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A high-fidelity first-order reliability analysis for shear deformable laminated composite plates

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Abstract

This paper presents the complete theoretical development and practical implementation of a first-order reliability analysis for shear deformable laminated composite plates. The choice of plate theory is initially presented as it is important to ensure that a formulation capable of representing realistic physics is used as the basis of the overall simulation tool to reduce epistemic uncertainty as far as possible. The first-order reliability method (FORM) is proposed for the reliability analysis and summarised in the paper, with comparisons made throughout the paper with Monte Carlo simulation. The sensitivities required as part of the FORM algorithm are presented and verified against finite difference approximations. The practical implementation of the computational framework is demonstrated by numerical examples in which the probability failure of plates is calculated with performance criteria based on deflection and stress and uncertainties associated with fibre orientation and ply thickness.

Keywords: Composite plates, reliability analysis, probabilistic analysis, FORM, Monte Carlo simulation

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1. Introduction

The growth in application of composite materials reflects the increasing importance of being able to design material properties consistent with mechanics performance metrics. Simulation is a key component of the engineering design process, at both material and component levels, and normally requires experimental validation. It is in this final stage that many deterministic studies have failed to simulate the mechanical behaviour of composite materials and components, with considerable observed differences between theoretical predictions and experimental measurements [1]. Variations in fibre volume fractions, matrix-fibre voids, damage, fibre misalignment, residual stresses, etc. (e.g. [2]-[5]) introduce uncertainty at a local level that then propagates to a larger scale and is reflected in the variability of stiffness and strength descriptors characterising material or component scale structural performance. In addition to the aleatoric uncertainty representing natural or intrinsic variability, epistemic uncertainty describes knowledge or information that is missing because, for example, quantities may not have been measured or may have been measured with insufficient accuracy, loading and boundary conditions have been inadequately represented, the numerical representation contains assumptions that results in certain phenomena being omitted or misrepresented, or the analysis method at both simulation and reliability levels are inappropriate or inadequate [6],[7]. The significance of these uncertainties is reflected in the use of high safety factors in deterministic structural analysis, and is particularly manifested as engineering conservatism in the presence of modelling or simulation uncertainty.

The complexity of laminated composites is partly reflected in the different approaches that are available to study these materials and structures. Single layer and discrete layer theories have been proposed in which the laminated structure is treated respectively as either a type a homogenised whole or a combination of individual layers (layer-wise). Plate theories are similarly divided into stress-based and displacement-based theories. Shear deformation may also contribute significantly to the behaviour of a composite plate or shell. Shear deformation theories are often considered to be those that are represented in an equivalent single layer formulation, whereas the theories that are "layer-wise" are not normally included in this category, even though shear effects are considered in these models. Transverse shear stress components are absent in classical laminated plate theory which may lead to errors for thick plates, especially where the transverse shear stiffness is low, as often found for advanced composites, making the inclusion of shear deformation a normal prerequisite for the analysis of composite plate and shell structures. We consider an equivalent single layer formulation in this paper (the sine approach of Touratier) in contrast to a layer-wise approach.

The reliability analysis of composites is challenging because it combines uncertainty quantification with the numerical estimation of behaviour and performance criteria that are themselves complex. Fundamental reliability analysis techniques have been developed and applied to a number of fields. The need for a framework into which to set this work has been identified [7]. A required component of this framework will be the identification of appropriate solid mechanics formulations that inherently minimise the epistemic uncertainty associated with the simulation. Recent research demonstrates the combination of reliability analysis with classical lamination theory [9], Mindlin theory [8], and higher-order (cubic) shear deformation theory [10]. In this paper we present a high-fidelity first-order reliability analysis for laminated composite plates.

2. Plate formulation

2.1. Selection

It is well known that with a ratio of elastic modulus to shear modulus of the order of 25-40, compared with a value of 2.6 for a typical isotropic material, classical Kirchhoff theory (CLT) is unable to simulate the behaviour of an advanced composite plate. First-order shear deformation plate theory (FSDT) assumes transverse shear strains that are, along with the through-thickness shear stresses, constant through the thickness, contradicting the physical behaviour. Whilst the shear stresses cannot be corrected within the limitations of the first-order shear deformation plate theory, corresponding shear forces may be modified by a shear correction factor. Therefore, research is always active on the development of new theoretical models for heterogeneous structures. In this context, two families can be identified: the Equivalent Single Layer Models (ESLM), where CLT, FSDT or high-order theories can be found and the Layer-Wise Models (LWM). According to [11], the number of unknowns remains independent of the number of constitutive layers in the ESLM, while the same set of variables is used in each layer for the LWM. Alternatively, new models may be formulated by introducing interface conditions into high-order ESLM or LWM models. This enables the number of unknowns to be reduced and can be viewed as a ZigZag model. Excellent reviews have been made in the following articles [12]-[17] or

in the more recent review [18].

Third-order shear deformation theories assume a quadratic shear stress distribution through the plate thickness (Fig.1(c)). Since the shear stresses vanish at the upper and lower surfaces of the plate the shear correction factor is no longer needed.

Figure 1: Undeformed and deformed geometry of an edge plate in various plate theories. (a) Undeformed plate, (b) Kirchhoff theory, (c) Mindlin theory, (d) Higher order theory

In this work, an efficient theory is needed but LWM is computationally too expensive. A ZigZag model can be viewed as a good compromise between cost and accuracy, as continuity at the layer interfaces and top and bottom free conditions are fulfilled while the number of unknowns remain independent of the number of layers. Touratier [19] proposed kinematics (referred to as the sine model) which assumed a shear strain distribution through the thickness of the plate in the form of a cosine function as;

$$\begin{aligned}
 u(x, y, z) &= u_0(x, y) - z \frac{\partial w_0(x, y)}{\partial x} + f(z) \gamma_x^0(x, y) \\
 v(x, y, z) &= v_0(x, y) - z \frac{\partial w_0(x, y)}{\partial y} + f(z) \gamma_y^0(x, y) \\
 w(x, y, z) &= w_0(x, y)
 \end{aligned} \tag{1}$$

where $f(z) = \frac{h}{\pi} \sin\left(\frac{\pi z}{h}\right)$, $\gamma_x^0(x, y) = \theta_x + \frac{\partial w}{\partial x}$, and $\gamma_y^0(x, y) = \theta_y + \frac{\partial w}{\partial y}$. With the existence of interlaminar stresses at geometric boundaries such as free-edges, cut-outs, notches, and holes of structural components made of composite laminates are important phenomena [20], Beakou & Touratier [21] developed displacement fields with the inclusion of interlayer continuity, such that;

$$\begin{aligned}
 u^{(k)}(x, y, z) &= u_0(x, y) - z \frac{\partial w(x, y)}{\partial x} + \left(f_1(z) + g_1^{(k)}(z)\right) \left(\frac{\partial w}{\partial x} + \theta_1\right) + g_2^{(k)}(z) \left(\frac{\partial w}{\partial y} - \theta_1\right) \\
 v^{(k)}(x, y, z) &= v_0(x, y) - z \frac{\partial w(x, y)}{\partial y} + g_3^{(k)}(z) \left(\frac{\partial w}{\partial x} + \theta_2\right) + \left(f_2(z) + g_4^{(k)}(z)\right) \left(\frac{\partial w}{\partial y} - \theta_1\right)
 \end{aligned}$$

$$w(x, y, z) = w_0(x, y) \quad (2)$$

where f_1 , f_2 , and $g_1^{(k)} - g_4^{(k)}$ are:

$$\begin{aligned} f_1(z) &= f(z) - \frac{e}{\pi} b_{55} f'(z) \\ f_2(z) &= f(z) - \frac{e}{\pi} b_{44} f'(z) \\ g_i^{(k)}(z) &= a_i^{(k)} z + d_i^{(k)} \end{aligned} \quad (3)$$

(k) is the layer number, h the thickness of the plate, and coefficients b_{44} , b_{55} , $a_i^{(k)}$, $d_i^{(k)}$ are determined from the boundary conditions on the top and bottom surfaces and from the continuity requirements at the layer interfaces for displacements and stresses.

For clarity, dropping the superscript (k), Eq(3) may be interpreted and used to represent existing general types of displacement field. For example, assuming:

1. $f_1(z) = f_2(z) = f(z)$, $g_i^{(k)}(z) = 0$, the Touratier sine model without interlayer continuity is obtained,
2. $f_1(z) = f_2(z) = z$, $g_i^{(k)}(z) = 0$, produces the Reissner-Mindlin model with shear correction factor,
3. $f_1(z) = f_2(z) = 0$, $g_i^{(k)}(z) = 0$, is the Kirchhoff-Love model,
4. $f_1(z) = f_2(z) = f(z) = z \left(1 - \frac{4z^2}{3e^2}\right)$, $g_i^{(k)}(z) = 0$, describes other higher order models.

The Touratier sine model kinematic, using five unknown functions as FSDT, has been selected for the research presented in this paper. It produces results that represent improvements over polynomial type models for thin laminated plate analysis. Polit & Touratier [23] presented a finite element (FE) implementation of this theory for semi-thick to thin composite plate modeling. It exhibited high convergence rates and accuracies for both displacements and stresses. It was shown [23] that the sine model produced an excellent approximation to the analytical analysis of Pagano [24] compared to other theories with the same complexity. This maybe explained by the fact that the series of sine function is very much richer than a polynomial function, because the sine function can be formed by an infinite series as:

$$f(z) = \frac{h}{z} \sin \frac{\pi z}{h} = \frac{h}{z} \left(\sum_{p=0}^{\infty} (-1)^p \left(\frac{\pi z}{h} \right)^{2p+1} / (2p+1)! \right) \quad (4)$$

It has been implemented using a discretisation comprising six node triangles based on the approximations of Argyris [25] and Ganev [26]. Argyris's FE is a fifth order polynomial interpolation function used for the deflection, while a fourth order polynomial interpolation is used with additional rotation and membrane displacement unknowns. Furthermore, the combined Argyris-Ganev FE displays no spurious energy modes or shear locking [23].

2.2. Stiffness description: linear analysis

The elementary linear elastic stiffness matrix is obtained from the bilinear form, and we have for an elementary bi-dimensional domain Ω_e (see [23]):

$$a(\vec{u}^h, \vec{u}^{*h})_{\Omega_e} = \int_{\Omega_e} \int_{-e/2}^{e/2} [\varepsilon_e^{*h}]^T [\bar{C}^{(k)}] [\varepsilon_e^h] dz d\Omega_e \quad (5)$$

where $[\varepsilon_e^h]$ and $[\varepsilon_e^{*h}]$ are the strain and virtual strain rate vector respectively, while $[\bar{C}^{(k)}]$ is the matrix associated with the bi-dimensional constitutive law of the k^{th} layer.

Eq.(5) can be rewrite as

$$a(\vec{u}^h, \vec{u}^{*h})_{\Omega_e} = \int_{\Omega_e} [E_e^{*h}]^T \left(\int_{-e/2}^{e/2} [B_e]^T [\bar{C}^{(k)}] [B_e] dz \right) [E_e^h] d\Omega_e \quad (6)$$

where,

$$[\varepsilon_e^h] = [B_e] [E_e^h] \quad (7)$$

$[E_e^h]$ is constant over the depth and can be viewed as the generalised strain vector. The inner integral is the material behaviour matrix for a multilayered finite element. Simplifying the notation of Eq.(6) then:

$$a(\vec{u}^h, \vec{u}^{*h})_{\Omega_e} = \int_{\Omega_e} [E_e^{*h}]^T [A_e] [E_e^h] d\Omega_e \quad (8)$$

in which,

$$[A_e] = \int_{-e/2}^{e/2} [B_e]^T [\bar{C}^{(k)}] [B_e] dz \quad (9)$$

$[B_e]$ can be computed using the displacement field of Eq.(2) differentiated with respect to the generalised displacement vector $[E_e^h]$.

The generalised strain vector is,

$$[E_e^h]^T = \begin{bmatrix} u_0^h,1 & u_0^h,2 & \vdots & v_0^h,1 & v_0^h,2 & \vdots & w_0^h,1 & w_0^h,2 & w_0^h,11 \\ w_0^h,12 & w_0^h,22 & \vdots & \theta_1^h & \theta_1^h,1 & \theta_1^h,2 & \vdots & \theta_2^h & \theta_2^h,1 & \theta_2^h,2 \end{bmatrix} \quad (10)$$

where the strain-displacement matrix $[B_e]$ has the components summarised in Eq.(106) in the Appendix, in which $f(z)$, $f_i(z)$ and $g_i^{(k)}$ are defined in Eq.(3).

In order to develop a conformed, efficient and accurate six-node triangular finite element, and also to avoid transverse shear locking, an Argyris interpolation [25] is chosen for out-of-plane deflections and the Ganey interpolation for the other in-plane generalised displacements and rotations. Since the Argyris interpolation is a polynomial of the fifth-order and the Ganey interpolation of the fourth-order, transverse shear locking is avoided as the field compatibility is automatically assured for the transverse shear strains. Note that the Argyris interpolation is precisely of continuity C^1 , and Ganey interpolation involve semi- C^1 continuity which is not required here but assures the field compatibility for the finite element approximation of the transverse shear strains.

The discrete form of $[E_e^h]$ and $[E_e^{*h}]$ can be written as,

$$[E_e^h] = [\Lambda T D_e] [Q_e] \quad [E_e^{*h}] = [\Lambda T D_e] [Q_e^*] \quad \text{with } [\Lambda T D_e] = [\Lambda k] [T k_e] [D_e] \quad (11)$$

where $[\Lambda k]$ is associated with the monomial terms of the barycentric coordinates, $[T k_e]$ is a matrix containing geometric constants and $[D_e]$ is a geometric transformation matrix between local and global degrees of freedom (dof). Finally, $[Q_e]$ is the dof vector in global coordinates. This vector contains the dof of the three corner nodes and of the three mid-side nodes, obtained from Fig 2 and 3, as

- for a corner node:

$$\begin{bmatrix} u_0 & u_{0,1} & u_{0,2} & v_0 & v_{0,1} & v_{0,2} \\ w_0 & w_{0,1} & w_{0,2} & w_{0,11} & w_{0,22} & w_{0,12} \\ \theta_1 & \theta_{1,1} & \theta_{1,2} & \theta_2 & \theta_{2,1} & \theta_{2,2} \end{bmatrix} \quad (12)$$

- for a mid-side node:

$$\begin{bmatrix} u_0 & u_{0,n} & v_0 & v_{0,n} \\ w_{0,n} \\ \theta_1 & \theta_{1,n} & \theta_2 & \theta_{2,n} \end{bmatrix} \quad (13)$$

where p_n is the derivative with respect to the normal direction of the edge.

Finally, the elementary stiffness matrix can be written as :

$$[K_e] = \int_{\Omega_e} [\Lambda T D_e]^T [A_e] [\Lambda T D_e] d\Omega_e \quad (14)$$

2.3. Stiffness description: geometrically non-linear analysis

Extending the formulation to non-linear plate analyses involving moderately large deformations, small rotations, then the von Karman strain-displacement relationships may be adopted as in [23] and [28],

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ 0 \\ \left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial w}{\partial y} \right) \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

Figure 2: The set Σ_K of local dof of a function p for an Argyris triangle

Figure 3: The set Σ_K of local dof of a function p for a Ganey triangle

which, following the previous compact form can be written as,

$$[\varepsilon_e^h] = [\varepsilon_{Le}^h] + [\varepsilon_{NLe}^h], \quad (16)$$

where $[\varepsilon_{Le}^h]$ and $[\varepsilon_{NLe}^h]$ are the linear and non-linear parts, respectively, with,

$$\begin{aligned} [\varepsilon_{Le}^h]^T &= [u^h_{,1} \quad v^h_{,2} \quad u^h_{,2} + v^h_{,1} \quad u^h_{,3} + w^h_{0,1} \quad v^h_{,3} + w^h_{0,2}] \\ [\varepsilon_{NLe}^h]^T &= [\frac{1}{2}(w^h_{0,1})^2 \quad \frac{1}{2}(w^h_{0,2})^2 \quad w^h_{0,1}w^h_{0,2} \quad 0 \quad 0]. \end{aligned} \quad (17)$$

Using standard linearization procedures for a displacement $\vec{u}^h = \vec{\bar{u}}^h + \Delta \vec{u}^h$ where $\vec{\bar{u}}^h$ refers to a known state, the corresponding tangent stiffness matrix for non-linear analyses can be written in the form,

$$[K_{Te}] = [K_e] + [K_{NLe}] \quad (18)$$

where $[K_{NLe}]$ is the non-linear term of stiffness matrix, contain bending and membrane terms, such that:

$$[K_{NLe}] = [K_e(\vec{v}^h)] + [K_e(\vec{\sigma}^h)] \quad (19)$$

For an elementary bi-dimensional domain Ω_e in the reference (fixed) configuration denoted (0), the matrices $[K_e]$, $[K_e(\vec{v}^h)]$ and $[K_e(\vec{\sigma}^h)]$ are obtained respectively from:

$$\begin{aligned} &\int_{\Omega_e(0)} [E_e^{*h}]^T [A_e] [\Delta E_e^h] d\Omega_e(0) \\ &\int_{\Omega_e(0)} [E_e^{*h}]^T [A_e(\vec{v}^h)] [\Delta E_e^h] d\Omega_e(0) \\ &\int_{\Omega_e(0)} [E_e^{*h}]^T [A_e(\vec{\sigma}^h)] [\Delta E_e^h] d\Omega_e(0) \end{aligned} \quad (20)$$

$[K_e]$ is the linear term of the stiffness matrix defined in Eq.(14). The non-linear terms $[K_e(\bar{v}^h)]$ and $[K_e(\bar{\sigma}^h)]$ are computed using Eq.(5) based on either $[A_e(\bar{u}^h)]$ or $[A_e(\bar{\sigma}^h)]$. $[A_e(\bar{u}^h)]$ is defined using only the nonlinear parts of the general strain given in Eq.(15). Using Eq.(9) $[A_e(\bar{v}^h)]$ is defined as:

$$[A_e(\bar{v}^h)] = \int_{-e/2}^{e/2} [B_{NLe}]^T [\bar{C}^{(k)}] [B_{NLe}] dz, \quad (21)$$

or,

$$[A_e(\bar{v}^h)] = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [a1] & [0] & [0] \\ & [0] & [a2] & [0] & [0] \\ & & [a3] & [a4] & [a5] \\ & Sym & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz \quad (22)$$

where the non-zero terms of the sub-matrices are given in Eqs.(107 - 109) in the Appendix.

Finally $[A_e(\bar{\sigma}^h)]$ is defined as:

$$[A_e(\bar{\sigma}^h)] = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [0] & [0] & [0] \\ & [0] & [0] & [0] & [0] \\ & & [a6] & [0] & [0] \\ & Sym. & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz, \quad (23)$$

with,

$$[a6] = \begin{bmatrix} \bar{\sigma}_{11}^{(k)} & \bar{\sigma}_{12}^{(k)} & 0 & 0 & 0 \\ & \bar{\sigma}_{22}^{(k)} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & Sym. & & 0 & 0 \\ & & & & 0 \end{bmatrix}.$$

3. Reliability analysis

3.1. Principles

Reliability analysis evaluates the probability of structural failure by determining whether the limit-state functions are exceeded. The structure will be considered unreliable if the failure probability of the structure limit-state exceeds the required value. For most structures, ultimate (related to collapse of part or all of the structure) and serviceability (related to disruption of the normal use - e.g. deflection, vibration, etc.) limit states are defined.

Generally, the limit-state indicates the margin of safety between the resistance of the structure and the load effect. The limit-state function, $g(X)$, and probability of failure, P_f , can be defined as:

$$g(X) = R(X) - S(X) \quad (24)$$

$$P_f = P[g(X) < 0] \quad (25)$$

where R is the resistance and S is the loading of the system. Both $R(X)$ and $S(X)$ are functions of random variables X . The notation $g(X) < 0$ denotes the failure region. Likewise, $g(X) = 0$ and $g(X) > 0$ indicate the failure surface and safe region, respectively.

The mean and standard deviation of the limit-state, $g(X)$, can be determined from the elementary definition of the mean and variance. The mean of $g(X)$ is

$$\mu_g = \mu_R - \mu_S \quad (26)$$

where μ_R and μ_S are the means of R and S , respectively. And the standard deviation of $g(X)$ is

$$\sigma_g = \sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S} \quad (27)$$

where, ρ_{RS} is the correlation coefficient between R and S , and σ_R and σ_S are the standard deviations of R and S , respectively.

The safety index or reliability index, β , is defined as

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 - 2\rho_{RS}\sigma_R\sigma_S}} \quad (28)$$

If the resistance and the loading are uncorrelated ($\rho_{RS} = 0$), the safety index becomes

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \quad (29)$$

Figure 4: Probability Density of Limit state $g(X)$

The safety index indicates the distance of the mean margin of safety from $g(X) = 0$. Figure 4 shows a geometrical illustration of the safety index in a one-dimensional case. The safety index is the distance from location measure μ_g to the limit-state surface and provides a measure of reliability. The distance is measured in units of the uncertainty scale parameter σ_g . The shaded area of Figure 4 identifies the probability of failure.

For the special case when the resistance, R , and loading, S , are assumed to be normally distributed and uncorrelated, the limit-state function is also normally distributed. Thus, the probability density function of the limit-state function in this case is;

$$f_g(g) = \frac{1}{\sigma_g \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{g - \mu_g}{\sigma_g} \right)^2 \right] \quad (30)$$

The probability of failure is

$$P_f = \int_{-\infty}^0 f_g(g) dg, \quad (31)$$

with the corresponding probability of failure computed as;

$$\begin{aligned} P_f &= \int_{-\infty}^0 \frac{1}{\sigma_g \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{0 - \mu_g}{\sigma_g} \right)^2 \right] dg \\ &= \int_{-\infty}^0 \frac{1}{\sigma_g \sqrt{2\pi}} \exp \left(-\frac{1}{2} \beta^2 \right) dg \\ &= 1 - \Phi(\beta) = \Phi(-\beta), \end{aligned} \quad (32)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

For the multi-dimensional case, the generalization of Equation 31 becomes

$$P_f = P[g(X) \leq 0] = \int \cdots \int f_X(x_1, \cdots, x_n) dx_1 \cdots dx_n \quad (33)$$

where $g(X)$ is the n-dimensional limit-state function and $f_X(x_1, \cdots, x_n)$ is the joint probability density function of all relevant random variables X . General closed-form analytical solutions are not possible for the convolution integral in Eq.33, requiring the use of numerical methods as a substitute.

3.2. Simulation methods

Monte Carlo Simulation (MCS) is a simple random sampling method or statistical trial method that make realisations based on randomly generated sampling sets for uncertain variables. It is a powerful numerical tool for determining the approximate probability of a specific event that is the outcome of a series of stochastic processes as in;

$$P_f \approx \frac{N_f}{N} \quad (34)$$

where N_f is the number of trials for which $g(X)$ is violated out of the N experiments conducted.

The accuracy of the estimate will depend on the number of realisations. For small failure probability and/or small N , the estimate of P_f given by Eq.34 may be subject to considerable error. The estimate of the probability of failure approaches the true value as N approach infinity. The accuracy of Eq.34 can be studied by estimating the 95% confidence interval of estimated probability of failure, such that;

$$P \left[-2\sqrt{\frac{(1 - P_f^T)P_f^T}{N}} < \frac{N_f}{N} - P_f^T < 2\sqrt{\frac{(1 - P_f^T)P_f^T}{N}} \right] = 0.95 \quad (35)$$

where, P_f^T is the true probability of failure. The percentage error can be defined as

$$\epsilon\% = \frac{N_f/N - P_f^T}{P_f^T} \times 100\% \quad (36)$$

Combining Eq.35 and 36, we obtain

$$\epsilon\% = \sqrt{\frac{1 - P_f^T}{N \times P_f^T}} \times 200\% \quad (37)$$

Eq.37 indicates that there will be about a 20% error if P_f^T is 0.001 for 10,000 simulations. It can also stated that there is 95% probability that the probability of failure will be in the range of 0.01 ± 0.002 with 10,000 simulations. Conversely, if desired error is 10% is 0.01, then from Eq.37, the required number of simulations $N = 39,600$.

The number of simulations to achieve a certain level of accuracy depends on the unknown probability of failure. In many engineering problems (e.g. buildings in accordance with Eurocode 0), the probability of failure could be of the order of 10^{-6} . Therefore, on average, only 3.4 of 10^6 simulations would indicate failure. Thus at least 3×10^5 simulations would be required to predict this behaviour. For a reliability estimate, at least 10 times this minimum is usually recommended. *A priori* knowledge of the expected P_f is required to estimate the number of required samples or simulations. A good random number generator is also required to avoid sampling non-uniqueness. Schemes such as importance sampling can be adopted to reduce the number of simulations, but require an assumed *importance function*, and convergence to the true estimate of the P_f is not guaranteed.

The First-order Second Moment Method (FOSM) simplifies the functional relationship involving the uncertain (or *basic*) variables and alleviates the difficulty of requiring *a priori* knowledge of the P_f to gauge the required number of simulations. In its basic form (denoted MVFOSM) the limit state function is written as a linear Taylor-series expansion about the mean as;

$$\tilde{g}(X) \approx g(\mu_X) + \nabla g(\mu_X)^T (X_i - \mu_{X_i}) \quad (38)$$

where $\mu_X = \mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n}$, and $\nabla g(\mu_X)$ is the gradient of g evaluated at μ_X , as in,

$$\nabla g(\mu_X) = \left\{ \frac{\partial g(\mu_X)}{\partial x_1}, \frac{\partial g(\mu_X)}{\partial x_2}, \dots, \frac{\partial g(\mu_X)}{\partial x_n} \right\}^T. \quad (39)$$

The mean value of the approximate limit-state function $\tilde{g}(X)$ is,

$$\mu_{\tilde{g}} \approx E[g(\mu_X)] = g(\mu_X) \quad (40)$$

Since,

$$\text{Var}[g(\mu_X)] = 0, \text{Var}[\nabla g(\mu_X)] = 0$$

$$\text{Var} [\nabla g(\mu_X)^T (X - \mu_X)] = \text{Var} [\nabla g(\mu_X)^T]^2 \text{Var}(X), \quad (41)$$

then the variance of the approximate limit state function is,

$$\text{Var} [\tilde{g}(X)] \approx \text{Var} [\nabla g(\mu_X)^T (X - \mu_X)], \quad (42)$$

such that the standard deviation is,

$$\sigma_{\tilde{g}} = \sqrt{\text{Var} [\tilde{g}(X)]} = \sqrt{[\nabla g(\mu_X)^T]^2 \text{Var}(X)} = \left[\sum_{i=1}^n \left(\frac{\partial g(\mu_X)}{\partial x_i} \right)^2 \sigma_{x_i}^2 \right]^{\frac{1}{2}} \quad (43)$$

The reliability index β is then computed as;

$$\beta = \frac{\mu_{\tilde{g}}}{\sigma_{\tilde{g}}} \quad (44)$$

If the limit state function is linear Eq.44 reverts to Eq.29. For a non-linear limit state function a linearised approximation about the mean values (MV) may be adopted leading to the the MVFOSM reliability index. In the general case with independent variables in n -dimensional space, the failure is surface is a hyper-plane that can be defined as a linear failure function;

$$\tilde{g}(X) = c_0 + \sum_{i=1}^n c_i x_i. \quad (45)$$

The reliability index defined in Eq.44 can still be used for the n -dimensional case, in which;

$$\begin{aligned} \mu_{\tilde{g}} &= c_0 + c_1 \mu_{x_1} + c_2 \mu_{x_2} + \cdots + c_n \mu_{x_n} \\ \sigma_{\tilde{g}} &= \sqrt{\sum_{i=1}^n c_i^2 \sigma_{x_i}^2} \end{aligned} \quad (46)$$

The MVFOSM directly establishes the relationship between the reliability index and the basic variable parameters (e.g. mean standard deviation). However, linearising the limit state function about the mean values of the basic variables can lead to significant errors for highly nonlinear limit state functions and large coefficients of variation, and a lack of mathematical invariance for not only nonlinear forms of $g(\cdot)$, but also for certain linear forms ([29]). These problems are circumvented by using the Most Probable failure Point (MPP) as the expansion point. Furthermore, to obtain a similar scale in the case of multiple variables, Hasofer & Lind [30] proposed a linear mapping of the basic variables into a set of normalised independent variable as in,

$$\mathbf{X}'_i = \frac{\mathbf{X}_i - \mu_{\mathbf{X}_i}}{\sigma_{\mathbf{X}_i}} \quad (i = 1, 2, \dots, n) \quad (47)$$

where $\mathbf{X}'_i$ is a random variable with zero mean and unit standard deviation. Equation 47 is used to transform the original limit state $g(\mathbf{X}) = 0$ to the reduced limit state, $g(\mathbf{X}') = 0$. In this reduced space the reliability index, β_{H-L} is defined as minimum distance from the origin to the design point on the limit state in the reduced coordinates, $\mathbf{X}'^*$.

For a non-linear limit state, the computation of the minimum distance becomes an optimisation problem:

$$\begin{aligned} &\text{Minimise} && D = \sqrt{\mathbf{X}'^t \mathbf{X}'} \\ &\text{Subject to the constraint} && g(\mathbf{X}) = g(\mathbf{X}') = 0 \end{aligned} \quad (48)$$

where $\mathbf{X}'$ represents the coordinates of the checking point on the limit state equation in the reduced coordinates to be estimated.

Using the method of Lagrange multipliers, we can obtain the minimum distance as,

$$\beta_{H-L} = - \frac{\sum_{i=1}^n x'_i{}^* \left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^*}{\sqrt{\sum_{i=1}^n \left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^{2*}}} \quad (49)$$

Figure 5: Hasofer-Lind reliability index, non-linear performance function

where $(\partial g / \partial \mathbf{X}'_i)^*$ is the i th partial derivative evaluated at the design point with coordinates $(x'^*_1, x'^*_2, \dots, x'^*_n)$. The asterisk after the derivative indicates that it is evaluated at $(x'^*_1, x'^*_2, \dots, x'^*_n)$. The design point in the reduced coordinates is given by

$$x'^*_i = -\alpha_i \beta_{H-L} \quad (i = 1, 2, \dots, n) \quad (50)$$

where

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^*}{\sqrt{\sum_{i=1}^n \left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^{2*}}} \quad (51)$$

are the direction cosines along the coordinate axes $\mathbf{X}'_i$. In the space of the original coordinates and using Eq.47, we find the design point to be,

$$x_i^* = \mu_{\mathbf{X}_i} - \alpha_i \sigma_{\mathbf{X}_i} \beta_{H-L}. \quad (52)$$

For problems with non-closed form solutions and several design variables Eq.48 is solved using an iterative numerical scheme (illustrated in Figure 6), as in,

- Step 1 Define the limit state function, $g(\mathbf{X})$;
- Step 2 Assume the initial design point $x_i^*, i = 1, 2, \dots, n$, and compute the corresponding limit state function $g(\mathbf{X})$. In the absence of any other information, the initial design point can be assumed to be at the mean values of the random variables;
- Step 3 Compute the mean and standard deviation at the design point of the equivalent normal distribution for those variables that are non-normal. The coordinates of the design point in the equivalent standard normal space are,

$$x'^*_i = \frac{x_i^* - \mu_{\mathbf{X}_i}^N}{\sigma_{\mathbf{X}_i}^N}; \quad (53)$$

- Step 4 Compute partial derivatives $\partial g / \partial \mathbf{X}_i$ evaluated at the design point x_i^* ;
- Step 5 Compute the partial derivatives $\partial g / \partial \mathbf{X}'_i$ in the equivalent standard normal space as,

$$\frac{\partial g}{\partial \mathbf{X}'_i} = \frac{\partial g}{\partial \mathbf{X}_i} \frac{\partial \mathbf{X}_i}{\partial \mathbf{X}'_i} = \frac{\partial g}{\partial \mathbf{X}_i} \sigma_{\mathbf{X}_i}^N \quad (54)$$

Figure 6: Graphical representation of Hasofer-Lind algorithm in the reduced coordinate systems. (Numbers in parentheses indicate the iteration number.)

The partial derivatives $\partial g / \partial \mathbf{X}'_i$ are the components of the gradient vector of the performance function in the equivalent standard normal space. The components of the corresponding unit vector are the direction cosines of the performance function, computed as,

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^*}{\sqrt{\sum_{i=1}^N \left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^{2*}}} = \frac{\left(\frac{\partial g}{\partial \mathbf{X}'_i} \right)^* \sigma_{\mathbf{X}_i}^N}{\sqrt{\sum_{i=1}^N \left(\frac{\partial g}{\partial \mathbf{X}'_i} \sigma_{\mathbf{X}_i}^N \right)^{2*}}} \quad (55)$$

Step 6 Compute the new values for the design point in the equivalent standard normal space ($\mathbf{X}'_i^*$) using recursive formula

$$\mathbf{X}'_{k+1} = \frac{1}{|\nabla g(\mathbf{X}'_k^*)|^2} [\nabla g(\mathbf{X}'_k^*)^t \mathbf{X}'_k^* - g(\mathbf{X}'_k^*)] \nabla g(\mathbf{X}'_k^*) \quad (56)$$

where $\nabla g(\mathbf{X}'_k^*)$ is the gradient vector of the performance function at $\mathbf{X}'_k^*$, the k th iteration, with $\mathbf{X}'_k^*$ a vector with components $\{x'_{1k}, x'_{2k}, \dots, x'_{nk}\}^t$, for n random variables. Similarly for $\mathbf{X}'_{k+1}$;

Step 7 Compute the distance to the new design point from the origin as,

$$\beta = \sqrt{\sum_{i=1}^n (x'_i)^2}, \quad (57)$$

and the new values for the design point in the original space (x_i^*) as,

$$x_i^* = \mu_{\mathbf{X}_i}^N + \sigma_{\mathbf{X}_i}^N x'_i. \quad (58)$$

Step 8 Compute the value of the performance function $g(\mathbf{X})$ for this new design point, and check the convergence criterion for both $g(\mathbf{X})$ and β before returning to Step 3 as required.

This algorithm requires the evaluation of the limit state function $G(X') = G(Y)$ for uncorrelated reduced normal variables Y , and its corresponding gradient $\nabla G(Y)$. Within a linear elastic finite element context, for example,

$$\mathbf{KU} = \mathbf{F}. \quad (59)$$

The vector of response quantities S (e.g., stresses) are computed using a transformation of the form,

$$\mathbf{S} = \mathbf{Q}^t \mathbf{U} + \mathbf{S}_0 \quad (60)$$

where $\mathbf{Q}^t$ is a transformation matrix relating $\mathbf{U}$ and $\mathbf{S}$, and $\mathbf{S}_0$ is the initial response vector for $\mathbf{U} = \mathbf{0}$. The performance (or limit state) function is then,

$$g(\mathbf{X}) = g\{\mathbf{R}(\mathbf{X}), \mathbf{S}(\mathbf{X})\} \quad (61)$$

where $\mathbf{R}$ is the vector of resistance variables, $\mathbf{S}$ is the vector of response quantities occurring in the performance function, and $\mathbf{X}$ is the vector of original random variables. Transforming $\mathbf{X}$ into the equivalent uncorrelated reduced normal variables $\mathbf{Y}$, then for a single random variable, Y_1 , the corresponding component of $\nabla G(Y)$ is,

$$\frac{\partial G}{\partial Y_1} = \frac{\partial g}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial X_1} \frac{\partial X_1}{\partial Y_1} + \frac{\partial g}{\partial \mathbf{S}} \frac{\partial \mathbf{S}}{\partial X_1} \frac{\partial X_1}{\partial Y_1}. \quad (62)$$

3.3. Sensitivities

The partial derivatives identified in Eq.62 are required to solve Eq.48. They also provide information about the dependence of the limit state function and with Eq.57, the safety index, on the basic variables. In this paper all sensitivities are derived analytically without recourse to finite difference approximations. Displacement and stress sensitivities are derived from the fundamental finite element formulation,

$$[K_T(b)] [Q(b)] = [R(b)] \quad (63)$$

where $[K_T(b)]$ is the structural tangent stiffness matrix as a function of the basic (uncertain) design variable b such as ply thickness or ply orientation angle, for example. $[Q(b)]$ is the global displacement vector, and $[R(b)]$ is the load vector. At the same time we may consider a performance measure F that may represent the objective function f such that,

$$F(b) = F([Q(b)], b) \quad (64)$$

where both the implicit and explicit dependence of the performance measure F on the design variables b are identified. Assuming sufficient smoothness, the design sensitivity of F with respect to the design variables b_i , $i = 1, 2, \dots, N$ is calculated as

$$\frac{dF}{db_i} = \left[\frac{\partial F}{\partial Q} \right] \left[\frac{\partial Q}{\partial b_i} \right] + \frac{\partial F}{\partial b_i}. \quad (65)$$

The difficulty in evaluating Eq.(65) is that there is no expression for the implicit response sensitivity $[\partial Q / \partial b_i]$, which is implicitly defined through the finite element equations. A pseudo problem is therefore formed for each design parameter by differentiating Eq.(63) with respect to each b_i , $i = 1, 2, \dots, N$, such that

$$\left[\frac{\partial K_T}{\partial b_i} \right] [Q] + [K_T] \left[\frac{\partial Q}{\partial b_i} \right] = \left[\frac{\partial R}{\partial b_i} \right]. \quad (66)$$

Rearranging gives

$$[K_T] \left[\frac{\partial Q}{\partial b_i} \right] = \left[\frac{\partial R}{\partial b_i} \right] - \left[\frac{\partial K_T}{\partial b_i} \right] [Q], \quad (67)$$

which may be used to solve for implicit response sensitivity $[\partial Q / \partial b_i]$. Pre-multiplying Eq(64) by $[K_T]^{-1}$ gives the displacement sensitivity in the form of

$$\left[\frac{\partial Q}{\partial b_i} \right] = [K_T]^{-1} \left(\left[\frac{\partial R}{\partial b_i} \right] - \left[\frac{\partial K_T}{\partial b_i} \right] [Q] \right). \quad (68)$$

The most challenging task in Eq.(68) is to evaluate the derivatives of the stiffness matrix and the force vector with respect to design parameters. It may also be noted that writing Eq.(68) at element level, then

$$\left[\frac{\partial Q}{\partial b_i} \right] = [K_T]^{-1} \sum_{l \in E} \left(\left[\frac{\partial R^e}{\partial b_i} \right] - \left[\frac{\partial K_T^e}{\partial b_i} \right] [Q_l^e] \right). \quad (69)$$

Eq(69) implies that the sensitivity of the system is obtained as the summation of the sensitivities of all elements in the system, or

$$\frac{\partial F}{\partial b_i} = \sum_{l=1}^N \frac{\partial F^e}{\partial b_i}. \quad (70)$$

Clearly uncertainties exist in a range of input variables at both geometric and material constant levels. To demonstrate the derivation of the sensitivities required for reliability analyses relevant to composites, ply angle and ply thickness are assumed to be uncertain, with limit state functions written as displacement and stress performance measures.

3.3.1. Displacement sensitivities

Displacement sensitivities with respect to (w.r.t) ply angle may be evaluated by substituting ply angle θ_i for the dummy parameter b_i in Eq(66) such that,

$$\left[\frac{\partial Q}{\partial \theta_i} \right] = [K_{Te}]^{-1} \sum_{l \in E} \left(\left[\frac{\partial R_e}{\partial \theta_i} \right] - \left[\frac{\partial K_{Te}}{\partial \theta_i} \right] [Q_l^e] \right). \quad (71)$$

Assuming that loading is independent of the ply angle, such that,

$$\frac{\partial R_e}{\partial \theta_i} = 0, \quad (72)$$

then the only term to be considered is the derivative of the stiffness matrix with respect to ply angle. The linear part of the stiffness matrix derivative can be derived from Eq.(14) and evaluated at element level, giving,

$$\left[\frac{\partial K_e}{\partial \theta_i} \right] = \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e]}{\partial \theta_i} \right) [\Lambda T D_e] d\Omega. \quad (73)$$

Using Eq.(9) the derivative of $[A_e]$ is of the form,

$$\frac{\partial [A_e]}{\partial \theta_i} = \int_{-e/2}^{e/2} \left([B_e]^T [\bar{C}^{(k)}] \frac{\partial [B_e]}{\partial \theta_i} + \left([B_e]^T [\bar{C}^{(k)}] \frac{\partial [B_e]}{\partial \theta_i} \right)^T + [B_e]^T \frac{\partial [\bar{C}^{(k)}]}{\partial \theta_i} [B_e] \right) dz \quad (74)$$

where

$$\frac{\partial [\bar{C}^{(k)}]}{\partial \theta_i} = T^{(k)} C^{(k)} \frac{\partial T^{(k)T}}{\partial \theta_i} + \left(T^{(k)T} C^{(k)} \frac{\partial T^{(k)}}{\partial \theta_i} \right)^T + T^{(k)T} \frac{\partial C^{(k)}}{\partial \theta_i} T^{(k)} \quad (75)$$

in which $T^{(k)}$ is k th lamina ply angle transformation matrix. The derivative of the transformation matrix is given in the Appendix (Eq.(110)) and $\frac{\partial [B_e]}{\partial \theta_i} = 0$.

Substituting Eq(110) into Eq(73) then Eq(73) reduces to,

$$\frac{\partial [A_e]}{\partial \theta_i} = \int_{-e/2}^{e/2} \left([B_e]^T \frac{\partial [\bar{C}^{(k)}]}{\partial \theta_i} [B_e] \right) dz \quad (76)$$

The displacement sensitivity of the non-linear stiffness is treated in a similar manner. Taking the basic form from Eq.(69),

$$\left[\frac{\partial Q}{\partial \theta_i} \right] = [K_{NLe}]^{-1} \sum_{l \in E} \left(\left[\frac{\partial R_e}{\partial \theta_i} \right] - \left[\frac{\partial K_{NLe}}{\partial \theta_i} \right] [Q_l^e] \right) \quad (77)$$

where

$$[K_{NLe}] = \int_{\Omega_e} [\Lambda T D_e]^T [A_e(\vec{u}^h)] [\Lambda T D_e] d\Omega_e + \int_{\Omega_e} [\Lambda T D_e]^T [A_e(\vec{\sigma}^h)] [\Lambda T D_e] d\Omega_e \quad (78)$$

or,

$$[K_{NLe}] = [K_e(\vec{u}^h)] + [K_e(\vec{\sigma}^h)]. \quad (79)$$

Then,

$$\left[\frac{\partial K_{NLe}}{\partial \theta_i} \right] = \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e(\vec{u}^h)]}{\partial \theta_i} \right) [\Lambda T D_e] d\Omega + \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e(\vec{\sigma}^h)]}{\partial \theta_i} \right) [\Lambda T D_e] d\Omega \quad (80)$$

with, making use of Eq.(21),

$$\begin{aligned} \frac{\partial [A_e(\vec{u}^h)]}{\partial \theta_i} &= \int_{-e/2}^{e/2} \left([B_{NLe}]^T [\bar{C}^{(k)}] \frac{\partial [B_{NLe}]}{\partial \theta_i} + \left([B_{NLe}]^T [\bar{C}^{(k)}] \frac{\partial [B_{NLe}]}{\partial \theta_i} \right)^T \right. \\ &\quad \left. + [B_{NLe}]^T \frac{\partial [\bar{C}^{(k)}]}{\partial \theta_i} [B_{NLe}] \right) dz \end{aligned} \quad (81)$$

From Eq.(22) the elements of Eq.(81) are

$$\frac{\partial [A_e (\vec{u}^h)]}{\partial \theta_i} = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [a1'] & [0] & [0] \\ & [0] & [a2'] & [0] & [0] \\ & & [a3'] & [0] & [0] \\ & Sym & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz \quad (82)$$

where the non-zero terms of the sub-matrix are detailed in Eqs.(111) - (113) in the Appendix.

The second term of the non-linear stiffness derivative in Eq.(80) requires variation of $[A_e (\bar{\sigma}^h)]$ w.r.t ply angle. This is defined as,

$$\frac{\partial [A_e (\bar{\sigma}^h)]}{\partial \theta_i} = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [0] & [0] & [0] \\ & [0] & [0] & [0] & [0] \\ & & [a4'] & [0] & [0] \\ & Sym. & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz \quad (83)$$

where,

$$[a4'] = \begin{bmatrix} -2 \sin(2\theta^{(k)}) \bar{\sigma}_{11}^{(k)} & 2 \cos(2\theta^{(k)}) \bar{\sigma}_{12}^{(k)} & 0 & 0 & 0 \\ & -2 \sin(2\theta^{(k)}) \bar{\sigma}_{22}^{(k)} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & Sym. & & 0 & 0 \\ & & & & 0 \end{bmatrix}.$$

Displacement sensitivities w.r.t. ply thickness follow the same principles when assuming ply angle as the basis. Assuming self-weight is negligible such that $\frac{\partial R^e}{\partial h_k} = 0$, then Eq.(69) is written as,

$$\left[\frac{\partial Q}{\partial h_k} \right] = [K_T]^{-1} \sum_{l \in E} \left(- \left[\frac{\partial K_{Te}}{\partial h_k} \right] [Q_l^e] \right) \quad (84)$$

For the linear part of the stiffness matrix, we have

$$\left[\frac{\partial K_e}{\partial h_k} \right] = \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e]}{\partial h_k} \right) [\Lambda T D_e] d\Omega \quad (85)$$

and where, assuming the material compliance matrix is independent of ply thickness such that $\frac{\partial [\bar{C}^{(k)}]}{\partial h_k} = 0$,

$$\frac{\partial [A_e]}{\partial h_k} = \int_{-e/2}^{e/2} \left([B_e]^T [\bar{C}^{(k)}] \frac{\partial [B_e]}{\partial h_k} + \left([B_e]^T [\bar{C}^{(k)}] \frac{\partial [B_e]}{\partial h_k} \right)^T \right) dz. \quad (86)$$

The non-zero terms in $\partial [B_e] / \partial h_k$ are given in Eq.(114) of the Appendix, in which $h^{(k)}$ and $h^{(k+1)}$ is the thickness of the k th and $(k+1)$ th laminae, respectively, and function f along with f' and f'' are defined in Eq.(3).

Variation of the nonlinear stiffness matrix with respect to ply thickness follows Eq.(80) in the form,

$$\left[\frac{\partial K_{NL_e}}{\partial h_k} \right] = \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e (\vec{u}^h)]}{\partial h_k} \right) [\Lambda T D_e] d\Omega + \int_{\Omega_e} [\Lambda T D_e]^T \left(\sum_{k=1}^l \frac{\partial [A_e (\bar{\sigma}^h)]}{\partial h_k} \right) [\Lambda T D_e] d\Omega \quad (87)$$

where, from Eq.(22),

$$\frac{\partial [A_e (\vec{u}^h)]}{\partial h_k} = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [0] & [0] & [0] \\ & [0] & [0] & [0] & [0] \\ & & [a3''] & [0] & [0] \\ & Sym & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz \quad (88)$$

with non-zero terms of $a3''$ being,

$$\begin{aligned} a3(1,3) &= \bar{C}_{11}^{(k)} \bar{w}_{0,1} (h^{(k+1)} - h^{(k)}) , & a3(1,4) &= \bar{C}_{66}^{(k)} \bar{w}_{0,2} (h^{(k+1)} - h^{(k)}) , \\ a3(1,5) &= \bar{C}_{12}^{(k)} \bar{w}_{0,1} (h^{(k+1)} - h^{(k)}) , & a3(2,3) &= \bar{C}_{12}^{(k)} \bar{w}_{0,2} (h^{(k+1)} - h^{(k)}) , \\ a3(2,4) &= \bar{C}_{66}^{(k)} \bar{w}_{0,1}^2 (h^{(k+1)} - h^{(k)}) , & a3(2,5) &= \bar{C}_{22}^{(k)} \bar{w}_{0,2} (h^{(k+1)} - h^{(k)}) , \end{aligned}$$

and,

$$\frac{[A_e(\bar{\sigma}^h)]}{\partial h_k} = \int_{-e/2}^{e/2} \begin{bmatrix} [0] & [0] & [0] & [0] & [0] \\ & [0] & [0] & [0] & [0] \\ & & [a4''] & [0] & [0] \\ Sym. & & & [0] & [0] \\ & & & & [0] \end{bmatrix} dz \quad (89)$$

with,

$$[a4''] = \begin{bmatrix} \frac{\partial \bar{\sigma}_{11}^{(k)}}{\partial h_k} & \frac{\partial \bar{\sigma}_{12}^{(k)}}{\partial h_k} & 0 & 0 & 0 \\ & \frac{\partial \bar{\sigma}_{22}^{(k)}}{\partial h_k} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ Sym. & & 0 & 0 & \\ & & & 0 & \end{bmatrix}, \quad (90)$$

in which $\frac{\partial \bar{\sigma}_{\alpha\beta}^{(k)}}{\partial h_k}$ is obtained from the material compliance matrix, $\bar{C}_{ij}^{(k)}$, and the components of $\partial B_e / \partial h_k$ defined in Eq.(114).

3.3.2. Stress sensitivities

Laminated composite stress limit states are frequently of the form of the well known Tsai-Wu or Hoffman's criteria, respectively [11],

$$\begin{aligned} \left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\sigma_{33}}{Z}\right)^2 - \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right) \sigma_{11} \sigma_{22} - \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2}\right) \sigma_{22} \sigma_{33} \\ - \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}\right) \sigma_{11} \sigma_{33} + \left(\frac{\sigma_{12}}{S}\right)^2 + \left(\frac{\sigma_{13}}{R}\right)^2 + \left(\frac{\sigma_{23}}{T}\right)^2 \geq 1, \end{aligned} \quad (91)$$

or,

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{Y_T Y_C} + \frac{1}{Z_T Z_C} - \frac{1}{X_T X_C} \right) (\sigma_{22} - \sigma_{33})^2 + \frac{1}{2} \left(\frac{1}{Z_T Z_C} + \frac{1}{X_T X_C} - \frac{1}{Y_T Y_C} \right) (\sigma_{33} - \sigma_{11})^2 \\ + \frac{1}{2} \left(\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} - \frac{1}{Z_T Z_C} \right) (\sigma_{11} - \sigma_{22})^2 + \left(\frac{1}{X_T} - \frac{1}{X_C} \right) \sigma_{11} + \left(\frac{1}{Y_T} - \frac{1}{Y_C} \right) \sigma_{22} \\ + \left(\frac{1}{Z_T} - \frac{1}{Z_C} \right) \sigma_{33} + \left(\frac{\sigma_{12}}{S} \right)^2 + \left(\frac{\sigma_{13}}{R} \right)^2 + \left(\frac{\sigma_{23}}{T} \right)^2 \geq 1. \end{aligned} \quad (92)$$

To include material strength uncertainty defined as variability in the coefficients X_T, Y_T, Z_T and X_C, Y_C, Z_C it is clear from the preceding expressions that this is relatively trivial. In contrast, uncertainty in ply thickness and orientation requires sensitivities of stresses arising from the load effects to be calculated.

For example, stress sensitivities of Hoffman's failure criterion with respect to uncertain ply angle and thickness variables are obtained by applying the chain rule Eq.(65) at element level to define the adjoint force vector $F_j^{e^l}$ for an arbitrary ply of element l as in,

$$\begin{aligned} F_j^{e^l} = \frac{dF_j}{dQ_e} = & \left(-\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} + \frac{1}{Z_T Z_C} \right) (\sigma_2 - \sigma_3) \left(\frac{\partial \sigma_2}{\partial Q_e} - \frac{\partial \sigma_3}{\partial Q_e} \right) \\ & + \left(\frac{1}{X_T X_C} - \frac{1}{Y_T Y_C} + \frac{1}{Z_T Z_C} \right) (\sigma_3 - \sigma_1) \left(\frac{\partial \sigma_3}{\partial Q_e} - \frac{\partial \sigma_1}{\partial Q_e} \right) \\ & + \left(\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} - \frac{1}{Z_T Z_C} \right) (\sigma_1 - \sigma_2) \left(\frac{\partial \sigma_1}{\partial Q_e} - \frac{\partial \sigma_2}{\partial Q_e} \right) \\ & + \left(\frac{1}{X_T} - \frac{1}{X_C} \right) \frac{\partial \sigma_1}{\partial Q_e} - \left(\frac{1}{Y_T} - \frac{1}{Y_C} \right) \frac{\partial \sigma_2}{\partial Q_e} + \left(\frac{1}{Z_T} - \frac{1}{Z_C} \right) \frac{\partial \sigma_3}{\partial Q_e} \\ & + 2 \frac{\sigma_{23}}{T^2} \frac{\partial \sigma_{23}}{\partial Q_e} + 2 \frac{\sigma_{13}}{R^2} \frac{\partial \sigma_{13}}{\partial Q_e} + \frac{\sigma_{12}}{S^2} \frac{\partial \sigma_{12}}{\partial Q_e}. \end{aligned} \quad (93)$$

The derivative $\partial\sigma/\partial Q_e$ is evaluated analytically from the constitutive relation,

$$\begin{aligned} [\sigma^k] &= [\bar{C}_{ij}^{(k)}] [\varepsilon^{(k)}] \\ \text{with } \begin{cases} \bar{C}_{ij}^{(k)} = C_{ij}^{(k)} - C_{i3}^{(k)} C_{j3}^{(k)} / C_{33}^{(k)} & \text{for } i, j = 1, 2, 6 \\ \bar{C}_{ij}^{(k)} = C_{ij}^{(k)} & \text{for } i, j = 4, 5 \end{cases} \end{aligned} \quad (94)$$

The stress sensitivity $\partial F_j / \partial b_i$ is obtained following Eq.(93), replacing the term $\partial\sigma/\partial Q_e$ with $\partial\sigma/\partial b_i$.

The stress sensitivity w.r.t. ply angle is, therefore,

$$\left(\frac{\partial \sigma_\alpha}{\partial \theta_i} \right)^{(k)} = [\bar{C}_{ij}^{(k)}] \frac{\partial [\varepsilon^{(k)}]}{\partial \theta_i} + \frac{\partial [\bar{C}_{ij}^{(k)}]}{\partial \theta_i} [\varepsilon^{(k)}], \quad (95)$$

in which,

$$\frac{\partial [\bar{C}_{ij}^{(k)}]}{\partial \theta_i} = \left[\frac{\partial T^{(k)}}{\partial \theta_i} \right]^T \left(C_{ij}^{(k)} T^{(k)} \right) + \left(T^{(k)} \right)^T C_{ij}^{(k)} \frac{\partial T^{(k)}}{\partial \theta_i}. \quad (96)$$

and $\frac{\partial T^{(k)}}{\partial \theta_i}$ is defined in Eq.(115) of the Appendix.

Noting that B_e is independent of the ply angle, then,

$$\frac{\partial [\varepsilon^{(k)}]}{\partial \theta_i} = B_e \frac{\partial Q_e}{\partial \theta_i} \quad (97)$$

where $\partial Q_e / \partial \theta_i$ is given in Eq.(71). Exactly equivalent expressions are obtained for the non-linear stress terms by substituting $\varepsilon_{NL}^{(k)}$ for $\varepsilon^{(k)}$ in Eq.(95) such that,

$$\frac{\partial [\varepsilon_{NL}^{(k)}]}{\partial \theta_i} = B_{NL} \frac{\partial Q_{NL}}{\partial \theta_i}, \quad (98)$$

with $\frac{\partial Q_{NL}}{\partial \theta_i}$ defined in Eq.(77) - Eq.(83).

The linear component of the stress sensitivity w.r.t ply thickness is similarly derived from Eq.(94) as,

$$\left(\frac{\partial \sigma_\alpha}{\partial h_k} \right)^{(k)} = [\bar{C}_{ij}^{(k)}] \frac{\partial [\varepsilon^{(k)}]}{\partial h_k} + \frac{\partial [\bar{C}_{ij}^{(k)}]}{\partial h_k} [\varepsilon^{(k)}], \quad (99)$$

which, assuming the material compliance is independent of ply thickness such that $\frac{\partial [\bar{C}_{ij}^{(k)}]}{\partial h_k} = 0$, then Eq.(99) becomes,

$$\left(\frac{\partial \sigma_\alpha}{\partial h_k} \right)^{(k)} = [\bar{C}_{ij}^{(k)}] \frac{\partial [\varepsilon^{(k)}]}{\partial h_k}, \quad (100)$$

with,

$$\frac{\partial \varepsilon^{(k)}}{\partial h_k} = \frac{\partial B_e}{\partial h_k} E^h, \quad (101)$$

and the components of $\frac{\partial B_e}{\partial h_k}$ given in Eq.(114).

The non-linear stress sensitivities follow Eq.(100) with the exception that the components of $\frac{\partial B_e}{\partial h_k}$ are derived from Eq.(21) and are obtained as given in Eq.(116) of the Appendix.

3.3.3. Sensitivities verification

To demonstrate the validity of the analytical sensitivities, a simply supported square laminate plate under uniform load of intensity 1 MPa is analysed (Fig 7). Two thickness to span ratios are chosen to resemble a moderately thick plate (0.06) and a thin plate (0.015). One quarter of the plate is modelled, taking advantage of symmetry, discretised by 32 triangular elements. The ply material properties are $E_L = 25$ GPa, $E_T = 1$ GPa, $G_{12} = 0.5$ GPa, $G_{13} = 0.5$ GPa, $G_{23} = 0.25$ GPa, and $\nu_{12} = 0.25$. The examples here use the same material properties and equal thicknesses for each ply, whilst the proposed methodology is entirely general. Three and five ply lay-ups are assumed as $[0/90]_s$ and $[0/90/0]_s$.

Figure 7: The meshes used for sensitivity evaluations.

Deflection and stress sensitivities with respect to ply angle and ply thickness are calculated at the mid-plate position of each ply. Linear and non-linear sensitivities are provided to illustrate the affects of large deformations. Values repeated as a result of symmetry are not included, meaning that details for two and three layers are provided for the three and five ply plates, respectively. The results of the analytical sensitivities are compared with forward finite different (FFD) approximations with perturbation values of 0.01° in the ply angle and 0.1% of the ply thickness. The sensitivities are presented in Table 1 - Table 8.

The comparison between the analytical and FFD approximations validate the mathematical derivatives in all cases. The errors are sufficiently small to make generalisations difficult, particularly when comparing the linear sensitivities. The effect of ply angle variation on the magnitude of the central plate displacement $\partial U / \partial \theta$ is essentially linear for the three ply plate in both thin and thick formats (Table 1). The effect of geometric non-linearity is more pronounced as the number of plies are increased for a thin plate (Table 2). The disparity between the analytical sensitivities and FFD approximations are more pronounced, whilst acceptably small, when the effects of non-linearity are included as may be expected. Similar trends are exhibited in the variability of displacement with respect to ply thickness, $\partial U / \partial e$, though in this case the non-linearity of the thickness in the displacement leads to increased error estimates for the linear sensitivities of similar magnitudes to the non-linear measures, without a significant increase in the latter or impact on the validity of the analytical solutions (Tables 3 and 4). The same type of comparative patterns and magnitudes are demonstrated by the stress sensitivities $\partial \sigma / \partial \theta$ and $\partial \sigma / \partial e$ (Tables 5 - 8). The analytical sensitivity formulations are shown to work well for both the thick and thin plates, and for alternative laminations. The additional effort required to compute the non-linear analytical sensitivities is justified with differences in comparison with the equivalent linear terms of around 30% for some components.

3.4. Reliability analysis of laminated plates - examples

The preceding probabilistic finite element method is applied to symmetrical plates comprising two a ply laminate. The mean material properties of each ply are $E_1 = 181$ GPa, $E_2 = 10.34$ GPa, $E_{12} = 7.17$ GPa, $\nu_1 = 0.28$. The material strengths for use in the failure criterion are $X_t = 1500$ MPa, $X_c = 1500$ MPa, $Y_t = 750$ MPa, $Y_c = 750$ MPa, $S = 450$ MPa [31]. The plates are assumed to be simply supported and uniformly loaded with an intensity of 100kPa. The total thickness of each of the $1\text{m} \times 1\text{m}$ plates is 10mm. The mean fibre angle orientations are $45^\circ / -45^\circ$.

Example 1

The reliability problem for the plate is taken to be a simple displacement restriction. The limit state function

is defined as:

$$g() = U_3 - U_{target} \quad (102)$$

where U_3 is mid-plate transverse displacement. The target displacement (U_{target}) is selected to be 50% higher than its deterministic equivalent. This value is based on fibre orientation and thickness random values, with the maximum deflection for the weakest orientation angle is less than 89% greater than the deterministic deflection of 95.8mm.

The limit state defined in Eq.(102) may be used to observe the reliability of the plate when random variables for both layers are, a) fibre orientation, and b) laminate thickness. Both uncertain variables are assumed to be normally distributed with coefficients of variation (COV) of 0.1. Outcomes of Monte Carlo and FORM analyses with fibre orientation as the only uncertainty is shown in Figure 8.

Within the initial simulations (fewer than 20,000), the Monte Carlo simulation is unstable. After around 23,000 simulations the solution is sufficiently stable to show the trend of the reliability value to this problem with a nominal probability of failure of around 0.0028. The FORM analysis provided a nominal probability of failure (P_f) of 0.00284 with only 8 iterations.

Based on Eq.(37), a Monte Carlo simulation represented by 30,000 analyses (e.g. Figure 8) will contain a computational error of around 11%. To achieve a 5% error it is estimated that 140,000 analyses will be required. It should be noted that the FORM calculation appears to be highly accurate with an apparent error of $<< 1\%$. However, with a sufficient number of analyses, Monte Carlo simulation provides the most accurate estimate of the probability of failure. FORM is limited by representing the limit state as a linear function of the uncertain variables, which may lead to an under- or over-estimate of the probability of failure. Therefore, when combining the apparent accuracy of the FORM probability measure with the estimated 5% error in the Monte Carlo simulation with 30,000 analyses, it may be conjectured that the FORM estimate of the probability of failure of the plate in Example 1 is both validated and satisfactory, particularly when considering the relatively small number of required iterations.

The result of taking only the thickness as the random variable is shown in Figure 9. The figure clearly shows that the probability of failure for the problem is an order of magnitude higher than problem (1a), in which the fibre orientation is uncertain. The probability of failure from a FORM analysis returned an estimate of 0.0372, reached in 7 iterations. As in the previous problem, at the initial stages of the Monte Carlo simulation (up to 15,000), the result is not stable. When the number of simulations is increased, the probability of failure increases and converges towards the FORM value after 18,000 simulation, remaining relatively constant up to 27,000 simulations. From 27,000 to 30,000 analyses the probability of failure is shown to show a tendency to decrease, with an instantaneous probability failure of 0.0369 (at 30,000 analyses). This characteristic of the Monte Carlo simulation to either show a decrease or increase in an apparently stable estimate probability of failure after several thousands of analyses can be understood as a temporary, but repeatable, trend and may be a function of the random number generator in which the random numbers are no longer independent (e.g. some numbers or cycles of numbers may be repeated). The quality of the random number generator used in a Monte Carlo simulation clearly has an impact on the accuracy of the probability estimate. It is well known that commonly found random number generators are in fact pseudo random number generators (PRNG) because of the repetition of the "random numbers" after several realisations. The issue of the quality of random number generators and the definition of a true set of random numbers is outside the scope of this paper.

Example 2

In this second example, similar uncertainties in ply angle orientation and ply thicknesses are assumed. The limit state function is used to examine the stress failure criterion, such that the limit state function is defined as:

$$g() = 1 - \text{Failure Index} \quad (103)$$

, where the Failure Index is based on the Tsai-Wu criterion with the specific form [11],

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\sigma_{33}}{Z}\right)^2 - \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right) \sigma_{11}\sigma_{22}$$

$$\begin{aligned}
& - \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \right) \sigma_{22}\sigma_{33} - \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \right) \sigma_{11}\sigma_{33} \\
& + \left(\frac{\sigma_{12}}{S} \right)^2 + \left(\frac{\sigma_{13}}{R} \right)^2 + \left(\frac{\sigma_{23}}{T} \right)^2 \geq 1.
\end{aligned} \tag{104}$$

It should be noted that in Tsai-Hill's criterion the stress terms do not appear as linear terms; therefore, the F_1, F_2 , and F_3 terms are zero. The values of X, Y, Z are taken as either X_T, Y_T, Z_T or as X_C, Y_C, Z_C , depending upon the sign of σ_{11}, σ_{22} , and σ_{33} , respectively. Strength tensors for this criterion are,

$$\begin{aligned}
F_i &= 0 \\
F_{11} &= \frac{1}{X^2}; \quad F_{22} = \frac{1}{Y^2}; \quad F_{33} = \frac{1}{Z^2} \\
F_{44} &= \frac{1}{R^2}; \quad F_{55} = \frac{1}{S^2}; \quad F_{66} = \frac{1}{T^2} \\
F_{12} &= -\frac{1}{2} \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \right) \\
F_{13} &= -\frac{1}{2} \left(\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \right) \\
F_{23} &= -\frac{1}{2} \left(\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \right).
\end{aligned} \tag{105}$$

Monte Carlo and FORM results assuming uncertain fibre orientations and ply thicknesses are shown in Figures 10 and 11. Comparing these two figures it is clear that the probability of failure associated with the randomised ply angle orientation is much higher than for randomised thicknesses. This can be explained by examining the nature of the problem given that the fibres are orientated at $45^\circ/-45^\circ$ angle. With these fibre orientations, shear stresses (σ_{12}) are produced in addition to the lateral (σ_{11}) and transverse stresses (σ_{22}). The in-plane shear stresses (σ_{12}) contribute significantly to the failure index since the shear strength of the material is low. When evaluated as deterministic about the mean values the Tsai-Wu failure index for the plates is 0.6822. With ply angle as the uncertainty variable, the Monte Carlo simulation estimated the probability of failure (P_f) of the plate under the stress criteria given in Eq.(103) of 0.0307 after 30,000 analyses, while FORM gave $P_f = 0.0314$ in 10 iterations.

Figure 11 shows the outcomes of the reliability analyses when the ply thicknesses are adopted as the uncertainty variables, with nominal failure probabilities estimated to be 0.0080 using Monte Carlo simulation (30,000 analyses) and 0.00792 after 9 iterations of FORM. Adopting the probability of failure achieved under uncertainty in fibre orientations as a design target, the only way to obtain equal probabilities of failure for both fibre orientation and ply thickness uncertainties for this plate is to make the ply thinner, leading to increased internal stresses.

Table 1: Displacement sensitivities ($\partial U/\partial \theta$) for moderately thick (denoted +) and thin (denoted -) laminated plates $[0/\overline{90}]_s$ with respect to ply angle.

Layer	linear sensitivities			non-linear sensitivities		
	Analytical	FFD	% Error	Analytical	FFD	% Error
1 ⁺	0.21028	0.21022	0.03	0.24732	0.24802	0.28
1 ⁻	0.11713	0.11717	0.03	0.09680	0.09763	0.86
2 ⁺	-0.01492	-0.01491	0.03	-0.01485	-0.01492	0.46
2 ⁻	-0.00593	-0.00593	0.04	-0.00327	-0.00329	0.75

Table 2: Displacement sensitivities ($\partial U/\partial\theta$) for moderately thick (denoted $+$) and thin (denoted $-$) laminated plates $[0/90/\bar{0}]_s$ with respect to ply angle.

linear sensitivities				non-linear sensitivities		
Layer	Analytical	FFD	% Error	Analytical	FFD	% Error
1 $^+$	0.14257	0.14256	0.01	0.14521	0.14593	0.50
1 $^-$	0.08702	0.08703	0.01	0.07312	0.07266	0.63
2 $^+$	-0.03722	-0.03722	0.01	-0.03972	-0.03983	0.28
2 $^-$	-0.02119	-0.02119	0.01	-0.01734	-0.01725	0.55
3 $^+$	0.00235	0.00235	0.04	0.00242	0.00244	0.83
3 $^-$	0.00175	0.00175	0.05	0.00071	0.00070	1.6

Table 3: Displacement sensitivities ($\partial U/\partial e$) for moderately thick (denoted $+$) and thin (denoted $-$) laminated plates $[0/90]_s$ with respect to ply thickness.

linear sensitivities				non-linear sensitivities		
Layer	Analytical	FFD	% Error	Analytical	FFD	% Error
1 $^+$	-0.35985	-0.36380	1.1	-0.33811	-0.33799	0.04
1 $^-$	-0.84928	-0.84801	0.15	-0.70425	-0.69583	1.2
2 $^+$	-0.35744	-0.36133	1.1	-0.33169	-0.32780	0.33
2 $^-$	-0.79155	-0.79049	0.14	-0.64962	-0.64362	0.92

Table 4: Displacement sensitivities ($\partial U/\partial e$) for moderately thick (denoted $+$) and thin (denoted $-$) laminated plates $[0/90/\bar{0}]_s$ with respect to ply thickness.

linear sensitivities				non-linear sensitivities		
Layer	Analytical	FFD	% Error	Analytical	FFD	% Error
1 $^+$	-0.20420	-0.20651	1.1	-0.19684	-0.19986	1.5
1 $^-$	-0.52128	-0.52058	0.13	-0.42345	-0.41977	0.87
2 $^+$	-0.21757	-0.22061	1.4	-0.20810	-0.21265	2.1
2 $^-$	-0.51200	-0.51135	0.13	-0.40998	-0.40569	1.0
3 $^+$	-0.21636	-0.21869	1.1	-0.20703	-0.20650	0.26
3 $^-$	-0.51296	-0.51233	0.12	-0.41223	-0.40826	0.96

Table 5: Stress sensitivities $\partial\sigma/\partial\theta$ for a moderately thick laminated plate $[0/90]_s$ wrt ply angle.

linear sensitivities					non-linear sensitivities		
Layer	Stress	Analytical	FFD	% Error	Analytical	FFD	% Error
1	σ_{11}	-0.85131×10^7	-0.85142×10^6	0.01	-0.98326×10^7	-0.98329×10^7	0.01
	σ_{22}	-0.63520×10^6	-0.63388×10^6	0.03	-0.73366×10^6	-0.73356×10^6	0.01
	σ_{12}	0.21773×10^{-11}	0.21774×10^{-11}	0.01	0.25148×10^{-11}	0.25150×10^{-11}	0.01
	σ_{13}	-0.71961×10^{-9}	-0.71982×10^{-9}	0.03	-0.83115×10^{-9}	-0.83122×10^{-9}	0.01
	σ_{23}	-0.35981×10^{-9}	-0.35975×10^{-9}	0.02	-0.41558×10^{-9}	-0.41578×10^{-9}	0.05
2	σ_{11}	-0.97803×10^4	-0.97819×10^4	0.02	-0.11306×10^5	-0.11311×10^5	0.01
	σ_{22}	-0.17065×10^6	-0.17068×10^6	0.02	-0.19271×10^6	-0.19731×10^6	0.02
	σ_{12}	-0.10062×10^{-10}	-0.10067×10^{-10}	0.05	-0.16317×10^{-10}	-0.11623×10^{-10}	0.08
	σ_{13}	0.70375×10^{-11}	0.70352×10^{-11}	0.02	0.81353×10^{-11}	0.81379×10^{-11}	0.03
	σ_{23}	0.14075×10^{-10}	0.14083×10^{-10}	0.06	0.16271×10^{-10}	0.16279×10^{-10}	0.06

Table 6: Stress sensitivities $\partial\sigma/\partial\theta$ for a thin laminated plate $[0/\overline{90}]_s$ wrt ply angle.

linear sensitivities					non-linear sensitivities		
Layer	Stress	Analytical	FFD	% Error	Analytical	FFD	% Error
1	σ_{11}	-0.85496×10^7	-0.85492×10^7	0.01	-0.10273×10^8	-0.10738×10^8	0.01
	σ_{22}	-0.63622×10^6	-0.63610×10^6	0.02	-0.79845×10^6	-0.79823×10^6	0.02
	σ_{12}	0.17983×10^{-11}	0.17976×10^{-11}	0.04	0.22568×10^{-11}	0.22578×10^{-11}	0.04
	σ_{13}	-0.72336×10^{-9}	-0.72317×10^{-9}	0.03	-0.90782×10^{-9}	-0.90772×10^{-9}	0.03
	σ_{23}	-0.36168×10^{-9}	-0.36150×10^{-9}	0.05	-0.45391×10^{-9}	-0.45382×10^{-9}	0.05
2	σ_{11}	-0.10230×10^5	-0.10227×10^5	0.03	-0.12839×10^5	-0.12846×10^5	0.03
	σ_{22}	-0.17614×10^6	-0.17608×10^6	0.03	-0.22106×10^6	-0.22114×10^6	0.03
	σ_{12}	-0.10505×10^{-10}	-0.10499×10^{-10}	0.06	-0.13184×10^{-10}	-0.13181×10^{-10}	0.06
	σ_{13}	-0.28251×10^{-11}	-0.28235×10^{-11}	0.01	-0.35794×10^{-11}	-0.35783×10^{-11}	0.01
	σ_{23}	0.42370×10^{-10}	0.42350×10^{-10}	0.05	0.53174×10^{-10}	0.53189×10^{-10}	0.05

Table 7: Stress sensitivities $\partial\sigma/\partial e$ for a moderately thick laminated plate $[0/\overline{90}]_s$ wrt ply thickness.

linear sensitivities					non-linear sensitivities		
Layer	Stress	Analytical	FFD	% Error	Analytical	FFD	% Error
1	σ_{11}	-0.34263×10^8	-0.34388×10^8	0.37	-0.39573×10^8	-0.39883×10^8	0.47
	σ_{22}	-0.14419×10^7	-0.14516×10^7	0.68	-0.16654×10^7	-0.16516×10^7	0.82
	σ_{12}	0.10393×10^{-9}	0.10489×10^{-9}	0.92	0.12004×10^{-9}	0.12083×10^{-9}	0.66
	σ_{13}	-0.54976×10^{-8}	-0.54596×10^{-8}	0.61	-0.63497×10^{-8}	-0.63196×10^{-8}	0.47
	σ_{23}	-0.63362×10^{-8}	-0.63762×10^{-8}	0.66	-0.73160×10^{-8}	-0.73762×10^{-8}	0.82
2	σ_{11}	-0.24192×10^7	-0.24242×10^7	0.21	-0.27742×10^7	-0.27542×10^7	0.72
	σ_{22}	-0.31904×10^6	-0.31997×10^6	0.29	-0.36997×10^6	-0.36849×10^6	0.40
	σ_{12}	-0.17816×10^{-8}	-0.17716×10^{-8}	0.56	-0.20716×10^{-8}	-0.20577×10^{-10}	0.67
	σ_{13}	0.58913×10^{-8}	0.58433×10^{-8}	0.82	0.38443×10^{-8}	0.38223×10^{-11}	0.57
	σ_{23}	-0.63834×10^{-8}	-0.63353×10^{-8}	0.75	-0.73728×10^{-8}	-0.73353×10^{-10}	0.51

Table 8: Stress sensitivities $\partial\sigma/\partial e$ for a thin laminated plate $[0/\overline{90}]_s$ wrt ply thickness.

linear sensitivities					non-linear sensitivities		
Layer	Stress	Analytical	FFD	% Error	Analytical	FFD	% Error
1	σ_{11}	-0.34144×10^{11}	-0.34433×10^{11}	0.87	-0.42851×10^{11}	-0.42455×10^{11}	0.92
	σ_{22}	-0.13741×10^{10}	-0.13832×10^{10}	0.66	-0.17245×10^{10}	-0.17339×10^{10}	0.55
	σ_{12}	0.11033×10^{-6}	0.11087×10^{-6}	0.49	0.13846×10^{-6}	0.13919×10^{-6}	0.53
	σ_{13}	0.32477×10^{-4}	0.32319×10^{-4}	0.49	0.40758×10^{-4}	0.40552×10^{-4}	0.51
	σ_{23}	0.16238×10^{-4}	0.16388×10^{-4}	0.92	0.20379×10^{-4}	0.20551×10^{-8}	0.85
2	σ_{11}	-0.23955×10^{10}	-0.23831×10^{10}	0.52	-0.29591×10^{10}	-0.29331×10^{10}	0.88
	σ_{22}	-0.30225×10^9	-0.30493×10^9	0.89	-0.37815×10^9	-0.37528×10^9	0.76
	σ_{12}	-0.16600×10^{-5}	-0.16763×10^{-5}	0.98	-0.20827×10^{-7}	-0.20637×10^{-7}	0.91
	σ_{13}	-0.46043×10^{-5}	-0.46347×10^{-5}	0.66	-0.57784×10^{-5}	-0.57403×10^{-5}	0.66
	σ_{23}	-0.92087×10^{-5}	-0.92773×10^{-5}	0.75	-0.11539×10^{-4}	-0.11443×10^{-4}	0.83

Figure 8: Probability of failure of a two ply laminate plate with uncertain fibre orientation under deflection criteria.

Figure 9: Probability of failure of a two ply laminate plate with uncertain laminate thickness under deflection criteria.

Figure 10: Probability of failure of a two ply laminate plate with uncertain fibre orientation under stress criteria.

Figure 11: Probability of failure of a two ply laminate plate with uncertain laminate thickness under stress criteria.

4. Conclusions

The analytical derivation of linear and non-linear sensitivities of a complex shear-deformable composite laminate plate model has been demonstrated in this paper. Displacement, strain, and stress components have been differentiated with respect to fibre orientation and ply thickness as examples and verified using finite difference approximations. The non-linear components of these sensitivities make a significant contribution for moderately thick laminates when considering changes in displacement with respect to both ply angle and thickness. Discounting the non-linear components may lead to an overestimate of the sensitivities by up to 30%. Similar findings are exhibited by the stress sensitivities. Once expressions for the analytical sensitivities have been obtained the first-order reliability method (FORM) is shown to be highly efficient and accurate at estimating the nominal probability of failure when compared with the computationally expensive Monte Carlo simulation alternative. The relative estimates of nominal failure probabilities obtained from FORM and Monte Carlo simulation demonstrate that a linear approximation of the limit states explored in this paper is adequate. The use of FORM is also demonstrated to be practicable through numerical examples. The reliability analysis approach based on analytical sensitivities presented in this paper is very well suited to new and emerging approaches for the analysis of composite plates and shells such as, for example, the unified framework proposed by Williams [32] and the layerwise approach of Batra [33].

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5. Appendix

$$[B_e]_{5 \times 15} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & f_1 + g_1^{(k)} - z & g_2^{(k)} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & g_3^{(k)} \\ 0 & 1 & 1 & 0 & 0 & 0 & g_3^{(k)} & f_1 + g_1^{(k)} + f_2 + g_4^{(k)} - 2z \\ 0 & 0 & 0 & 0 & f_1' + g_1^{(k)'} & g_2^{(k)'} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_3^{(k)'} & f_2' - g_4^{(k)'} & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_2^{(k)} & 0 & 0 & f_1 + g_1^{(k)} \\ f_2 + g_4^{(k)} - z & 0 & 0 & 0 & -f_2 - g_4^{(k)} & 0 & 0 & g_3^{(k)} \\ g_2^{(k)} & 0 & 0 & -f_2 - g_4^{(k)} & -g_2^{(k)} & 0 & g_3^{(k)} & f_1 + g_1^{(k)} \\ 0 & g_2^{(k)'} & 0 & 0 & 0 & f_1' + g_1^{(k)'} & 0 & 0 \\ 0 & -f_2' - g_4^{(k)'} & 0 & 0 & 0 & 0 & 0 & g_3^{(k)'} \end{bmatrix} \quad (106)$$

$$[a1] = \begin{bmatrix} \bar{C}_{11}^{(k)} \bar{w}_{0,1} & \bar{C}_{12}^{(k)} \bar{w}_{0,2} \\ \bar{C}_{66}^{(k)} \bar{w}_{0,2} & \bar{C}_{66}^{(k)} \bar{w}_{0,1} \end{bmatrix} \quad [a2] = \begin{bmatrix} \bar{C}_{66}^{(k)} \bar{w}_{0,2} & \bar{C}_{66}^{(k)} \bar{w}_{0,1} \\ \bar{C}_{12}^{(k)} \bar{w}_{0,1} & \bar{C}_{22}^{(k)} \bar{w}_{0,2} \end{bmatrix} \quad (107)$$

$$\begin{aligned} a3(1,1) &= \bar{C}_{11}^{(k)} \bar{w}_{0,1}^2 + \bar{C}_{66}^{(k)} \bar{w}_{0,2}^2, & a3(1,2) &= (\bar{C}_{11}^{(k)} + \bar{C}_{66}^{(k)}) \bar{w}_{0,1} \bar{w}_{0,2}, \\ a3(1,3) &= \bar{C}_{11}^{(k)} h_1^{(k)} \bar{w}_{0,1}, & a3(1,4) &= \bar{C}_{66}^{(k)} (h_1^{(k)} + h_2^{(k)}) \bar{w}_{0,2}, & a3(1,5) &= \bar{C}_{12}^{(k)} h_2^{(k)} \bar{w}_{0,1} \\ a3(2,2) &= \bar{C}_{66}^{(k)} \bar{w}_{0,1}^2 + \bar{C}_{22}^{(k)} \bar{w}_{0,2}^2, & a3(2,3) &= \bar{C}_{12}^{(k)} h_1^{(k)} \bar{w}_{0,2} \\ a3(2,4) &= \bar{C}_{66}^{(k)} (h_1^{(k)} + h_2^{(k)}) \bar{w}_{0,1}, & a3(2,5) &= \bar{C}_{22}^{(k)} h_2^{(k)} \bar{w}_{0,2} \end{aligned} \quad (108)$$

Sub-matrix $[a3]$ is symmetric with $h_1^{(k)} = (f_1 + g_1^{(k)} - z)$ and $h_2^{(k)} = (f_2 + g_4^{(k)} - z)$ and,

$$[a4] = - (f_2 + g_4^{(k)}) \begin{bmatrix} \bar{C}_{66}^{(k)} \bar{w}_{0,2} & \bar{C}_{12}^{(k)} \bar{w}_{0,1} \\ \bar{C}_{66}^{(k)} \bar{w}_{0,1} & \bar{C}_{22}^{(k)} \bar{w}_{0,2} \end{bmatrix}, \quad [a5] = (f_1 + g_1^{(k)}) \begin{bmatrix} \bar{C}_{11}^{(k)} \bar{w}_{0,1} & \bar{C}_{66}^{(k)} \bar{w}_{0,2} \\ \bar{C}_{12}^{(k)} \bar{w}_{0,2} & \bar{C}_{66}^{(k)} \bar{w}_{0,1} \end{bmatrix}. \quad (109)$$

$$\frac{\partial T^{(k)}}{\partial \theta_i} = \begin{bmatrix} -2 \sin(2\theta^{(k)}) & 2 \cos(2\theta^{(k)}) & 0 & 0 & 2 \cos(2\theta^{(k)}) \\ 2 \cos(2\theta^{(k)}) & -2 \sin(2\theta^{(k)}) & 0 & 0 & -2 \cos(2\theta^{(k)}) \\ 0 & 0 & -\sin(\theta^{(k)}) & -\cos(\theta^{(k)}) & 0 \\ 0 & 0 & \cos(\theta^{(k)}) & -\sin(\theta^{(k)}) & 0 \\ -2 \cos(2\theta^{(k)}) & 2 \cos(2\theta^{(k)}) & 0 & 0 & -2 \cos(2\theta^{(k)}) \end{bmatrix}. \quad (110)$$

$$[a1'] = \begin{bmatrix} -2 \sin(2\theta^{(k)}) \bar{C}_{11}^{(k)} \bar{w}_{0,1} & 2 \cos(2\theta^{(k)}) \bar{C}_{12}^{(k)} \bar{w}_{0,2} \\ -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,2} & -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,1} \end{bmatrix} \quad (111)$$

$$[a2'] = \begin{bmatrix} -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,2} & -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,1} \\ 2 \cos(2\theta^{(k)}) \bar{C}_{12}^{(k)} \bar{w}_{0,1} & -2 \sin(2\theta^{(k)}) \bar{C}_{22}^{(k)} \bar{w}_{0,2} \end{bmatrix} \quad (112)$$

$$\begin{aligned}
a3'(1, 1) &= -2 \sin(2\theta^{(k)}) \bar{C}_{11}^{(k)} \bar{w}_{0,1}^2 - 2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,2}^2, \\
a3'(1, 2) &= \left(-2 \sin(2\theta^{(k)}) \bar{C}_{11}^{(k)} - 2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \right) \bar{w}_{0,1} \bar{w}_{0,2}, \\
a3'(1, 3) &= -2 \sin(2\theta^{(k)}) \bar{C}_{11}^{(k)} h_1^{(k)} \bar{w}_{0,1}, \\
a3'(1, 4) &= -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \left(h_1^{(k)} + h_2^{(k)} \right) \bar{w}_{0,2}, \\
a3'(1, 5) &= 2 \cos(2\theta^{(k)}) \bar{C}_{12}^{(k)} h_2^{(k)} \bar{w}_{0,1} \\
a3'(2, 2) &= -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \bar{w}_{0,1}^2 - 2 \sin(2\theta^{(k)}) \bar{C}_{22}^{(k)} \bar{w}_{0,2}^2, \\
a3'(2, 3) &= 2 \cos(2\theta^{(k)}) \bar{C}_{12}^{(k)} h_1^{(k)} \bar{w}_{0,2} \\
a3'(2, 4) &= -2 \cos(2\theta^{(k)}) \bar{C}_{66}^{(k)} \left(h_1^{(k)} + h_2^{(k)} \right) \bar{w}_{0,1}, \\
a3'(2, 5) &= -2 \sin(2\theta^{(k)}) \bar{C}_{22}^{(k)} h_2^{(k)} \bar{w}_{0,2}.
\end{aligned} \tag{113}$$

$$\begin{aligned}
\partial B_e / \partial h_k(1, 7) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)} - (h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(1, 8) &= a_2^{(k)}; \\
\partial B_e / \partial h_k(1, 11) &= -a_2^{(k)}; \\
\partial B_e / \partial h_k(1, 14) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)}; \\
\partial B_e / \partial h_k(2, 8) &= a_3^{(k)}; \\
\partial B_e / \partial h_k(2, 9) &= f'(h^{(k)} - h^{(k-1)}) + b_{44} f(h^{(k)} - h^{(k-1)}) + a_4^{(k)} - (h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(2, 12) &= -f'(h^{(k)} - h^{(k-1)}) - b_{44} f(h^{(k)} - h^{(k-1)}) + a_4^{(k)}; \\
\partial B_e / \partial h_k(2, 15) &= a_3^{(k)}; \\
\partial B_e / \partial h_k(3, 7) &= a_3^{(k)}; \\
\partial B_e / \partial h_k(3, 8) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)} \\
&\quad + f'(h^{(k)} - h^{(k-1)}) + b_{44} f(h^{(k)} - h^{(k-1)}) + a_4^{(k)} - 2(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(3, 9) &= a_2^{(k)}; \\
\partial B_e / \partial h_k(3, 11) &= -f'(h^{(k)} - h^{(k-1)}) - b_{44} f(h^{(k)} - h^{(k-1)}) - a_4^{(k)}; \\
\partial B_e / \partial h_k(3, 12) &= -a_2^{(k)}; \\
\partial B_e / \partial h_k(3, 14) &= a_3^{(k)}; \\
\partial B_e / \partial h_k(3, 15) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)}; \\
\partial B_e / \partial h_k(4, 5) &= f''(h^{(k)} - h^{(k-1)}) + b_{55} f'(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(4, 13) &= f''(h^{(k)} - h^{(k-1)}) + b_{55} f'(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(5, 6) &= f''(h^{(k)} - h^{(k-1)}) + b_{44} f'(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k(5, 10) &= -f''(h^{(k)} - h^{(k-1)}) - b_{44} f'(h^{(k)} - h^{(k-1)}).
\end{aligned} \tag{114}$$

$$\frac{\partial T^{(k)}}{\partial \theta_i} = \begin{bmatrix} -2 \sin(2\theta^{(k)}) & 2 \cos(2\theta^{(k)}) & 0 & 0 & 2 \cos(2\theta^{(k)}) \\ 2 \cos(2\theta^{(k)}) & -2 \sin(2\theta^{(k)}) & 0 & 0 & -2 \cos(2\theta^{(k)}) \\ 0 & 0 & -\sin(\theta^{(k)}) & -\cos(\theta^{(k)}) & 0 \\ 0 & 0 & \cos(\theta^{(k)}) & -\sin(\theta^{(k)}) & 0 \\ -2 \cos(2\theta^{(k)}) & 2 \cos(2\theta^{(k)}) & 0 & 0 & -2 \cos(2\theta^{(k)}) \end{bmatrix}. \tag{115}$$

$$\begin{aligned}
\partial B_e / \partial h_k (1, 7) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)} - (h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k (1, 8) &= a_2^{(k)}; \quad \partial B_e / \partial h_k (1, 11) = -a_2^{(k)} \\
\partial B_e / \partial h_k (2, 8) &= a_3^{(k)}; \\
\partial B_e / \partial h_k (3, 8) &= f'(h^{(k)} - h^{(k-1)}) + b_{55} f(h^{(k)} - h^{(k-1)}) + a_1^{(k)} + \\
&\quad f'(h^{(k)} - h^{(k-1)}) + b_{44} f(h^{(k)} - h^{(k-1)}) + a_4^{(k)} - 2(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k (4, 5) &= f''(h^{(k)} - h^{(k-1)}) + b_{55} f'(h^{(k)} - h^{(k-1)}); \\
\partial B_e / \partial h_k (5, 6) &= f''(h^{(k)} - h^{(k-1)}) + b_{44} f'(h^{(k)} - h^{(k-1)}).
\end{aligned} \tag{116}$$