first theorem gives subgroups in the main Aschbacher classes. It is the second theorem which addresses the question of almost simple subgroups.

Lemma 5 *Suppose that M is a maximal subgroup of $Sp(4, q)$ with $q \geq 8$ even such that M does not lie in one of the Aschbacher classes $C_1 - C_8$. Then $M_0 \leq M \leq Aut(M_0)$ for some subgroup $M_0 \approxeq PSL(2, q')$ or $Sz(q')$, where q' is a power of 2.*

Proof. Suppose that M is a maximal subgroup of $Sp(4, q)$ (q even and greater than 2), not lying in one of the Aschbacher classes $C_1 - C_8$. Then M is almost simple and doesn't appear amongst the subgroups listed in Flesner's first theorem. Thus M contains normal subgroups M_1 and M_2 such that $M \geq M_1 \geq M_2 \geq 1$, where M/M_1 and M_2 are of odd order, and M_1/M_2 is isomorphic to either $PSL(2, q')$ or $Sz(q')$ for some power q' of 2. The term "almost simple" means that there is a non-abelian simple group M_0 such that $M_0 \leq M \leq Aut(M_0)$; in consequence M_0 is the unique minimal normal subgroup of M and any non-trivial normal subgroup of M contains M_0. The subgroup M_2 has odd order so cannot contain M_0 (any non-abelian simple group has even order) and therefore $M_2 = 1$ and $M_0 \leq M_1$. Now M_1/M_2 is simple so $M_1 = M_0 = PSL(2, q')$ or $Sz(q')$ for some power q' of 2 and $M \leq Aut(M_0)$.

In arguments that follow we refer to the list of subgroups of $PSL(2, p^a)$ established by Dickson nearly a century ago. The version stated here is for $PSL(2, q^2)$ where $q \geq 8$ is even.

Theorem 6 *(Dickson [4])*
The subgroups of $PSL(2,q^2)$ are as follows:
Cyclic subgroups of order d, where d divides q^2-1; Dihedral subgroups of order $2d$, where d divides q^2-1; Cyclic subgroups of order d, where d divides q^2+1; Dihedral subgroups of order $2d$, where d divides q^2+1; Elementary abelian subgroups of order q^2, together with all subgroups; Subgroups of order cd, where $c=2^m$ and $d=2^k-1$ for some divisor k of both m and $2e$ (here $q=2^e$); Subgroups isomorphic to $PSL(2,q')$ or $PGL(2,q')$ where q^2 is a power of q' (an even power in the case of $PGL(2,q')$).

We have an immediate corollary:

Corollary 7 *If $q\geq 8$ is an odd power of 2 and if G is a subgroup of $PSL(2,q^2)$ divisible by q^2-1, then G is one of: $PSL(2,q^2)$; of order $q^2(q^2-1)$; cyclic of order q^2-1; dihedral of order $2(q^2-1)$.*

We find that subgroups of $PSp(4,q)$ isomorphic to $PSL(2,q^2)$ arise. It is helpful to know that these are the types in classes C_3 and C_8. The following argument establishes this using Singer cyclic subgroups and results from Flesner.

Proposition 8 *If $q\geq 8$ is an odd power of 2, then a subgroup of $PSp(4,q)$ isomorphic to $PSL(2,q^2)$ either stabilizes an elliptic quadric or a regular spread.*

Proof. Let J be a subgroup of $PSp(4,q)$ isomorphic to $PSL(2,q^2)$ and let C be a cyclic subgroup of J of order q^2+1. A reducible subgroup of $PSp(4,q)$ has order dividing $q^4(q^2-1)(q-1)$ so C is irreducible and it follows from ([7], Lemma 3) that C is a Singer cyclic subgroup of $PSp(4,q)$. The orbits of C on the points of $PG(3,q)$ are $q+1$ elliptic quadrics; dually the orbits of C on the totally isotropic lines of $PG(3,q)$ are $q+1$ regular spreads. Now let D be a cyclic subgroup of J of order q^2-1. The same theorem of Flesner implies that D must be reducible, so D stabilizes one of a point, a totally istropic line or a non-isotropic line. Note, however, that the stabilizer of a non-isotropic line is isomorphic to $PSL(2,q)\times PSL(2,q)$ and that a cyclic subgroup of this group having order q^2-1 must fix a point on either the line or its orthogonal complement. Thus D stabilizes either a point or a totally isotropic line. Let us suppose that D stabilizes a point x and suppose that the orbit of J on $PG(3,q)$ containing J has length $a(q^2+1)$. Then $1\leq a\leq q+1$ and J_x (the stabilizer of x in J) has order $q^2(q^2-1)/a$. We have chosen x so that $D\leq J_x$ so a divides q^2. The Corollary to Dickson's list of subgroups of

$PSL(2,q^2)$ demonstrates that a must be $1, q^2/2$ or q^2, but we also have the restriction that $a \leq q+1$. Hence $a = 1$ and J stabilizes an elliptic quadric. Since J is perfect, it must be the group $PSO^-(4,q)$ of an elliptic quadric. The second possibility for D, that it stabilizes a totally isotropic line, gives the dual conclusion: J stabilizes a regular spread and must be the group $PSO^-(4,q)$ of a regular spread.

□

Theorem 9 *If M is a maximal subgroup of $Sp(4,q)$ with $q \geq 8$ even such that q^2+1 divides the order of M, then one of the following occurs: (i) $M \approxeq PSL(2,q^2).2$ and stabilizes a regular spread of lines in $PG(3,q)$, (ii) $M \approxeq PSL(2,q^2).2$ and stabilizes an elliptic ovoid in $PG(3,q)$; (iii) $M \approxeq Sz(q)$ and stabilizes a Suzuki-Tits ovoid.*

Proof. Suppose that M is a maximal subgroup of $Sp(4,q)$ ($q \geq 8$ even), with order divisible by q^2+1, not stabilizing a regular spread of lines in $PG(3,q)$ or an elliptic ovoid in $PG(3,q)$. Then by the previous lemma $M_0 \leq M \leq Aut(M_0)$ for some subgroup $M_0 \approxeq PSL(2,q')$ or $Sz(q')$, where q' is a power of 2. We write $q = 2^e$ and $q' = 2^f$.

Consider first the case where $M_0 \approxeq PSL(2,q')$. The order of M is $q'(q'^2-1)g$ for some divisor g of f. Let $s = (2e, 2f)$. One possibility is that $2e/s$ is even or $2f/s$ is odd, in which case $(2^{2e}+1, 2^{2f}-1) = 1$. Here q'^2-1 divides $(q^2-1)^2$ and so $q' < q^2$, but also q^2+1 divides $g < q'-1$ so $q^2 < q'$ giving a contradiction. Therefore $2e/s$ is odd and $2f/s$ is even, so $(2^{2e}+1, 2^{2f}-1) = 2^s+1$, $(2^{2e}-1, 2^{2f}-1) = 2^s-1$. We have q'^2-1 divides $(q^2-1)^2(q^2+1)$ so $q'^2-1 \leq (2^s-1)^2(2^s+1) < 2^{3s}-1$ and therefore $2f < 3s$. Given that $2f/s$ is even, we can only have $f = s$ and then $2e/f$ is odd. If $2e \geq 3f$, then $q^2+1 > q'^3 > (q'^2-1)g$, a contradiction. Thus we are left with just one possibility: $2e/f = 1$, i.e., $q' = q^2$ and $M_0 \approxeq PSL(2,q^2)$. However, any subgroup of $PSp(4,q)$ isomorphic to $PSL(2,q^2)$ fixes an elliptic ovoid or a regular spread of lines (dual to an elliptic ovoid). It follows that in this case M fixes an elliptic ovoid or a regular spread of lines, a contradiction.

Thus we may suppose that $M_0 \approxeq Sz(q')$. The automorphism group of $Sz(q')$ is $Sz(q').f$ so the order of M is $q'^2(q'^2+1)(q'-1)g$ for some divisor g of f. The significant facts are that q^2+1 divides $q'^2(q'^2+1)(q'-1)f$ and $q'^2(q'^2+1)(q'-1)$ divides $q^4(q^2-1)^2(q^2+1)$; we immediately deduce that q'^2 divides q^4, so that $f \leq 2e$, and that q'^2+1 divides either q^2+1 or $(q^2-1)^2$. Let $t = (2e, f)$. Then $(2e, 2f) = t$ or $2t$. If $(2e, 2f) = t$, then $2f/t$ is even, so $(2^{2e}+1, 2^{2f}+1) = 1$ and $(2^{2e}-1, 2^{2f}+1) = 1$, but then q'^2+1 cannot divide q^2+1 or $(q^2-1)^2$, a contradiction. On the other hand if $(2e, 2f) = 2t$, then $2e/t$ is even and therefore $(2^{2e}+1, 2^f-1) = 1$ and so q^2+1 divides $(q'^2+1)g$. However,

$q > e$ so $q^2 + 1 > f \geq g$ and therefore $(2^{2e} + 1, 2^{2f} + 1) \neq 1$. It follows that $(2^{2e} - 1, 2^{2f} + 1) = 1$ and so $q'^2 + 1$ divides $q^2 + 1$. Hence $f = t$ and e/f is odd. Finally $q' > g$ so $q^2 + 1 < (q'^2 + 1)^{3/2}$ and therefore $e/f = 1$, i.e., $q' = q$. We have shown that $M_0 \approxeq Sz(q)$. By [18], there is only one subgroup of $PSp(4, q)$ isomorphic to $Sz(q)$ up to conjugacy, and such a subgroup is maximal. Thus in this case $M \approxeq Sz(q)$ and stabilizes a Suzuki-Tits ovoid.

□

Corollary 10 *Suppose that G is a subgroup of $PSp(4, q)$ (with $q \geq 8$ even) such that $q^2 + 1$ divides the order of G. Then either G fixes an elliptic ovoid or a regular spread (dual to an elliptic ovoid) or $G \approxeq Sz(q)$ and fixes a unique (Suzuki-Tits) ovoid.*

Proof. Let M be a maximal subgroup of $PSp(4, q)$ containing G. Then by the previous theorem, either M fixes an elliptic ovoid or a regular spread (dual to an elliptic ovoid) or $M \approxeq Sz(q)$ and fixes a Suzuki-Tits ovoid. Suppose that $M \approxeq Sz(q)$. From the list of subgroups of $Sz(q)$ given by [12] it is clear that $G = M$. It is well known that $Sz(q)$ has two orbits of points on $PG(3, q)$, the points on a Suzuki-Tits ovoid forming one orbit, so G fixes a unique ovoid.

□

Corollary 11 *Suppose that a subgroup G of $PSp(4, q)$ has order divisible by $q^2 + 1$ (with $q \geq 8$ even) and stabilizes an ovoid Ω in $PG(3, q)$. Then the ovoid is classical.*

Proof. By the theorems above, either $G \approxeq Sz(q)$ and G fixes a unique (Suzuki-Tits) ovoid, or $G \leq M \approxeq PSL(2, q^2).2$. We need only consider the latter case.
Let $G_1 = G \cap PSL(2, q^2)$. Then $q^2 + 1$ divides the order of G_1. Consulting the list of subgroups of $PSL(2, q^2)$, we see that the only subgroups whose order is divisible by $q^2 + 1$ are cyclic groups of order $q^2 + 1$ and dihedral groups of order $2(q^2 + 1)$. In any case G contains a cyclic subgroup of order $q^2 + 1$ and by [7] this is a subgroup of a Singer subgroup of $PSp(4, q)$. Moreover the orbits of this cyclic subgroup are all elliptic quadrics. It follows that the orbits of G are unions of elliptic quadrics and, in particular, Ω must be an elliptic quadric.

□

We are now in a position to state three theorems, all essentially proven.

Theorem 12 *If Ω is an ovoid in $PG(3, q)$, where $q \geq 8$ is even, such that the stabilizer of Ω in $PSp(4, q)$ acts transitively on Ω, then Ω is classical.*

Theorem 13 *If Ω is an ovoid in $PG(3,q)$ (with $q \geq 8$ even) admitting a group containing cyclic subgroups of orders $q+2r+1$, $q-2r+1$, where $r=\sqrt{2q}$, then Ω is classical.*

Theorem 14 *Suppose that G is a subgroup of $PSp(4,q)$ (with $q \geq 8$ even) containing elements of orders $q+2r+1$ and $q-2r+1$. Then one of the following occurs: (i) $G \leq PSL(2,q^2).2$ and $S \leq G \leq N_{PSp(4,q)}(S)$, where S is a Singer cyclic subgroup of $PSp(4,q)$ of order q^2+1 with normalizer having order $4(q^2+1)$; (ii) $G \approxeq PSL(2,q^2)$; (iii) $G \approxeq PSL(2,q^2).2$; (iv) $G \approxeq Sz(q)$; (v) $G = PSp(4,q)$.*

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