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Lukasz Mikulski, Marcin Piatkowski and Wojciech Rytter

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About the authors

Lukasz Mikulski obtained his PhD in 2012 from the University of Warsaw, Poland. He now works as a Lecturer in the Faculty of Mathematics and Computer Science at the Nicolaus Copernicus University in Torun, Poland. He is a Visiting Researcher in the School of Computing Science at the Newcastle University.

Marcin Piatkowski obtained his PhD in 2011 from the University of Warsaw, Poland. He now works as a Lecturer in the Faculty of Mathematics and Computer Science at the Nicolaus Copernicus University in Torun, Poland.

Wojciech Rytter is a Professor in the Faculty of Mathematics, Informatics and Mechanics at the University of Warsaw, Poland. His research focuses on the design and analysis of algorithms, in particular on stringology.

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Square-free words over partially commutative alphabets

Łukasz Mikulski^{1,2}, Marcin Piątkowski^{1,3}, and Wojciech Rytter^{4,1}

¹ Faculty of Mathematics and Computer Science
Nicolaus Copernicus University, Toruń, Poland

² School of Computing Science
Newcastle University, Newcastle upon Tyne, United Kingdom

³ Department of Computer Science,
University of Helsinki, Helsinki, Finland

⁴ Faculty of Mathematics, Informatics and Mechanics
Warsaw University, Warsaw, Poland

lukasz.mikulski@mat.umk.pl
marcin.piatkowski@mat.umk.pl
rytter@mimuw.edu.pl

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1 Introduction

Problems related to repetitions are crucial in the combinatorics on words due to many practical application, for instance in data compression, pattern matching, text indexing and so on (see [16]). On the other hand, in some cases it is important to consider words avoiding regularities and repetitions. Example applications can be found in such research areas as cryptography and bioinformatics. Languages of words over partially commutative alphabets are fundamental tools for concurrent systems investigation, see [9]. Therefore, the study of repetitions and their avoidability in such languages is significant.

The simplest form of repetition is a square – the factor of the form $x \cdot x$, where x is not empty. Therefore, to show that a word w contains no repetitions, it is sufficient to show that w does not contain squares. Another interesting type

of repetition is the abelian square – a factor of the form $x \cdot y$, where x can be obtained from y by permutation of the letters. For example, $baca \cdot caab$ is an abelian square, whereas $bcca \cdot cbba$ is not. A word that contains no abelian squares is called abelian square-free.

Square-free and abelian square-free words have been extensively studied. In 1906 Thue showed that squares are avoidable over three-letter alphabet (see [18]), i.e. there exist infinitely many ternary words without a square. In 1961 Erdős raised the question whether abelian squares are avoidable (see [13]). First attempt to answer this question was made by Evdokimov in 1968 (see [14]), who showed that abelian squares are avoidable over alphabets consisting of at least 25 letters. Then, in 1970, the required size of the alphabet was decreased to 5 by Pleasants (see [17]), and finally, in 1992, to 4 by Keränen (see [15]). Moreover, it can be easily shown that abelian squares cannot be avoided over three-letter alphabet.

A one step further is to study repetitions and their avoidability in words over partially commutative alphabets, see for instance [6–8, 10, 11]. In contrast to the abelian case, only some fixed pairs of letters from the alphabet are allowed to commute. It complicates considerably the analysis of repetitions in such classes of words.

In this paper we deal with the avoidability of repetitions in words over three-letter alphabet Θ with one pair of commuting letters. We start with the analysis of possible shapes of three-letter alphabets. Then we describe an infinite language of increasing square-free words and investigate their combinatorial properties. Further we use this language, utilising the results of [10], to define the infinite language of partially abelian square-free words over Θ .

Our results. As a final result, we define the infinite lexicographically minimal Θ -square-free word $\tilde{\mathbf{v}}$ and give an efficient construction of this word. The n -th letter of $\tilde{\mathbf{v}}$ can be computed in logarithmic time with respect to n . The first 228 letters of $\tilde{\mathbf{v}}$ are:

$$\begin{aligned} \tilde{\mathbf{v}} = & abacabcbacabacbcabacabcbacbcabcbacabcbacabcbacabcbacabcb \\ & acbcabacabcbacabacbcabacabcbacbcabcbacabcbacbcabcbacabcb \\ & acbcabcbacabacbcabacabcbacabcbacbcabcbacabcbacabcbacabcb \\ & abcbacbcabcbacabacbcabacabcbacabcbacbcabcbacabcbacabcbac \dots \end{aligned}$$

2 Basic notions

Throughout the paper we use the standard notions of the formal language theory (see [16] for a more detailed introduction). By Σ we denote a finite set, called the *alphabet*. Elements of the alphabet are called *letters*. A finite word over Σ is a finite sequence of letters. The length of a word w is defined as the number of its letters and denoted $|w|$. The set of all finite words over Σ is denoted by Σ^* and is equipped with a binary associative concatenation operation $\cdot$, where $a_1 \dots a_n \cdot b_1 \dots b_m$ is simply $a_1 \dots a_n b_1 \dots b_m$. An empty sequence of letters, called the *empty word* and denoted by ε , is the neutral element of the concatenation

operation. Thus for any word w we have $\varepsilon \cdot w = w \cdot \varepsilon = w$. An infinite word over Σ is a sequence of letters indexed by non-negative integers. On the other hand, it can be also defined as a limit of infinite sequence of finite words.

A word u is called a *factor* of a word w if there exist words x and y such that $w = xuy$. If $y = \varepsilon$ then u is called a *prefix* of w and if $x = \varepsilon$ then u is called a *suffix* of w . For a word $w = a_1a_2 \dots a_n$ and $1 \leq i, j \leq n$ by $w[i..j]$ we denote its factor of the form $a_ia_{i+1} \dots a_j$.

We assume that the alphabet Σ is given together with a strict total order $<$, called the *lexicographical order*. This notion is extended in a natural way to the level of words. For any two words x and y we have $x < y$ if x is a proper prefix of y or we have $x = uav_1$ and $y = ubv_2$, where a, b are letters and $a < b$.

A mapping $\phi : \Sigma_1^* \rightarrow \Sigma_2^*$ is called a *morphism* if we have $\phi(u \cdot v) = \phi(u) \cdot \phi(v)$ for every $u, v \in \Sigma_1^*$. A morphism ϕ is uniquely determined by its values on the alphabet. Moreover, ϕ maps the neutral element of Σ_1^* into the neutral element of Σ_2^* .

A *partially commutative alphabet* is a pair $\Theta = (\Sigma, ind)$, where Σ is an ordered alphabet and $ind \subseteq \Sigma \times \Sigma$ is a symmetric *commutation* relation. Such an alphabet defines an equivalence relation $\equiv_\Theta$ identifying words, which differ only by the ordering of commuting letters. Two words $w, v \in \Sigma^*$ satisfy $w \equiv_\Theta v$ if there exists a finite sequence of commutations of adjacent commuting letters transforming w into v . For example let us consider the partially commutative alphabet $\Theta = (\{a, b, c\}, \{(b, c), (c, b)\})$. Then the word $w = acbcacb$ is equivalent to $v = accbabc$, but is not equivalent to $u = baccacb$. Words over a partially commutative alphabet $\Theta = (\Sigma, ind)$ are called *partially commutative words*. Note that it is usually assumed that for each $a \in \Sigma$ we have $(a, a) \notin ind$, but in the case of this paper such an assumption is not essential and it does not affect the presented results.

A square in a word w is a factor of the form $x \cdot x$, where x is not empty. A word w is called *square-free* if none of its factors is a square. If we consider a partially commutative alphabet $\Theta = (\Sigma, ind)$ there are two possible (not equivalent) ways to define a square. Such a square is called a partially commutative square or a Θ -square in short.

Definition 1 ([10]). Let $\Theta = (\Sigma, ind)$ be a partially commutative alphabet. A Θ -square is a factor of the form $u \cdot v$ such that $u \equiv_\Theta v$.

Definition 2. Let $\Theta = (\Sigma, ind)$ be a partially commutative alphabet. A Θ -square is a factor u such that we have $u \equiv_\Theta v \cdot v$ for some word v .

Note that, if a word is a square according to Definition 1, it is also a square according to Definition 2, but not vice-versa. The above definitions lead to two different notions of Θ -square-freeness.

Definition 3 ([10]). Let $\Theta = (\Sigma, ind)$ be a partially commutative alphabet. A word w is Θ -square-free if it does not contain any factor of the form $u \cdot v$ such that $u \equiv_\Theta v$.

Definition 4. Let $\Theta = (\Sigma, \text{ind})$ be a partially commutative alphabet. A word w is Θ -square-free if it does not contain any factor u such that we have $u \equiv_{\Theta} v \cdot v$ for some v .

One can also define the square-freeness yet in another way as follows.

Definition 5 ([8]). Let $\Theta = (\Sigma, \text{ind})$ be the partially commutative alphabet. A word w is Θ -square-free if no word $v \equiv_{\Theta} w$ contain a square $u \cdot u$.

Observe that if a word w is Θ -square-free according to Definition 4, then it is also Θ -square-free according to Definition 3, and if w is Θ -square-free according to Definition 5, then it is also Θ -square-free according to both Definition 4 and Definition 3. Moreover, the inversion of any part of the above statement does not hold. Therefore, Definition 3 specifies the strongest notion of Θ -square-freeness.

Note that in the case of the full commutation relation (i.e. any pair of letters can commute) Θ -squares are called the *abelian squares*, and words avoiding them – the *abelian square-free* words.

Example 1. Let $\Theta = (\{a, b, c\}, \{(b, c), (c, b)\})$ be a partially commutative alphabet. The word $w_1 = abc \cdot acb$ is Θ -square according both to Definition 1 and Definition 2, and the word $w_2 = bab \cdot cac$ is Θ -square only according to Definition 2. Moreover, the word w_2 is Θ -square-free according to Definition 3, whereas according to Definition 4 and Definition 5 it is not. On the other hand, the word $w_3 = bcbabc$ is Θ -square-free according to Definition 3 and Definition 4, but not according to Definition 5 (w_3 is equivalent to $w_4 = bbcabc$, which contains a square bb). Finally the word $w_5 = bcacb$ is Θ -square-free according to all definitions presented above. $\square$

Remark 1. Thorough the rest of this paper we consider the notion of Θ -squares as stated in Definition 1 and Θ -square-freeness as stated in Definition 3.

3 Partially abelian square-free words over three-letter alphabets

It is easy to see that any binary word consisting of at least four letters must contain a square. In 1906 Thue shown that three letters are sufficient to construct an infinite square-free word (see [18]). Moreover, in 1992 Keränen proved that to avoid abelian squares (i.e. factors of the form $x \cdot y$, where x and y differ only by permutation of their letters) four letters are sufficient (see [15]). It follows immediately that any four-letter alphabet with more restricted commutation relation also allows to avoid partially commutative squares. Therefore, the alphabets of size three are the most interesting boundary case.

In partially commutative alphabets consisting of three letters one can consider four distinct commutation relations as depicted on Fig. 1. We start with the most restricted case of Θ_4 . Observe that the concepts of Θ_4 -square freeness and ordinary square-freeness are equivalent. Due to the results of Thue (see [18]),

Fig. 1. The possible shapes of the commutation relation over three-letter alphabet. The pairs of letters connected by an edge can commute.

the number of Θ_4 -square-free words is infinite. Moreover, the number of finite square-free words of a given length is exponential with respect to this length (see [5] for more details).

On the other hand, the concepts of abelian square-freeness and Θ_1 -square-freeness are equivalent. We have only 117 words without Θ_1 -square and the longest of them consists of 7 letters. Similarly, the number of Θ_2 -square-free words is finite. In this case we have 289 such words with the longest having 15 letters (see [10] for more details).

The most interesting case is the remaining alphabet Θ_3 . Similarly as in the case of Θ_4 , the number of finite Θ_3 -square-free words is infinite, however it is polynomially proportional to the length of the word (see [11]). In what follows, we focus on this alphabet and investigate the combinatorial structure of Θ_3 -square-free words in more details.

Remark 2. From now on we only consider the alphabet Θ_3 and denote it by Θ . Thus, by Θ -square and Θ -square-freeness we mean the Θ_3 -square and Θ_3 -square-freeness.

Conditions for Θ -square-freeness

We start with giving some necessary and sufficient conditions for a word to be Θ -square-free. The more detailed study of their combinatorial structure is presented in the subsequent sections. We follow here the results of [10] presented below. Initially we present a statement, which is used further to formulate the conditions for Θ -square-freeness.

Definition 6 (Condition (F), see [10]). *The word $v \in \Sigma^*$ satisfies the condition (F) if neither $abca$ nor $acba$ is a factor of v , where $a, b, c \in \Theta = (\Sigma, \text{ind})$.*

The possible structure of a finite word containing a Θ -square is established by the following fact (see Proposition 3.2 in [10]).

Proposition 1 (see [10]). *Let w be a finite square-free word satisfying the condition (F) and containing a Θ -square as a factor. Then w admits one of the following decompositions:*

- (i) $w = w_1bcw_2bcbw_2bw_3$

- (ii) $w = w_1cbw_2cbcw_2cw_3$
- (iii) $w = w_1bw_2bcw_2cbw_3$
- (iv) $w = w_1cw_2cbcw_2bcw_3$

where $w_1, w_2, w_3 \in \Sigma^*$. Moreover in such a decomposition one of the factors w_1 or w_3 is of length at most 1.

As a corollary to Proposition 1 we can formulate the following fact characterizing the possible building blocks of Θ -square-free words (see the proof of Proposition 2.1 and Proposition 3.2 in [10]). It will be utilized further in construction an infinite Θ -square-free word.

Corollary 1. *Any infinite Θ -square-free word w starting with a consists of the factors belonging to the following: $\mathcal{B} = \{aba, aca, abcba, acbca, abca, acba\}$. Moreover, the factors $acba$ and $abca$ can appear only as a prefix of w .*

Remark 3. Note that no two different words created by concatenating factors (without ending a) from the set $\mathcal{B}$ defined in Corollary 1 are equivalent under the relation $\equiv_\Theta$.

Finally, the following theorem gives a sufficient condition for the Θ -square-freeness of an infinite word, see Corollary 3.3 in [10] for the proof.

Theorem 1 (see [10]). *Any infinite square-free word over Σ starting with a and satisfying the (F) condition is Θ -square-free.*

4 The structure of Θ -square-free words

In the preceding section we presented the necessary and sufficient conditions for a word to be Θ -square-free. Below we investigate the combinatorial structure of such words in more detail.

Recall that due to Corollary 1 any infinite Θ -square-free word w consists only of the factors $aba, aca, abcba, acbca, acba$ and $abca$, where the last two can appear only as a prefix w . It can be easily proven that neither abc nor acb could be a prefix of the lexicographically minimal infinite Θ -square-free word. Therefore we have to consider only the factors $aba, aca, abcba, acbca$.

The above observations are the basis of the idea of encoding the possible building blocks of Θ -square-free words as the symbols of a four-letter meta-alphabet $\Delta = \{A, B, C, D\}$.

Definition 7. *Let $\Sigma = \{a, b, c\}$ (alphabet) and $\Delta = \{A, B, C, D\}$ (meta-alphabet). We define a morphism $M : \Delta^* \rightarrow \Sigma^*$ as follows:*

$$M = \begin{cases} A \longrightarrow ab \\ B \longrightarrow ac \\ C \longrightarrow abcb \\ D \longrightarrow acbc \end{cases}.$$

It is worth to note that the morphism defined above is a code with finite deciphering delay. This fact is utilized in operations described further in this paper.

In what follows, if a word w over Σ is an image of a word u over Δ we call u an M -reduction of w and w is called M -reducible⁵.

The alphabet Δ consists of four letters, hence it allows us to construct words without repetitions. However, not all such words over Δ lead to words with no repetitions over Θ . Since we are interested in Θ -square-free words, the considered words over Δ must satisfy additional conditions presented further.

Lemma 1. *Let w be an infinite, M -reducible and Θ -square-free word starting with $abacabcabaca$. Then M -reduction of w does not contain any of the factors: AC , CA , BD , DB , ABA , BAB , CBC , DAD , $ADCB$, $BCDA$.*

Proof. The main idea of the proof is to show that each of the above presented meta-factors lead to a square either over the meta-alphabet Δ or over the original alphabet Σ .

AC: By Definition 7 the image of A is a prefix of the image of C , therefore the factor AC over the meta-alphabet Δ leads to a square over the alphabet Σ . The proof of the case BD is similar.

CA: Observe that by Definition 7 the image of CA ends with bab . Moreover, the image of each meta-letter from Δ starts with a . Therefore, any meta-letter directly following the factor CA leads to a square over the alphabet Σ . The proof of the case DB is similar.

ABA: It is clear that ABA cannot be followed neither by A nor by B . Moreover, the factor AC is forbidden, hence ABA could not be followed by C . Finally, the image of B is a prefix of the image of D , therefore ABA could not be followed by D , and this concludes the proof of this case. The proof of the case BAB is similar.

CBC: It is clear that CBC cannot be followed neither by B nor by C . Moreover, the factor CA is forbidden, hence CBC could not be followed by A . Finally, the image of B is a prefix of the image of D , therefore CBC could not be followed by D , and this concludes the proof of this case. The proof of the case DAD is similar.

ADCB: Note that $M(w)$ starts with AB , hence it cannot start with AD . It is clear that $ADCB$ cannot be preceded by A . Moreover, the factors CA , DB , DAD and ABA are forbidden, hence $ADCB$ could not be preceded neither by C nor by D nor by AB nor by BB nor by DB . Note that BD , AC , CBC and BAB are also forbidden, hence $ADCB$ cannot be followed neither by D nor by C nor by AB nor by AA nor by AC . Finally, if $ADCB$ is preceded by CB and followed by AD (the only possibilities that are not excluded), then it basically forms a square $CBADCBAD$, which concludes the proof of this case. The proof of the case $BCDA$ is similar.

⁵ In the approach presented in this paper the morphism M is in fact used as a translation of an infinite word over four-letter alphabet into an infinite word over three-letter alphabet.

Remark 4. Let $w \in \Sigma^*$ be a Θ -square-free word satisfying the condition (F) stated in Definition 6. Then w consists of blocks, which are images of letters from the alphabet Δ by the morphism M defined above, hence it is always M -reducible and we can apply the inverse mapping M^{-1} (M -reduction) to w . Moreover, the obtained result is a square-free word. On the other hand, the image by M of a square-free word over Δ does not have to be Θ -square-free word. For instance AC is a square-free word over Δ , but $M(AC) = ababcb$ is not Θ -square-free.

As a corollary to Lemma 1 we can describe the structure of Θ -square-free words in the terms of meta-alphabet.

Corollary 2. *Each M -reduction of a Θ -square-free word starting with aba is an element of the set defined by a following regular expression $\mathcal{T} = ((A|C)(B|D))^*$.*

Remark 5. The inverse of Corollary 2 is not true, i.e. there exist words whose M -reductions are in the set $\mathcal{T}$, but they are not Θ -square-free, for instance $abacabacbc$ which is an image of $ABAD$.

Lemma 2. *Let $w < v \in \Sigma^*$ be two M -reducible words starting with aba , which M -reductions have equal length and are contained in $\mathcal{T}$, and let $u \in \Sigma^*$ be the longest M -reducible word such that $M^{-1}(w) = M^{-1}(u)M^{-1}(w')$ and $M^{-1}(v) = M^{-1}(u)M^{-1}(v')$. Then $M^{-1}(v')$ starts with C and $M^{-1}(w')$ starts with A or $M^{-1}(v')$ starts with D and $M^{-1}(w')$ starts with B .*

Proof. The first meta-letters of $M(w')$ and $M(v')$ have to be different, as otherwise u could be extended by M -image of the common first meta-letter. Moreover, if the last meta-letter of $M^{-1}(u)$ is B or D , then, by being contained in $\mathcal{T}$, the first meta letters of $M^{-1}(w')$ and $M^{-1}(v')$ have to be contained in the set $\{A, C\}$ (note that u starts with $M(A)$). Since those first meta-letters are different and $w < v$, we conclude that the first meta-letter of $M^{-1}(w')$ is A , while the first meta-letter of $M^{-1}(v')$ is C . For the same reason, if $M^{-1}(u)$ ends with A or C , then $M^{-1}(w')$ starts with B and $M^{-1}(v')$ starts with D .

Theorem 2. *An infinite word w starting with the letter a and not starting with the prefix $abca$ or $acba$ is Θ -square-free if and only if w is square-free and M -reducible.*

Proof.

$\Rightarrow$: Let w be an infinite Θ -square-free word starting with the letter a and not starting with the prefix $abca$ or $acba$. Due to Corollary 1, w in the concatenation of the following factors: $ab, ac, abcb, acbc$, hence is M -reducible. Note that each ordinary square is also a Θ -square. Therefore, since w is Θ -square-free, it is also square-free.

$\Leftarrow$: Let w be an infinite M -reducible square-free word (hence it consist only of the following factors starting and ending with a : $aba, aca, abcba, acbca$). Then w obviously satisfies the condition (F) stated in Definition 6. Moreover, since w is square-free and satisfies the condition (F), due to Theorem 1, w is Θ -square-free, and this concludes the proof.

5 Equivalent definitions of the infinite word $\tilde{v}$

In this section we present two alternative definitions of the language of square-free words over the meta-alphabet $\Delta = \{A, B, C, D\}$ introduced in the previous section. We start with a definition using a morphism, then we show a recurrent procedure, which generates the same class of words.

5.1 Morphic definition

Let us define the sequence of words $\{X_i\}_{i \geq 0}$ over Δ together with the languages L_m and L_{dep} similar to $M(L_m)$. Both of them are based on the following morphism.

Definition 8. We define a morphism $m : \{A, B, C, D\}^* \rightarrow \{A, B, C, D\}^*$ as:

$$m = \begin{cases} A \rightarrow BCB \\ B \rightarrow ADA \\ C \rightarrow BCDCB \\ D \rightarrow ADCDA \end{cases}.$$

Definition 9. Let $\{X_i\}_{i \geq 0}$ be defined as:

$$X_i = \begin{cases} AB & \text{for } i = 0 \\ A \cdot m(X_{i-1}) \cdot B & \text{for } i > 0 \end{cases}.$$

We define the languages

$$L_m = \{m^i(AB) : i \geq 0\}; \quad \text{and} \quad L_{dep} = \{M(X_i) : i \geq 0\}.$$

The subsequent fact describes the combinatorial structure of words contained in the language L_{dep} .

Lemma 3. For every two words $u, v \in L_{dep}$ either u is the prefix of v or v is the prefix of u .

Proof. By definition every word $u \in L_{dep}$ is of the form $u = M(X_i)$ for some $i \geq 0$, where $X_{i+1} = A \cdot m(X_i) \cdot B$. We prove by induction that X_i is a prefix of X_{i+1} for each $i \geq 0$. For the smallest values of i we have:

$$\begin{aligned} X_0 &= \underline{AB} \\ X_1 &= \underline{ABCBADAB} \\ X_2 &= \underline{ABCDADABCD}CBADABCDABCBADAB \end{aligned}$$

Assume now that $i > 2$ and X_{i-1} is a prefix of X_i , i.e $X_i = X_{i-1} \cdot u$. We show that X_i is a prefix of X_{i+1} .

By definition of L_{dep} we have

$$X_{i+1} = A \cdot m(X_i) \cdot B = A \cdot m(X_{i-1} \cdot u) \cdot B = A \cdot m(X_{i-1}) \cdot m(u) \cdot B.$$

Moreover, the last letter of X_{i-1} is B . Therefore, due to Corollary 2, the first letter of u is either A or C . Hence, by definition of the morphism m , in both cases we have $m(u) = B \cdot v$. Finally, we have

$$X_{i+1} = A \cdot m(X_{i-1}) \cdot m(u) \cdot B = A \cdot m(X_{i-1}) \cdot B \cdot v \cdot B = X_i \cdot v \cdot B.$$

Due to definition of the morphism M and the fact above we obtain that $M(X_i)$ is a prefix of $M(X_{i+1})$ for each $i > 0$, and this concludes the proof.

Note that L_{dep} is an infinite set of words with strictly growing lengths. Therefore, for any $k > 0$ there exists a word $w \in L_{dep}$ such that $|w| > k$. This observation, together with Lemma 3, constitutes the correctness of the definition of the infinite word $\tilde{\mathbf{v}}$.

Definition 10. We define the infinite word $\tilde{\mathbf{v}} = \lim_{i \rightarrow \infty} M(X_i)$, or equivalently as $\tilde{\mathbf{v}} = \sup(L_{dep})$.

The study of combinatorial properties of the word defined above (including its Θ -square-freeness) is the matter of the following sections.

5.2 Recurrent definition

In this subsection we define two sequences of words using recurrence. Furthermore, at the end of this section, we show that one of them is equivalent to the sequence $\{X_i\}_{i \geq 0}$ defined previously. We start with defining the operation of so-called complement for letters.

Definition 11. We define the operation $\wedge: \{A, B, C, D\} \rightarrow \{A, B, C, D\}$ as follows:

$$\wedge = \begin{cases} A \rightarrow B, & B \rightarrow A \\ C \rightarrow D, & D \rightarrow C \end{cases}.$$

The mapping defined above is in a natural way extended to the level of words. It allows us to define two recurrent sequences of words.

Definition 12. We define the sequences of words $\{Y_i\}_{i \geq -1}$ and $\{S_i\}_{i \geq 0}$ over the alphabet $\Delta = \{A, B, C, D\}$ as follows:

$$Y_{-1} = \varepsilon, \quad Y_0 = AB, \quad S_0 = C,$$

$$Y_{n+1} = Y_n S_n \hat{Y}_n \hat{S}_n Y_n, \quad S_{n+1} = S_n \hat{Y}_{n-1} \hat{S}_n Y_{n-1} S_n.$$

Example 2. The first few elements of sequences defined above are as follows:

$$\begin{aligned} Y_0 &= AB, & S_0 &= C, & Y_1 &= ABCBADAB, & S_1 &= CDC, \\ Y_2 &= ABCBADABCD CBADABCBADCD ABCBADAB, \\ S_2 &= CDCBADCD ABCDC. \end{aligned}$$

□

The following facts describe some of the combinatorial properties of the sequences defined above.

Lemma 4. *For each i , the word S_i is a palindrome and the word Y_i is a pseudo-palindrome, i.e. for each $1 \leq k \leq l$ we have $Y[l - k + 1] = \widehat{Y}[k]$, where $l = |Y_i|$.*

Proof. We prove the above lemma by induction on i . For $i \leq 2$ the statement is true, see Example 2 for details.

Let us assume now that the statement is true for each $j \leq i$, and denote $l_Y = |Y_{n+1}|$, $l_S = |S_{n+1}|$, $l_1 = |Y_n| = |\widehat{Y}_n|$, $l_2 = |S_n| = |\widehat{S}_n|$, $l_3 = |Y_{n-1}| = |\widehat{Y}_{n-1}|$. By Definition 12, we have $Y_{n+1} = Y_n S_n \widehat{Y}_n \widehat{S}_n Y_n$. From the induction assumption $\widehat{Y}_n$ is a pseudo-palindrome. Moreover, for $1 \leq k \leq l_1$ we have

$$Y_{n+1}[k] = Y_n[k] = \widehat{Y}_n[l_1 - k + 1] = \widehat{Y}_{n+1}[l_Y - k + 1],$$

and for $l_1 + 1 \leq k \leq l_1 + l_2$ we have

$$Y_{n+1}[k] = S_n[k - l_1] = \widehat{S}_n[l_Y - k - l_1 + 1] = \widehat{Y}_{n+1}[l_Y - k + 1].$$

Hence, Y_{n+1} is a pseudo-palindrome.

Similarly, by Definition 12, we have $S_{n+1} = S_n \widehat{Y}_{n-1} \widehat{S}_n Y_{n-1} S_n$. From the induction assumption S_n is a palindrome, hence $\widehat{S}_n$ is also a palindrome. Moreover, for $1 \leq k \leq l_2$ we have

$$S_{n+1}[k] = S_n[k] = S_n[l_2 - k + 1] = S_{n+1}[l_S - k + 1],$$

and for $l_2 + 1 \leq k \leq l_1 + l_2$ we have

$$S_{n+1}[k] = \widehat{Y}_n[k - l_2] = Y_n[l_S - k - l_2 + 1] = S_{n+1}[l_S - k + 1].$$

Hence, S_{n+1} is a palindrome, and this completes the proof.

Lemma 5. *Let Y_i be as in Definition 12 and X_i as in Definition 9. Then for each $i \geq 0$ we have $X_i = Y_i$.*

Proof. Note that, directly from Definition 7 and Definition 11, $m(\widehat{X}) = \widehat{m(X)}$. We prove by induction two equalities, $X_i = Y_i$ and $BS_{i+1}B = m(S_i)$. First, observe that $X_0 = AB = Y_0$ and $BS_1B = BCDCB = m(S_0)$. Assume now that for all $k \leq i$ we have $X_k = Y_k$ and $BS_{k+1}B = m(S_k)$. Then

$$\begin{aligned} \mathbf{X}_{i+1} &= A \cdot m(X_i) \cdot B = A \cdot m(Y_i) \cdot B \\ &= A \cdot m\left(Y_{i-1} \cdot S_{i-1} \cdot \widehat{Y_{i-1}} \cdot \widehat{S_{i-1}} \cdot Y_{i-1}\right) \cdot B \\ &= A \cdot m(Y_{i-1}) \cdot m(S_{i-1}) \cdot m(\widehat{Y_{i-1}}) \cdot m(\widehat{S_{i-1}}) \cdot m(Y_{i-1}) \cdot B \\ &= A \cdot m(Y_{i-1}) \cdot B \cdot S_i \cdot B \cdot m(\widehat{Y_{i-1}}) \cdot A \cdot \widehat{S_i} \cdot A \cdot m(Y_{i-1}) \cdot B \\ &= A \cdot m(X_{i-1}) \cdot B \cdot S_i \cdot B \cdot m(\widehat{X_{i-1}}) \cdot A \cdot \widehat{S_i} \cdot A \cdot m(X_{i-1}) \cdot B \\ &= X_i \cdot S_i \cdot \widehat{X_i} \cdot \widehat{S_i} \cdot X_i = Y_i \cdot S_i \cdot \widehat{Y_i} \cdot \widehat{S_i} \cdot Y_i \\ &= \mathbf{Y}_{i+1} \end{aligned}$$

and

$$\begin{aligned}
\mathbf{m}(\mathbf{S}_{i+1}) &= m(S_i \cdot \widehat{Y_{i-1}} \cdot \widehat{S_i} \cdot Y_{i-1} \cdot S_i) \\
&= m(S_i) \cdot m(\widehat{Y_{i-1}}) \cdot m(\widehat{S_i}) \cdot m(Y_{i-1}) \cdot m(S_i) \\
&= B \cdot S_{i+1} \cdot B \cdot m(\widehat{X_{i-1}}) \cdot A \cdot \widehat{S_{i+1}} \cdot A \cdot m(X_{i-1}) \cdot B \cdot S_{i+1} \cdot B \\
&= B \cdot S_{i+1} \cdot \widehat{X_i} \cdot \widehat{S_{i+1}} \cdot X_i \cdot S_{i+1} \cdot B \\
&= B \cdot S_{i+1} \cdot \widehat{Y_i} \cdot \widehat{S_{i+1}} \cdot Y_i \cdot S_{i+1} \cdot B \\
&= \mathbf{B} \cdot \mathbf{S}_{i+2} \cdot \mathbf{B}
\end{aligned}$$

and this completes the proof.

The above fact allows us to formulate the following corollary.

Corollary 3. *Let us define a language $L_Y = \{M(Y_i) : i \geq 0\}$. Then the word $\tilde{\mathbf{v}}$ can be equivalently defined as $\tilde{\mathbf{v}} = \sup(L_Y)$.*

Taking into account the images of the words Y_i and S_i by morphism M we can formulate the following fact, which will be very useful further.

Proposition 2 (Block lengths). *Let Y_i and S_i be as defined above. Then for each $i \geq 1$ we have:*

$$|M(Y_i)| = \frac{4(4^{i+1} - 1)}{3} \quad \text{and} \quad |M(S_i)| = \frac{4(2 \cdot 4^i + 1)}{3}.$$

Proof. Let us denote $y_i = |M(Y_i)|$ and $s_i = |M(S_i)|$. By Definition 7 we have $M(A) = ab$, $M(B) = ac$ and $M(C) = abcb$. Therefore, by Definition 12, to prove the statement of the proposition it is sufficient to solve a recurrent equation:

$$\begin{cases} y_{n+1} = 3y_n + 2s_n \\ s_{n+1} = 3s_n + 2y_{n-1} \end{cases}$$

with starting conditions given as $y_{-1} = 0$, $y_0 = 4$ and $s_0 = 4$.

6 Combinatorial properties of the word $\tilde{\mathbf{v}}$

In this section we formulate and prove the main results of the paper. Namely, we show the Θ -square-freeness (Theorem 3) and lexicographical minimality (Theorem 4) of the word $\tilde{\mathbf{v}}$ and the time complexity of the computing the n -th letter of $\tilde{\mathbf{v}}$ (Theorem 5). The proof of the latter yields in fact a very efficient procedure.

We start with a series of facts which lead to the proof of Θ -square-freeness of $\tilde{\mathbf{v}}$. Let us recall languages L_m and $L_{dep} = M(L_m)$ from Definition 9.

Lemma 6. *Let $v \in L_m$. Then v does not contain a factor of the form $y \cdot w \cdot x \cdot w \cdot y$, where $w \in \Delta^*$, and $(x = A \wedge y = C)$ or $(x = B \wedge y = D)$.*

Proof. Let us assume that there exist a word $v \in L_m$ containing a factor of the form $u = y \cdot w \cdot x \cdot w \cdot y$. Without the loss of generality, we can assume that $(x = A \wedge y = C)$ (the case $(x = B \wedge y = D)$ is symmetric).

Note that, by definition of the set L_m , there exist a word $v_0 \in L_m$ such that $m(v_0) = v$. Moreover, the images of letters from v_0 may contain the letter A only as the first or the last element.

We consider two cases: (1) $x = A$ is the first letter of an image of a letter from v_0 and (2) $x = A$ is the last letter of an image of a letter from v_0 .

In the first case the suffix of w is a full m -image of a letter from v_0 . Hence, by the fact that m is a bifix code, $C \cdot \beta$ is an image of a factor from v_0 , which is in contradiction with the definition of the morphism m , because no such image starts with C .

In the second case the prefix of w is a full m -image of a letter from v_0 . Hence, by the fact that m is a bifix code, $\alpha \cdot C$ is an image of a factor from v_0 , which is in contradiction with the definition of the morphism m , because no such image ends with C , and this completes the proof.

Proposition 3. *The language L_m consists of square-free words only.*

Proof. Suppose that w is the shortest word from the language L_m containing a square, i.e. $w = \alpha \cdot u \cdot u \cdot \beta$.

First, note that u cannot consist of full images of the morphism m . Otherwise, $m^{-1}(w)$ would contain a square $m^{-1}(u) \cdot m^{-1}(u)$, which contradicts the minimality of the length of w .

(*) Note that u cannot start in the same way as β and cannot end in the same way as α (forming full images of the morphism m). Otherwise w contains a square consisting of full images, which is impossible due to observation above.

We consider two cases: (1) u starts with A and (2) u starts with C . The remaining cases (u starts with B and u starts with D) can be considered similarly.

(1): In this case u ends either with AD or with $ADCD$, hence, by (*), β starts with CDA , which excludes $ABCD$ as a suffix of u . Using (*) once more, we conclude that α ends with $ADCD$. This implies that

$$w = \alpha_1 \cdot ADCDA \cdot u_1 \cdot ADA \cdot u_1 \cdot ADCDA \cdot \beta_1.$$

Hence, $m^{-1}(w)$ contains a factor $D \cdot m^{-1}(u_1) \cdot B \cdot m^{-1}(u_1) \cdot D$, which is in contradiction with Lemma 6.

(2): Observe that u cannot start with $CDCB$, because in such a case both u and α ends with B , which is in contradiction with (*). Therefore, u starts with CB . Moreover, u cannot end with DCB , because in such a case β starts with CB , which is in contradiction with (*). Hence, u ends with B and β starts with $CDCB$. This implies that

$$w = \alpha_1 \cdot BCDCB \cdot u_1 \cdot BCB \cdot u_1 \cdot BCDCB \cdot \beta_1.$$

Hence, $m^{-1}(w)$ contains a factor $C \cdot m^{-1}(u_1) \cdot A \cdot m^{-1}(u_1) \cdot C$, which is in contradiction with Lemma 6.

Lemma 7. $L_m \subseteq \mathcal{T} = \left((A|C)(B|D)\right)^*$.

Proof. First note that directly by definition of the morphism m , if $x \in \mathcal{T}$ then $m(x) \in \mathcal{T}$ (the images of letters A and C starts and ends with the letter B , while the images of the letters B and D starts and ends with the letter A).

We have $m^0(AB) = AB \in \mathcal{T}$, and the rest of the proof is a simple induction.

Lemma 8. Let $v_1 \in L_m$. Suppose that the word v_1 contains the factor of the form $u_1 = w_1x_1w_1y_1$, where $w_1 \in \Delta^*$ and $(x_1 = A \wedge y_1 = C)$ or $(x_1 = B \wedge y_1 = D)$. Then there exists a word $v_0 \in L_m$ containing the factor of the form $u_0 = w_0x_0w_0y_0$, where $w_0 \in \Delta^*$ and $(x_0 = B \wedge y_0 = D)$ or $(x_0 = A \wedge y_0 = C)$ respectively, and $|w_0| < |w_1|$.

Proof. Let us assume that there exist a word $v_1 \in L_m$ containing a factor of the form $u_1 = w_1x_1w_1y_1$. Without the loss of generality, we can assume that $(x_1 = A \wedge y_1 = C)$ (the case $(x_1 = B \wedge y_1 = D)$ is symmetric).

Note that, by definition of the set L_m , there exist a word $v_0 \in L_m$ such that $m(v_0) = v_1$. Moreover, the images of letters from v_0 may contain the letter A only as the first or the last element.

We consider two cases: (1) $x_1 = A$ is the first letter of an image of a letter from v_0 and (2) $x_1 = A$ is the last letter of an image of a letter from v_0 .

(1): In this case, the last three or five letters of w_1 are the image of a letter from v_0 . Hence, due to the fact that m is a bifix code, Aw_1 is an image of factor of v_0 . As a result we obtain that $y = C$ is the first letter of an image of a letter from v_0 , which is in contradiction with the definition of the morphism (Definition 9).

(1): In this case, the first three or five letters of w are the image of a letter from v_0 . Hence, due to fact that m is a bifix code, wA is an image of factor of v_0 . Now we consider two situations.

(2a): If w_1Aw_1C is not a prefix of v_1 , then by Lemma 7 and Proposition 3, the factor w_1Aw_1C is preceded by C , which is in contradiction with Lemma 6.

(2b): If w_1Aw_1C is a prefix of v_1 , then following similar reasoning as in (1) we have that w_1 ends with AD ($w_1 = w'AD$). By Definition 9, the factor w_1Aw_1C has to be appended by DA (the extension of ADC to the full image of the morphism m). Finally, $w'ADAw'ADCDA$ is a factor (prefix) of v_1 which is an image of a word $u_0 = w_0Bw_0D$, which is a factor (prefix) of v_0 , and this completes the proof.

Proposition 4. Let $v \in L_m$. Then v does not contain a factor of the form wxy , where $w \in \Delta^*$, and $(x = A \wedge y = C)$ or $(x = B \wedge y = D)$.

Proof. Suppose that $v_1 \in L_m$ contain the factor of the form w_1xw_1y , where $w_1 \in \Delta^*$, and $(x = A \wedge y = C)$ or $(x = B \wedge y = D)$. By Lemma 8, there exists another word $v_2 \in L_m$ containing the factor of the form $u_2 = w_2x_2w_2y_2$, where $w_2 \in \Delta^*$ and $(x_2 = B \wedge y_2 = D)$ or $(x_2 = A \wedge y_2 = C)$, and $|w_2| < |w_1|$. Iterating this reasoning we obtain an infinite sequence of strictly decreasing positive integers, which is an obvious contradiction. It means that our initial supposition was wrong and completes the proof.

Proposition 5. *The language $M(L_m)$ consists of square-free words only.*

Proof. Note that, by definition, each word from $M(L_m)$ is M -reducible. Suppose that a word $w \in M(L_m)$ contains a square uu (i.e. $w = v_1 u u v_2$).

First we show that a has to be the first letter of a square. Suppose that u does not start with a and consider the first occurrence of a inside u . It implies immediately non-emptiness of v_1 . Without the loss of generality we can assume that the leftmost a in u is preceded by the letter b . Due to M -reducibility of w , we obtain that u starts with one of the factors (1) ba , (2) cba or (3) $bcba$.

(1): In this case u may end either with a or with abc (since an M -reducible block has to appear on the intersection of two consecutive u 's). Note that this implies that the factor v_2 cannot be empty. By definition of the morphism m , the first letter of v_2 has to be b , and we can find another square $(u'bu'b$, where $u = bu')$ shifted one position to the right and starting with the letter a .

(2): In this case both u and v_1 have to end with ab , and we can find another square by shifting the factor uu two positions to the left.

(3): In the case both u and v_1 have to end with a , and we can find another square by shifting uu one position to the left.

Since u starts with the letter a , we obtain that u is M -reducible. Therefore, in the M -reduction of w we can find a square $M^{-1}(u) \cdot M^{-1}(u)$, if v_2 starts with the letter a , or one of the factors $u'Au'C$ or $u'Bu'D$, if v_2 does not start with a . Due to Proposition 4 and Proposition 3 we obtain a contradiction which concludes the proof.

Proposition 6. *The language L_{dep} consists of square-free words only.*

Proof. Let $L_{dep} = \{u_i : i \geq 0\}$ and $M(L_m) = \{v_i : i \geq 0\}$. The key observation is that for each $i > 0$ we have $u_i = \alpha_i \cdot v_i \cdot \beta_i$, where $|\alpha| = |\beta| \neq 0$. Therefore, by Definition 12 and Definition 9 we have

$$u_{i+1} = \alpha_{i+1} \cdot v_{i+1} \cdot \beta_{i+1} = u_i \cdot z_i \cdot \hat{u}_i \cdot \hat{z}_i \cdot u_i,$$

where $z_i = M(S_i)$. Suppose that u_i contains a square. Then also $\hat{u}_i$ contains a square. Due to Definition 9 we have $|\alpha_{i+1}| < |u_i \cdot z_i|$. This implies that v_{i+1} contains $\hat{u}_i$ as a factor and this contradicts its square-freeness (Proposition 5). Therefore, we conclude that each word u_i for $i \geq 0$ is square-free.

Theorem 3. *The word $\tilde{\mathbf{v}}$ is an infinite Θ -square-free word.*

Proof. The statement follows from Theorem 1 and Proposition 6.

Theorem 4. *The word $\tilde{\mathbf{v}}$ is the lexicographically minimal infinite Θ -square-free word.*

Proof. The Θ -square-freeness of $\tilde{\mathbf{v}}$ follows from Theorem 3.

Suppose that there exists an infinite Θ -square-free word $\tilde{\mathbf{w}}$ that is lexicographically smaller than $\tilde{\mathbf{v}}$. Then, by the analysis of short Θ -square-free words, $\tilde{\mathbf{w}}$ has to start with $abacabcabaca$. Moreover, due to Theorem 2 the word $\tilde{\mathbf{w}}$ is

M -reducible. Let us consider $u \in \Sigma^*$ – the longest common M -reducible prefix of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$. Moreover, let $X, Y \in \Delta$ be such that $v = uM(X)$ and $w = uM(Y)$ are prefixes of $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{w}}$, respectively. We have that $M(X) > M(Y)$. Potentially there are 6 cases for X, Y , however from our previous results it follows that the only cases to consider are:

$$(X, Y) = (C, A), \text{ or } (X, Y) = (D, B).$$

Precisely, w, v and u satisfy all assumptions of Lemma 2, hence indeed, one of the following conditions holds:

1. $M^{-1}(v) = M^{-1}(u)C$ and $M^{-1}(w) = M^{-1}(u)A$,
2. $M^{-1}(v) = M^{-1}(u)D$ and $M^{-1}(w) = M^{-1}(u)B$.

Without the loss of generality, we can assume the first case. According to Lemma 1, the last meta-letter of $M^{-1}(u)$ is either B or D . We deal with each of those cases separately.

1° ($M^{-1}(u)$ ends with B): Due to the morphic definition of $\tilde{\mathbf{v}}$, the last but one letter of $M^{-1}(u)$ is A . Hence, the word $M^{-1}(w)$ contains a forbidden factor ABA . Therefore, by Lemma 1, the infinite word $\tilde{\mathbf{w}}$ cannot be Θ -square-free.

2° ($M^{-1}(u)$ ends with D): Following similar reasoning as above, we obtain that $M^{-1}(u)$ ends with BCD . Hence, the word $M^{-1}(w)$ contains a forbidden factor $BCDA$. Therefore, due to Lemma 1, the infinite word $\tilde{\mathbf{w}}$ cannot be Θ -square-free.

The contradictions obtained above prove that the initial assumption concerning the existence of $\tilde{\mathbf{w}}$ was wrong. Therefore, $\tilde{\mathbf{v}}$ is indeed the lexicographically minimal infinite Θ -square-free word.

Theorem 5. *For each $n > 0$ the n -th letter of the word $\tilde{\mathbf{v}}$ can be determined in time $O(\log n)$.*

Proof. To prove the above lemma we present a simple recurrent procedure.

For given $n \geq 0$ we find the shortest word Y_i of length $l \geq n$, where Y_i 's are as in Definition 12. Due to Proposition 2, both the index i and length l are given by simple arithmetic formulas. By Definition 12, the word Y_i consists of five factors (either Y_{i-1} or S_{i-1} or their complements) with lengths given by Proposition 2. Thus, we can determine the factor $F \in \{Y_{i-1}, \widehat{Y}_{i-1}, S_{i-1}, \widehat{S}_{i-1}\}$ containing the considered position k in a constant time. It remains to determine (using a recurrent call) the letter in F on a position n' , which is obtained by subtracting from k the starting position of F . The recurrence stops when $i = 1$ and we have F equal to one of the words $M(Y_1)$, $M(S_1)$, $M(\widehat{Y}_1)$ or $M(\widehat{S}_1)$.

Note that at each call of the recurrence it is necessary to memorize whether we are looking for a letter in one of words ($M(Y_1)$ or $M(S_1)$) or their complements ($M(\widehat{Y}_1)$ or $M(\widehat{S}_1)$). It could be done by using a single boolean variable.

It is easy to see that the number of iterations performed by the procedure described above is logarithmic with respect to n . Moreover, the required computations on each level of recurrence can be performed in a constant time.

7 Final remarks

In partially commutative alphabets of size three with one pair of commuting letters one can consider three commutation relations:

From the point of view of the partially abelian square-freeness all above alphabets are equivalent. We considered in this paper square-free words over the partially commutative alphabet Θ_3 , but any Θ_3 -square-free word could be transformed by a morphism (precisely an isomorphism) to a Θ_1 -square-free or Θ_2 -square-free word and almost all the results follow.

However, if we are interested in construction of the lexicographically minimal partially abelian square-free word, the choice of the alphabet is very important. Such a choice determines the blocking letter and the structure of the lexicographically minimal word. In the case of alphabets Θ_1 and Θ_2 it requires further investigation.

In [2] Allouche and Shallit presented an open problem of characterizing the lexicographically minimal square-free word over three-letter alphabet without any commutation allowed. The construction of Thue (see [18]) leads to a word which is not lexicographically minimal.

On the other hand, there is a procedure, proposed by Currie [12], which allows to determine if a given finite word is a prefix of an infinite word avoiding some repetitions. It immediately gives an algorithm computing arbitrary long prefix of the lexicographically least infinite word. However, generating n -th letter is definitely not logarithmic with respect to n . Moreover, in the case of square-freeness, it seems to be directly applicable for alphabets consisting more than four letters.

Another problem related to square-freeness is the overlap-freeness (i.e., avoiding pattern $axaxa$, where a is a letter and x is a word). Berstel proved [3], (see also [4]), that the lexicographically *greatest* infinite overlap-free word on the binary alphabet $\Sigma = \{0, 1\}$ that begins with 0 is the Thue-Morse overlap-free sequence τ .

Moreover, it has been shown in [1] that the lexicographically least infinite overlap-free binary word is $001001\bar{\tau}$, where $\bar{\tau}$ is the negation of overlap-free Thue-Morse word τ . This makes the problem of extremal cases for overlap-freeness closed. However, its solution relies on Thue-Morse word, which is a fix point of a morphism. This supports the claim that there is also an efficient construction of the lexicographically least square-free word over a ternary alphabet without commutation. We believe that the techniques utilized in this paper might be helpful in finding such a construction.

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