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Translation ovoids of unitary polar spaces

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Abstract

An *ovoid* of a finite classical polar space is a set of points having exactly one point in common with every generator. An ovoid is a *translation ovoid* with respect to one of its points if it admits a collineation group fixing all totally isotropic lines through the point and acting regularly on the remaining points of the ovoid. Lunardon and Polverino proved in [11] that $Q^+(3, q)$, $Q(4, q)$ and $Q^+(5, q)$ are the only finite orthogonal polar spaces having translation ovoids. In this paper we prove that $H(3, q^2)$ is the only finite unitary polar space having translation ovoids. We also prove that translation groups of ovoids of $H(3, q^2)$ contain only linear collineations.

Keywords: Finite polar spaces, Hermitian varieties, ovoids, translation ovoids

1 Introduction

In the projective space $\text{PG}(r, q)$ coordinatized by the finite field $\text{GF}(q)$ let $\mathcal{P}$ denote a (finite) classical polar space. The *generators* of $\mathcal{P}$ are the subspaces of maximal dimension contained in it. By Witt's theorem, the maximal totally isotropic subspaces have the same dimension, which is called the *rank* of the polar space. A line L of $\text{PG}(r, q)$ not contained in the polar space $\mathcal{P}$ is said to be *external*, *tangent* or *hyperbolic* if L meets $\mathcal{P}$ in 0, 1 or $s > 1$ points, respectively. More precisely, if $\mathcal{P}$ is an orthogonal polar space in $\text{PG}(r, q)$ then $s = 2$ whereas if $\mathcal{P}$ is a unitary polar space in $\text{PG}(r, q^2)$ then $s = q + 1$.

An *ovoid* of $\mathcal{P}$ is a set of points of $\mathcal{P}$ which meets every generator in a point. The size of an ovoid (when it exists) depends on the rank of $\mathcal{P}$. More precisely, ovoids of $H(2n + 1, q^2)$, $H(2n, q^2)$, $Q^+(2n - 1, q)$, $Q(2n, q)$, $Q^-(2n + 1, q)$ have size $q^{2n+1} + 1$, $q^{2n+1} + 1$, $q^{n-1} + 1$, $q^n + 1$, $q^{n+1} + 1$, respectively.

The current position regarding the existence of ovoids of finite classical polar spaces is summarized in Table 1 of [6]. In particular, when $n \geq 1$ the unitary polar space $H(2n, q^2)$ has no ovoid [17]. In [7], De Beule and Metsch proved the non-existence of ovoids in $H(5, 4)$; in [8], Klein proved a similar result for $H(2n + 1, q^2)$ in the case $n > q^3$. On the other hand, it is well known that $H(3, q^2)$ has many ovoids.

An ovoid $\mathcal{O}$ of $\mathcal{P}$ is a *translation ovoid* with respect to a point P of $\mathcal{O}$ if there is a collineation group of $\mathcal{P}$ fixing all totally isotropic lines through P and acting regularly on points of the ovoid different from P . Such a group is called the *translation group (about P)* of $\mathcal{O}$. We call a translation ovoid $\mathcal{O}$ *semilinear* if it has a translation group containing non-linear collineations; we call $\mathcal{O}$ *linear* otherwise. An ovoid $\mathcal{O}$ is said to be *locally hermitian* with respect to a point P if $\mathcal{O}$ is the union of q^{2n} hyperbolic lines through P . From [1] a linear translation ovoid of $H(3, q^2)$ is locally hermitian.

Translation ovoids of finite orthogonal polar spaces have been intensively studied by many authors. Examples of translation ovoids of $Q^+(3, q)$ are non-degenerate conics contained in it. In the polar space $Q(4, q)$ there exist three infinite families and one sporadic example of translation ovoids (for more details see [2] and [13]). According to a construction given by Lunardon [9], translation ovoids of $Q(4, q)$ correspond to semifield flocks of the quadratic cone in $\text{PG}(3, q)$, which, by a well-known construction, correspond to generalized quadrangles with parameters (q^2, q) [18]. Translation ovoids of $Q^+(5, q)$ exist for all values of q and they are equivalent to semifield spreads of $\text{PG}(3, q)$.

In [11], Lunardon and Polverino proved that $Q^+(3, q)$, $Q(4, q)$ and $Q^+(5, q)$ are the only finite orthogonal polar spaces having translation ovoids.

The understanding of translation ovoids of finite unitary polar spaces is not as deep as that of translation ovoids of orthogonal spaces. Examples of translation ovoids of $H(3, q^2)$ are non-degenerate hermitian curves contained in it. Several other infinite

families of translation ovoids of $H(3, q^2)$ have been constructed in [4]. The intimate connection between linear translation ovoids of $H(3, q^2)$ and semifield spreads of $\text{PG}(3, q)$ was highlighted in [1], [5] and [10].

Our purpose is to broaden the understanding of translation ovoids of finite unitary polar spaces.

In [10], Lunardon represented $H(3, q^2)$ as a cone of $\text{PG}(6, q)$ projecting onto an orthogonal polar space $Q^+(5, q)$ from its vertex by using the Barlotti-Cofman representation of $\text{PG}(3, q^2)$. In this representation, every locally hermitian ovoid of $H(3, q^2)$ (with respect to a point P) corresponds to an ovoid of $Q^+(5, q)$ (containing a specified point), and conversely. The above result can be extended to every finite unitary polar space as shown in [16]. In particular, the unitary space $H(2n + 1, q^2)$ can be represented as a cone of $\text{PG}(4n + 2, q)$ projecting an orthogonal space $Q^+(4n + 1, q)$ from its vertex. As above, every locally hermitian ovoid of $H(2n + 1, q^2)$ (with respect to a point P) corresponds to an ovoid of $Q^+(4n + 1, q)$ (containing a specified point), and conversely.

In this paper we prove that a translation ovoid of $H(2n + 1, q^2)$ corresponds to a translation ovoid $Q^+(4n + 1, q)$. By applying the result of Lunardon and Polverino, we conclude that $H(3, q^2)$ is the only finite unitary polar space having translation ovoids. Finally, we also prove that translation groups of ovoids of $H(3, q^2)$ contain only linear collineations. The authors would like to thank the referees for their helpful comments.

2 On the structure of the translation group of a translation ovoid of $H(2n + 1, q^2)$

We use the following concise notation: for $\mathbf{a} = (a_1, \dots, a_n) \in \text{GF}(q^2)^n$ we set $\bar{\mathbf{a}} := (a_1^q, \dots, a_n^q)$, $\mathbf{a}'$ will denote the transpose of $\mathbf{a}$.

In the projective space $\text{PG}(2n + 1, q^2)$, $q = p^h$, p a prime, let $H(2n + 1, q^2)$ be a unitary polar space. We can choose the homogeneous projective coordinates $(X, \mathbf{X}, Y, \mathbf{Y})$ in such a way that $H(2n + 1, q^2)$ has equation

$$XY^q - X^qY + \mathbf{X}\bar{\mathbf{Y}}' - \bar{\mathbf{X}}\mathbf{Y}' = 0 \quad (1)$$

We shall denote by $\perp$ the polarity defined by $H(2n + 1, q^2)$.

As usual we denote by $\text{PGU}(2n + 2, q^2)$ and $\text{PTU}(2n + 2, q^2)$, respectively, the groups of linear and semilinear collineations of $\text{PG}(2n + 1, q^2)$ leaving $H(2n + 1, q^2)$ invariant. Then $\text{PTU}(2n + 2, q^2)$ has the structure $\text{PGU}(2n + 2, q^2) \rtimes \text{Aut}(\text{GF}(q^2))$.

Let P be a point of $H(2n + 1, q^2)$ and $\hat{E}$ be the subgroup of $\text{PGU}(2n + 2, q^2)$ fixing P and all totally isotropic lines of $H(2n + 1, q^2)$ on P .

Lemma 2.1. *The group $\widehat{E}$ acts transitively on points of $H(2n+1, q^2)$ not in $P^\perp$ and has order $(q^2-1)q^{4n+1}$. Further $\widehat{E}$ has a unique subgroup E of order q^{4n+1} , and E acts regularly on points of $H(2n+1, q^2)$ not in $P^\perp$.*

Proof. Let $V(2n+2, q^2)$ be the underlying vector space of $\text{PG}(2n+1, q^2)$ and h a skew-hermitian form corresponding to $H(2n+1, q^2)$. We take a basis $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_{n+1}, \mathbf{f}_1, \dots, \mathbf{f}_{n+1})$ consisting of mutually orthogonal hyperbolic pairs $(\mathbf{e}_i, \mathbf{f}_i)$, $i = 1, \dots, n+1$, such that $P = \langle \mathbf{e}_1 \rangle$. It follows that the matrix of h is $\begin{pmatrix} O_{n+1} & I_{n+1} \\ -I_{n+1} & O_{n+1} \end{pmatrix}$ (where O_k denotes the $k \times k$ zero matrix and I_k denotes the $k \times k$ identity matrix) and $P^\perp \subset \langle \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}, \mathbf{f}_2, \dots, \mathbf{f}_{n+1} \rangle$.

Since the group $\widehat{E}$ preserves the form h and induces the identity on $P^\perp/P$, a tedious but straightforward calculation shows that an element of $\widehat{E}$ can be represented, with respect to the basis $\mathcal{B}$, by a matrix having the form

$$\begin{pmatrix} a & -a\bar{\mathbf{a}} & a(c - \bar{\mathbf{a}}\mathbf{b}') & a\bar{\mathbf{b}} \\ \mathbf{0}' & I_n & \mathbf{b}' & O_n \\ 0 & \mathbf{0} & a^{-q} & \mathbf{0} \\ \mathbf{0}' & O_n & \mathbf{a}' & I_n \end{pmatrix}$$

with $\mathbf{a}, \mathbf{b} \in \text{GF}(q^2)^n$, $a \in \text{GF}(q^2)^*$ and $c \in \text{GF}(q)$; here multiplication is on the left, i.e., points are regarded as column vectors. All combinations of $\mathbf{a}, \mathbf{b}$, a and c are possible, so $\widehat{E}$ has order $(q^2-1)q^{4n+1}$. It is then clear that $\widehat{E}$ has a unique Sylow p -subgroup E of order q^{4n+1} consisting precisely of the matrices with $a = 1$.

Let Q be a point of $H(2n+1, q^2)$ not in $P^\perp$. Then we can write $Q = (b, \mathbf{b}, 1, \mathbf{a})$ with $c = b + \bar{\mathbf{a}}\mathbf{b}' \in \text{GF}(q)$. Let R be the point $(0, \mathbf{0}, 1, \mathbf{0})$. Then there is a unique element g of E such that $g(R) = Q$, given by the matrix

$$\begin{pmatrix} 1 & -\bar{\mathbf{a}} & c - \bar{\mathbf{a}}\mathbf{b}' & \bar{\mathbf{b}} \\ \mathbf{0}' & I_n & \mathbf{b}' & O_n \\ 0 & \mathbf{0} & 1 & \mathbf{0} \\ \mathbf{0}' & O_n & \mathbf{a}' & I_n \end{pmatrix}.$$

It follows that E acts regularly (and $\widehat{E}$ transitively) on points of $H(2n+1, q^2)$ not in $P^\perp$. $\square$

When $n = 1$, the group E is called an *elation group about P* (see Section 4). Noting that the E -orbit of R is $\{(c - \bar{\mathbf{a}}\mathbf{b}', \mathbf{b}, 1, \mathbf{a}) : \mathbf{a}, \mathbf{b} \in \text{GF}(q^2)^n, c \in \text{GF}(q)\}$, it is often more convenient to represent the group E as the set of triples

$$E = \{[\mathbf{a}, \mathbf{b}, c] : \mathbf{a}, \mathbf{b} \in \text{GF}(q^2)^n, c \in \text{GF}(q)\}$$

with group operation

$$[\mathbf{a}_1, \mathbf{b}_1, c_1] * [\mathbf{a}_2, \mathbf{b}_2, c_2] = [\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}_1 + \mathbf{b}_2, c_1 + c_2 + \bar{\mathbf{a}}_2\mathbf{b}'_1 + \mathbf{a}_2\bar{\mathbf{b}}'_1].$$

The set $K = \{[\mathbf{0}, \mathbf{0}, c] : c \in \text{GF}(q)\}$ is an elementary abelian subgroup (of order q) of E fixing each hyperbolic line of $H(2n+1, q^2)$ through P ; by the Orbit-Stabilizer Theorem, K is the stabilizer in E of each hyperbolic line through P . Further, E/K is an elementary abelian group of order q^{4n} . In fact E is a Heisenberg group.

Notice that the matrices we have specified for elements of E actually form a subgroup E_0 of $\text{GU}(2n+1, q^2)$ isomorphic to E . In fact E is the image of E_0 in $\text{PGU}(2n+1, q^2)$.

Proposition 2.2. *If $n > 1$ then $H(2n+1, q^2)$ has no semilinear translation ovoids.*

Proof. We denote by φ the Frobenius automorphism of $\text{GF}(q^2)$ over $\text{GF}(p)$:

$$\begin{array}{ccc} \varphi : & \text{GF}(q^2) & \longrightarrow \text{GF}(q^2) \\ & x & \longmapsto x^p \end{array}$$

Then $\text{Aut}(\text{GF}(q^2))$ is generated by φ and each automorphism φ^j , $j = 1, \dots, 2h$, induces the collineation

$$\begin{array}{ccc} \Phi^j : & \text{PG}(2n+1, q^2) & \longrightarrow \text{PG}(2n+1, q^2) \\ & (X, \mathbf{X}, Y, \mathbf{Y}) & \longmapsto (X^{\varphi^j}, \mathbf{X}^{\varphi^j}, Y^{\varphi^j}, \mathbf{Y}^{\varphi^j}) \end{array}$$

which preserves $H(2n+1, q^2)$ and fixes P ; here $\mathbf{a}^{\varphi^j}$ denotes the n -tuple $(a_1^{p^j}, \dots, a_n^{p^j})$. Note that Φ^j is semilinear if $j < 2h$ and the identity if $j = 2h$.

Further, the set of fixed points of Φ^j is the canonical subgeometry $\text{PG}(2n+1, p^m)$ generated by $\mathbf{e}_i, \mathbf{f}_i$, $i = 1, \dots, n+1$, over $\text{GF}(p^m)$ for $m = \text{GCD}(j, 2h)$.

Let $R = \langle \mathbf{f}_1 \rangle$ and consider the unitary space $H(2n-1, q^2) = P^\perp \cap R^\perp$ with equation $\mathbf{X}\overline{\mathbf{Y}}' - \overline{\mathbf{X}}\mathbf{Y}' = 0$. This contains the points $P_i = \langle \mathbf{e}_i \rangle$, $R_i = \langle \mathbf{f}_i \rangle$, $i = 2, \dots, n+1$.

Next let G be a translation group (about P) of an ovoid $\mathcal{O}$ of $H(2n+1, q^2)$ containing the collineation $g\Phi^j$ for some $1 \leq j \leq 2h$. Since Φ^j fixes all of the points Q in $\text{PG}(2n+1, p^m) \cap H(2n-1, q^2)$, it fixes all the lines PQ , and then so does g . Consequently g fixes each totally isotropic line through P . Suppose $j < 2h$. Then $H(2n-1, q^2)$ is not contained in the subgeometry $\text{PG}(2n-1, p^m)$ and therefore there are points of $H(2n-1, q^2)$ which are not fixed by Φ^j . If T is such a point, then the line PT is fixed by g but not by $g\Phi^j$. This shows that G cannot contain elements of the form $g\Phi^j$ with $j < 2h$, in other words the translation ovoid is linear. $\square$

The following extends arguments used in [1].

Lemma 2.3. *If $\mathcal{O}$ is a linear translation ovoid with respect to P , with translation group G , then $K \leq G$ and $H(2n+1, q^2)$ is locally hermitian.*

Proof. First observe that the subgroup $L = \{[\mathbf{0}, \mathbf{b}, 0] : \mathbf{b} \in \text{GF}(q^2)^n\}$ of E fixes the generator $X = 0, \mathbf{Y} = \mathbf{0}$ of $H(2n+1, q^2)$ and therefore $L \cap G = 1_G$. Let $(\mathbf{a}, c) \in \text{GF}(q^2)^n \times \text{GF}(q)$ and suppose that G contains elements $g = [\mathbf{a}, \mathbf{b}, c]$ and $h = [\mathbf{a}, \widehat{\mathbf{b}}, c]$. Then G contains $g^{-1}h = [\mathbf{0}, \widehat{\mathbf{b}} - \mathbf{b}, 0]$ which implies that $\widehat{\mathbf{b}} = \mathbf{b}$. Cardinality considerations now imply that for each $(\mathbf{a}, c) \in \text{GF}(q^2)^n \times \text{GF}(q)$, there is exactly one $\mathbf{b} \in \text{GF}(q^2)^n$ such that $[\mathbf{a}, \mathbf{b}, c] \in G$.

The commutator $[\mathbf{a}_1, \mathbf{b}_1, c_1]^{-1} * [\mathbf{a}_2, \mathbf{b}_2, c_2]^{-1} * [\mathbf{a}_1, \mathbf{b}_1, c_1] * [\mathbf{a}_2, \mathbf{b}_2, c_2]$ is given by

$$= [\mathbf{0}, \mathbf{0}, \bar{\mathbf{a}}_2 \mathbf{b}'_1 + \mathbf{a}_2 \bar{\mathbf{b}}'_1 - \bar{\mathbf{a}}_1 \mathbf{b}'_2 - \mathbf{a}_1 \bar{\mathbf{b}}'_2].$$

For each $c_1 \in \text{GF}(q)$, with $\mathbf{b}_1 \in \text{GF}(q^2)^n$ such that $[\mathbf{0}, \mathbf{b}_1, c_1] \in G$, we conclude that $[\mathbf{0}, \mathbf{0}, \bar{\mathbf{a}}_2 \mathbf{b}'_1 + \mathbf{a}_2 \bar{\mathbf{b}}'_1] \in G$ for every $[\mathbf{a}_2, \mathbf{b}_2, c_2] \in G$. If $\mathbf{b}_1 \neq \mathbf{0}$ for some $c_1 \in \text{GF}(q)$, then there would exist $\mathbf{a}_2 \in \text{GF}(q^2)^n$ such that $\bar{\mathbf{a}}_2 \mathbf{b}'_1 + \mathbf{a}_2 \bar{\mathbf{b}}'_1 = c_1$, giving a contradiction. Hence $[\mathbf{0}, \mathbf{0}, c_1] \in G$ for each $c_1 \in \text{GF}(q)$, i.e., $K \leq G$.

The group K fixes every hyperbolic line through P and each orbit of K on points of $H(2n+1, q^2)$ distinct from G has length q . In particular K has an orbit of length q on each hyperbolic line through P meeting $\mathcal{O}$. It follows that $\mathcal{O}$ is the union of q^{2n} such lines, i.e., that $\mathcal{O}$ is locally hermitian. $\square$

Corollary 2.4. *If $n > 1$ then every translation ovoid of $H(2n+1, q^2)$ is locally hermitian.*

3 Translation ovoids of unitary spaces and translation ovoids of orthogonal polar spaces

The geometry of unitary polar spaces under the Barlotti-Cofman representation of $\text{PG}(n, q^2)$, introduced in [3], has been intensively studied by many authors; see e.g., [10], [12], [16]. In the following we assume the reader is familiar with notation and results from [10] and [16].

Lunardon proved in [10] that every translation ovoid of $H(3, q^2)$ defines a translation ovoid of $Q^+(5, q)$. In the following we prove a result which generalizes Theorem 6 in [10].

Theorem 3.1. *Every linear translation ovoid of $H(2n+1, q^2)$ determines a translation ovoid of $Q^+(4n+1, q)$.*

Proof. Let $\mathcal{O}$ be a translation ovoid of $H(2n+1, q^2)$ with respect to P with translation group G . We use the Barlotti-Cofman representation of the points of $\text{PG}(2n+1, q^2)$ as points and lines of a $\text{PG}(4n+2, q)$. The following is described in [10, 16]. The points of $\text{PG}(2n+1, q^2)$ are represented by $(2n+2)$ -tuples $\mathbf{x} \in V(2n+2, q^2)$, and points of $\text{PG}(4n+3, q^2)$ are then represented by $(4n+4)$ -tuples $(\mathbf{x}, \mathbf{y})$. A semi-linear transformation σ of $\text{PG}(4n+3, q^2)$ is defined by $(\mathbf{x}, \mathbf{y}) \mapsto (\bar{\mathbf{y}}, \bar{\mathbf{x}})$, the fixed

points of which form a $\text{PG}(4n+3, q)$. To each point $\mathbf{x}$ of $\text{PG}(2n+1, q^2)$, there corresponds a line $\ell(\mathbf{x})$ of $\text{PG}(4n+3, q)$ consisting of the fixed points of σ on the line $\langle(\mathbf{x}, \mathbf{0}), (\mathbf{0}, \bar{\mathbf{x}})\rangle$. The lines $\ell(\mathbf{x})$ corresponding to points $\mathbf{x} \in H(2n+1, q^2)$ form a line spread of a hyperbolic quadric $Q^+(4n+3, q)$. We let P be the point corresponding to $\mathbf{e}_1$ as before, and let T be a point of $\ell(\mathbf{e}_1)$. Let α be the tangent hyperplane to $H(2n+1, q^2)$ at P and let Σ' be the tangent hyperplane to $Q^+(4n+3, q)$ at T . Then the lines $\ell(\mathbf{x})$ for $\mathbf{x} \in \alpha$ form a line spread of a hyperplane Σ of Σ' and the lines $\ell(\mathbf{x})$ for $\mathbf{x} \notin \alpha$ meet Σ' in a point. A line of α through P corresponds to a line spread of a 3-space on $Q^+(4n+3, q)$ (contained in Σ). In contrast, a hyperbolic line L of $\text{PG}(2n+1, q^2)$ through P corresponds to a 3-space of $\text{PG}(4n+3, q)$ that meets Σ' in a plane which in turn meets $Q^+(4n+3, q)$ in two lines through T : one is $\ell(\mathbf{e}_1)$ and the other we denote by $t(L)$. Let $\mathcal{K}$ be the tangent cone $Q^+(4n+3, q) \cap \Sigma'$, then the quotient $\mathcal{K}/T$ is a hyperbolic quadric $Q^+(4n+1, q)$ containing points corresponding to $\ell(\mathbf{e}_1)$ and $t(L)$. We denote by X the point of $\mathcal{K}/T$ corresponding to $\ell(\mathbf{e}_1)$.

Given $A \in \text{GU}(2n+2, q^2)$, we have an action on $V(4n+4, q^2)$ by mapping $(\mathbf{x}, \mathbf{y})$ to $(A\mathbf{x}, \bar{A}\mathbf{y})$ (where $\bar{A}$ is the matrix obtained from A by raising each entry to the power q). Under this action $\text{GU}(2n+2, q^2)$ acts as a linear group on $\text{PG}(4n+3, q)$ preserving the quadric $Q^+(4n+3, q)$. Notice that scalar matrices in $\text{GU}(2n+1, q^2)$ do not necessarily act as scalar matrices on $V(4n+4, q)$. Recall the subgroup E_0 of $\text{GU}(2n+2, q^2)$, isomorphic to E , as defined in Section 2. The elements of E_0 fix $\mathbf{e}_1$ and therefore fix every point of $\ell(\mathbf{e}_1)$. In particular they fix T and therefore fix (globally) both the cone $\mathcal{K}$ and the quotient $\mathcal{K}/T$. Let G_0 and K_0 denote the subgroups of E_0 corresponding to G and K respectively. For each point $\mathbf{x} \in \alpha$ and each $A \in E_0$, we observe that $(A\mathbf{x}, \bar{A}\bar{\mathbf{x}})$ lies in the plane spanned by $\ell(\mathbf{e}_1)$ and $(\mathbf{x}, \bar{\mathbf{x}})$, which in Σ'/T corresponds to a line of $\mathcal{K}/T$ through X and $(\mathbf{x}, \bar{\mathbf{x}})$. Every line of $\mathcal{K}/T$ through X arises in this way, and so every line of $\mathcal{K}/T$ through X is fixed by E_0 . As noted in Section 2 (in terms of K), the group K_0 fixes every hyperbolic line L through P and therefore fixes $t(L)$. Also K_0 fixes every $(2n+2)$ -tuple $\mathbf{x}$ corresponding to a point of α , so fixes each point of Σ ; every point of $\mathcal{K}/T$ is either a line of Σ through T or has the form $t(L)$ for some hyperbolic line L of $\text{PG}(2n+1, q^2)$ through P , and in either case the point of $\mathcal{K}/T$ is fixed by K_0 . Hence K_0 acts trivially on Σ'/T .

Let $\mathbb{O}$ be the set of points of $\mathcal{K}/T$ determined by the q^{2n} hyperbolic lines of $\text{PG}(2n+1, q^2)$ through P that contain the points of $\mathcal{O}$, together with the point X . In [16] it was shown that $\mathbb{O}$ is an ovoid of $Q^+(4n+1, q)$. We now show that $\mathbb{O}$ is a translation ovoid of $Q^+(4n+1, q)$ with respect to X . It follows from Lemma 2.3 that G_0 contains K_0 . Given that G_0 acts transitively on the points of $\mathcal{O} \setminus \{P\}$, we conclude that G_0 acts transitively on the points of $\mathbb{O} \setminus \{X\}$. Hence G_0/K_0 also acts transitively on this set, and since G_0/K_0 has order q^{2n} , the action is regular. The (isomorphic) image of G_0/K_0 in the linear collineation group of $\mathcal{K}/T$ is therefore a translation group for $\mathbb{O}$ with respect to X . $\square$

Combining the previous Theorem and Theorem 4 in [11] gives the following result.

Theorem 3.2. *The only finite unitary polar space having translation ovoids is $H(3, q^2)$.*

4 On semilinear translation ovoids of $H(3, q^2)$

In this section, we restrict our attention on the unitary polar space $H(3, q^2)$. We shall prove that $H(3, q^2)$ has no semilinear translation ovoids.

Let $\mathcal{O}$ be a translation ovoid of $H(3, q^2)$ with translation group G . Then $|\mathcal{O}| = q^3 + 1$ and G has order q^3 . Examples of translation ovoids of $H(3, q^2)$ are non-degenerate hermitian curves. Several other infinite families of translation ovoids of $H(3, q^2)$ have been constructed in [4]. We note that all the exhibited translation ovoids are linear (and thus locally hermitian).

As in Section 2 we represent the group $E \leq \text{PGU}(4, q^2)$ as the set of triples

$$E = \{[a, b, c] : a, b \in \text{GF}(q^2), c \in \text{GF}(q)\}$$

with group operation

$$[a_1, b_1, c_1] * [a_2, b_2, c_2] = [a_1 + a_2, b_1 + b_2, c_1 + c_2 + \bar{a}_2 b_1 + a_2 \bar{b}_1].$$

In $\text{PG}(3, q^2)$, $q = p^h$, the semilinear collineation Φ^j different from identity fixes each totally isotropic line through the point $P \in H(3, q^2)$ precisely when $j = h$. If we adjoin the collineation $\phi = \Phi^h$ to the group E , we form the group

$$W = \{[a, b, c]\phi^i : a, b \in \text{GF}(q^2), c \in \text{GF}(q), i = 0, 1\}$$

with the group operation being composition of maps as defined in Section 2 together with $\phi[a, b, c]\phi = [\bar{a}, \bar{b}, c]$ and $\phi^2 = 1$. Clearly $W \leq \text{PGU}(4, q^2)$, it has order $2q^5$ and it fixes P . Moreover it fixes every totally isotropic line through P and acts transitively on points of $H(3, q^2)$ not in $P^\perp$. A *whorl* about P is a collineation fixing every totally isotropic line through P . Thus the full *whorl group about P* is the group $\hat{W} = \langle \hat{E}, \phi \rangle$, and W is a group of whorls. An *elation* (in the generalized quadrangle sense) is either the identity mapping or a whorl that fixes no point of $H(3, q^2) \setminus P^\perp$. An *elation group about P* is a group of elations acting regularly on the set of points of $H(3, q^2) \setminus P^\perp$. Thus an elation group has order q^5 . We see that E is an elation group about P . When q is odd, E is the unique Sylow p -subgroup of $\hat{W}$ leading to the following result.

Theorem 4.1. *If q is odd, then every translation ovoid of $H(3, q^2)$ is linear.*

Proof. The translation group G is contained in E , so is necessarily linear. $\square$

Whereas for odd q there is a unique elation group about P , indeed a unique subgroup of W of order q^5 , the same is not true for even q : there are many elation groups (classified in [15]) and many subgroups of W of order q^5 . This means that the non-existence of semilinear ovoids is less evident when q is even. For even q , W is the unique Sylow 2-subgroup of $\hat{W}$ (having order $2q^5$), an elation group is a subgroup of W and G is a subgroup of W . Note that W contains elements that are not elations (ϕ for example) and it is not immediately clear that G is a subgroup of an elation group about P .

Let $\text{PG}(3, q^2)$, be the three-dimensional projective space over $\text{GF}(q^2)$ with homogeneous projective coordinates x_1, x_2, x_3, x_4 . The skew-hermitian form h is given by $h(X, Y) = x_1y_3^q + x_2y_4^q - x_3y_1^q - x_4y_2^q$. It is easy to check that $H(3, q^2)$ contains the points of the canonical subgeometry $\text{PG}(3, q)$ which is actually the symplectic geometry with respect the alternating form $x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$ obtained by restricting h to $\text{PG}(3, q)$. In fact, given a subgeometry $\text{PG}(3, q)$ on $H(3, q^2)$, the co-ordinate system may be chosen so that the subgeometry is the canonical one.

An *i-tight set* $\mathcal{T}$ of $H(3, q^2)$ is a set of points such that every point in $\mathcal{T}$ is collinear with $q^2 + i$ points of $\mathcal{T}$, while every point not in $\mathcal{T}$ is collinear with i points of $\mathcal{T}$. A result on *i-tight sets* enables us to conclude that an ovoid meets a subgeometry in $q + 1$ points, a property that proves useful subsequently.

Lemma 4.2. *Every symplectic subgeometry contained in $H(3, q^2)$ is a $(q + 1)$ -tight set.*

Proof. Let $\mathcal{T}$ be a symplectic subgeometry. A point in $\mathcal{T}$ is collinear with $q^2 + q + 1$ points of $\mathcal{T}$, namely the points of the tangent plane at the point. Let us turn to points of $\mathcal{X} = H(3, q^2) \setminus \mathcal{T}$. Two lines of $\mathcal{T}$ either meet in a point of $\mathcal{T}$ or are disjoint in $\text{PG}(3, q^2)$, and therefore a point of $\mathcal{X}$ lies on at most one line of $\mathcal{T}$. Each line of $\mathcal{T}$ contains $q^2 - q$ points of $\mathcal{X}$, so the $q^3 + q^2 + q + 1$ lines of $\mathcal{T}$ contain $q^5 - q$ points of $\mathcal{X}$. But this accounts for all the points of $\mathcal{X}$. Hence each point of $\mathcal{X}$ lies on exactly one line of $\mathcal{T}$ and is collinear with the $q + 1$ points of $\mathcal{T}$ on that line. Therefore $\mathcal{T}$ is a $(q + 1)$ -tight set. $\square$

Proposition 4.3. *Let $\mathcal{O}$ be an ovoid of $H(3, q^2)$ and $\mathcal{T}$ a symplectic subgeometry contained in $H(3, q^2)$. Then $\mathcal{O}$ and $\mathcal{T}$ intersect in $q + 1$ points.*

Proof. Since $H(3, q^2)$ is a generalized quadrangle, we can directly apply Theorem 4.3 of [2]. $\square$

We now assume that q is even, say $q = 2^h$. It is shown in [15] that the derived group of W is given by

$$W' = \{[a, b, c] : a, b, c \in \text{GF}(q)\}$$

and W/W' is a vector space of dimension $2h + 1$ over $\text{GF}(2)$. Since every p -group is nilpotent and W is a 2-group, it follows that W' is a subgroup of the Frattini subgroup $\mathcal{F}(W)$ (c.f., [14]). Thus each maximal subgroup of W has order q^5 and contains W' . Note that $K \leq W'$.

Lemma 4.4. *If q is even and $\mathcal{O}$ is a translation ovoid of $H(3, q^2)$ with translation group G , then GW' is a maximal subgroup of W and $|G \cap W'| = q$.*

Proof. We may assume that $\mathcal{O}$ contains the point $R = (0, 0, 1, 0)$. Let $\mathcal{W}$ be the canonical symplectic subgeometry of $H(3, q^2)$ with respect to the given coordinate system. Then, by Proposition 4.3, $\mathcal{O}$ meets $\mathcal{W}$ in $q + 1$ points. Then $R \in \mathcal{W}$ and $g(R) \in \mathcal{W}$ for exactly q elements in G . Since W' fixes $\mathcal{W}$, it follows that $|G \cap W'| \leq q$.

A group isomorphism theorem tells us that since $W' \trianglelefteq E$ and $G \leq W$ we have $\frac{GW'}{W'} \cong \frac{G}{G \cap W'}$. Now $\left| \frac{GW'}{W'} \right| \leq 2q^5/q^3 = 2q^2$. Given that $|G \cap W'| \leq q$ we have $\left| \frac{G}{G \cap W'} \right| \geq q^3/q = q^2$. Hence $\left| \frac{GW'}{W'} \right| = \left| \frac{G}{G \cap W'} \right| = q^2$ or $2q^2$.

Now consider that G lies in some maximal subgroup, H say, and we know that H contains W' , so H contains GW' . Hence $|GW'| \leq |H| = q^5$. This tells us that $|GW'| = q^5$ and $|G \cap W'| = q$. $\square$

The following Lemma lists results that will be used to show that $|G \cap W'| = q$ is not possible if $\mathcal{O}$ is semilinear. For the reader's convenience, we first list some calculations. Suppose that $u, v, w \in W$ with $u = [a_1, b_1, c_1], v = [a_2, b_2, c_2]\phi, w = [a_3, b_3, c_3]$. Then:

- $u^{-1} = [a_1, b_1, c_1 + \overline{a_1}b_1 + a_1\overline{b_1}], v^{-1} = [\overline{a_2}, \overline{b_2}, c_2 + \overline{a_2}b_2 + a_2\overline{b_2}]\phi.$
- $u^2 = [0, 0, \overline{a_1}b_1 + a_1\overline{b_1}], v^2 = [a_2 + \overline{a_2}, b_2 + \overline{b_2}, a_2b_2 + \overline{a_2}b_2].$
- $u^{-1}w^{-1}uw = [0, 0, \overline{a_1}b_3 + a_1\overline{b_3} + \overline{a_3}b_1 + a_3\overline{b_1}].$

Lemma 4.5. *Suppose that q is even and $\mathcal{O}$ is a semilinear translation ovoid of $H(3, q^2)$ with translation group G and let $G_L = G \cap E$. Let $\Delta = \{a \in \text{GF}(q^2) : [a, b, c] \in G \text{ for some } b \in \text{GF}(q)^2, c \in \text{GF}(q)\}, \Gamma = \{a \in \text{GF}(q^2) : [a, b, c]\phi \in G \text{ for some } b \in \text{GF}(q)^2, c \in \text{GF}(q)\}$ and $G_a = \{[a, b, c] : b \in \text{GF}(q)^2, c \in \text{GF}(q)\} \cap G_L$. Then:*

- (a) G contains no elements of $W'\phi$.
- (b) $|G_L| = q^3/2$.
- (c) Δ is an additive subgroup of $\text{GF}(q^2)$, $|\Delta| = |\Gamma| = q^2/2$ and $|G_a| = q$ for each $a \in \Delta$.

(d) $|G_0 \cap W'| = q$.

Proof. (a) Given that $|G \cap W'| = q$ and $|\mathcal{O} \cap \mathcal{W}| = q + 1$ with G acting regularly on $\mathcal{O} \setminus \{P\}$, the image of R under $G \cap W'$ is precisely $(\mathcal{O} \cap \mathcal{W}) \setminus \{P\}$. This means that G does not contain any elements of $W'\phi$ (since such elements also fix $\mathcal{W}$).

(b) This follows from the fact that E has index 2 in W and $G \not\leq E$.

(c) Let ρ be the projection $G_L \rightarrow GF(q^2)$, $[a, b, c] \mapsto a$, then ρ has image Δ and kernel G_0 . Thus $|\Delta| \times |G_0| = q^3/2$.

Observe that any element of the form $[0, b, 0]$ or $[0, b, 0]\phi$ (with $b \in GF(q^2)$) maps R to a point $(0, b, 1, 0)$ that lies in a generator of $H(3, q^2)$ containing R . Such a point can only lie in $\mathcal{O}$ if $b = 0$, and we have already seen that $\phi \notin G$. Thus the only element in G of this form is the identity $[0, 0, 0]$.

If $[a, b, c], [a, \hat{b}, c] \in G$, then G contains $[a, b, c]^{-1}[a, \hat{b}, c] = [0, b + \hat{b}, 0]$, so $b = \hat{b}$. Similar considerations for $[a, b, c], [a, \hat{b}, c]\phi \in G$ and $[a, b, c]\phi, [a, \hat{b}, c]\phi \in G$ lead to the conclusion that for each $(a, c) \in GF(q^2) \times GF(q)$, there is exactly one $b \in GF(q^2)$ such that G contains either $[a, b, c]$ or $[a, b, c]\phi$ (and G cannot contain both).

Suppose G contains an element $[0, b, c]\phi$ ($b \in GF(q^2), c \in GF(q)$). Then the square of this element is $[0, b + \bar{b}, 0]$ and it follows that $b \in GF(q)$. But then $[0, b, c]\phi \in W'\phi$, a contradiction to (a) above. Thus for each $c \in GF(q)$, there is at exactly one element $[0, b, c] \in G_L$, so $|G_0| = q$ and $|\Delta| = |\Gamma| = q^2/2$. It follows that $|G_a| = q$ for each $a \in \Delta$.

(d) We need to show that $G_0 \leq W'$. Suppose $G_0 \not\leq W'$: we obtain a contradiction as follows. Observe that G_0 contains some $[0, b, c]$ with $b \notin GF(q)$. If $u = [0, b, c], v = [a_1, b_1, c_1]$, then $u^{-1}v^{-1}uv = [0, 0, \bar{a}_1b + a_1\bar{b}]$, so G_0 contains $[0, 0, a_1(b + \bar{b})] \in K$ for every $a_1 \in GF(q) \cap \Delta$, and so $|G_0 \cap W'| \geq |G_0 \cap K| \geq q/2$. If $GF(q) \leq \Delta$, this argument shows that $|G_0 \cap W'| = q$. If $GF(q) \not\leq \Delta$, then G contains an element $v = [a_2, b_2, c]\phi$ with $a_2 \in GF(q)$, and $b_2 \notin GF(q)$ by (a) above. Then $v^2 = [0, b_2 + \bar{b}_2, a_2(b_2 + \bar{b}_2)]$ lies in $G_0 \cap W'$ but not in K . It follows that $|G_0 \cap W'| = q$ in this case also.

□

Theorem 4.6. *If q is even, then every translation ovoid of $H(3, q^2)$ is linear.*

Proof. Suppose that there exists a semilinear translation ovoid. By Lemmas 4.4 and 4.5 we have $G \cap W' = G_0 \cap W'$. However Γ must contain at least one $a \notin GF(q)$ (for the case $q = 2$, note that $0 \in \Delta$), with $[a, b, c]\phi \in G$ for some $b \in GF(q^2), c \in GF(q)$, and then $G \cap W'$ contains the square $[a + \bar{a}, b + \bar{b}, ab + \bar{a}\bar{b}] \notin G_0$. We have a contradiction. □

By Lemma 2.3 and Theorems 4.1 and 4.6, we have:

Corollary 4.7. *Every translation ovoid of $H(3, q^2)$ is locally hermitian.*

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