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Feynman Diagrams for Stochastic Inflation and Quantum Field Theory in de Sitter Space

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Abstract

We consider a massive scalar field with quartic self-interaction $\lambda/4! \phi^4$ in de Sitter spacetime and present a diagrammatic expansion that describes the field as driven by stochastic noise. This is compared with the Feynman diagrams in the Keldysh basis of the Amphichronous (Closed-Time-Path) Field Theoretical formalism. For all orders in the expansion, we find that the diagrams agree when evaluated in the leading infrared approximation, *i.e.* to leading order in m^2/H^2 , where m is the mass of the scalar field and H is the Hubble rate. As a consequence, the correlation functions computed in both approaches also agree to leading infrared order. This perturbative correspondence shows that the stochastic Theory is exactly equivalent to the Field Theory in the infrared. The former can then offer a non-perturbative resummation of the Field Theoretical Feynman diagram expansion, including fields with $0 \leq m^2 \ll \sqrt{\lambda} H^2$ for which the perturbation expansion fails at late times.

1 Introduction

The stochastic approach to Inflation [1, 2] is a simple and effective framework that can be used in order to evaluate correlation functions of scalar fields in de Sitter space on scales exceeding the horizon. It can be derived from the underlying Field Theoretical formulation, by treating the short-wavelength modes as quantum noise to the horizon-size field which is described as a classical random variable. This is justified by the fact that the canonical commutator (between the field and the canonical momentum) estimated within the stochastic framework is small compared to the anti-commutator, *i.e.* by the usual criterion for the classical behaviour of a dynamic system. The resulting random walk of the scalar field (on top of the solution to the deterministic equation of motion) does not only offer valuable intuition for understanding the field evolution and the emergence of classical stochastic perturbations in the Universe, it is also useful in

order to derive quantitative results [3–7]. Nevertheless, the stochastic field dynamics – defined by the Starobinsky Equation (1) – is an approximation to the underlying formulation in terms of Quantum Field Theory (QFT) that has remained somewhat obscure, despite previous works [8–10].

Some questions that one might still raise in the stochastic formulation are:

- The fact that the canonical commutator is much smaller than the anticommutator is required for the self-consistency of the stochastic approach. This has been shown for the modes of linearized (non-interacting) fields [1, 11] but can this be validated within a QFT calculation that systematically includes at least the leading non-linearities?
- How can the effective stochastic approach be identified in terms of a truncation of the QFT calculation at a certain order in an expansion parameter? Can this procedure be strictly justified?
- The separation into horizon-size and short-wavelength modes breaks de Sitter invariance. Can the stochastic results be confirmed in a framework that treats all field modes on the same footing?

A powerful method to approach these questions is to set up the problem in Euclidean de Sitter space. For a massless scalar field with quartic interactions, the leading infrared (IR) expansion for long-wavelength correlators corresponds to functionally integrating over the constant mode (*i.e.* the zero mode) of the field. Since Euclidean de Sitter is compact, this reduces to a one-dimensional integration [12]. It can also be observed that thus simplified functional integrals coincide with the integrals over the probability distribution functions in Stochastic Inflation, see Ref. [13]. Moreover, it has been demonstrated in [13] that for Schwinger-Dyson equations derived from a two-particle-irreducible effective action, the solutions for the two-point functions to leading IR order take the form of free propagators with a dynamical mass. The resummation of an infinite class of self-energy diagrams is then necessary in order to recover the results from Stochastic Inflation or the functional integration of the constant mode. Beyond the leading IR approximation, two-point functions for massive scalar fields on Euclidean de Sitter space to all orders in perturbation theory have been investigated in Refs. [14, 15], where it is found that these are well defined and that in particular, the field correlations exhibit an exponentially decaying behaviour for large separations. However, the decay at large Euclidean distances could not yet be proved for the massless case [16]. Therefore, calculations of leading IR effects in the massless theory presently have to rely on the assumption that the correlation functions computed without truncations are well-defined after all.

While the aforementioned investigations in Euclidean space provide some substantial insights into interacting scalar field theory on de Sitter, calculations in a spacetime with Lorentzian signature, as performed in the present work, are crucial to address the following important points which are beyond the scope of Euclidean methods:

- Depending on the initial conditions, the correlation functions of light or massless fields exhibit a transient growth or decay, *i.e.* the two point function evolves

proportional to the number of e-folds, before reaching an “equilibrium state”. This can have important consequences on the evolution of light fields during inflation and subsequently, in the Early Universe. Clearly, this important feature requires calculations in the Lorentzian spacetime.

- The analytic continuation between Euclidean and Lorentzian spacetimes requires that the expansion of the latter is exactly of the de Sitter form. Cosmic inflation does however break de Sitter invariance, due to its definite end along a spatial hypersurface, and potentially due to a beginning at a finite time. Also while inflation takes place, there is a difference from de Sitter expansion, indicated by the observed deviation of the spectral index of the scalar power spectrum from unity [17].
- Even though there are theoretical arguments supporting the assertion that the late-time limit of correlations in Lorentzian de Sitter space can be obtained by analytic continuation of the Euclidean results [18], it is desirable to demonstrate and to understand this agreement through a direct calculation of the Lorentzian observables within a QFT framework.

The appropriate tool for computing the time evolution of quantum correlation functions is the Closed-Time-Path (CTP) formalism of QFT [19, 20] (see *e.g.* [21] or [22, 23] for functional integral developments of the formalism and *e.g.* [24–29] for cosmological applications), also known as the Schwinger-Keldysh or the “in-in” formalism. Since it involves two branches of dynamical evolution, forward and backward in time, we find it appropriate to also refer to it as Amphichronous QFT.

Perturbation theory on inflationary backgrounds with Amphichronous QFT can become complicated and, most importantly, breaks down at late times for (almost) massless fields due to the growth of individual terms in the series. On the other hand, stochastic methods address both these problems because calculations in that framework are comparably simple, and they offer a particularly convenient way of performing the necessary resummations [8]. Thus, a refined picture of the stochastic dynamics and its relation to the underlying Lorentzian Amphichronous QFT is essential for understanding the behaviour of scalar fields during inflation. In a number of computations reported in the literature, various results for interacting Field Theory on de Sitter space in the presence of IR enhanced correlations are derived. Some of the different truncations and also resummation strategies that have been applied are *e.g.* usual perturbation theory [30–38], the Hartree approximation [39–41], a $1/N$ expansion in $O(N)$ -symmetric theories [5, 36, 42–45], the Wigner-Weisskopf method [46], functional renormalization group techniques [47] or other partial resummation schemes [48]. These approaches however do not appear to recover the non-perturbative resummation that is readily performed in the stochastic approach.

Here, we therefore aim to elucidate the nature of the stochastic description by demonstrating its perturbative equivalence to the underlying QFT in the IR. For this purpose, we choose the simple setting of a self interacting scalar field in de Sitter space, but our

arguments should hold for more general inflationary spacetimes and Field Theories. Our results imply that indeed, stochastic inflation fully captures the leading IR behaviour of the underlying QFT and is safe for field theorists to use. From this we can also conclude that the stochastic resummation applies in the IR to QFT on de Sitter space as well, thus proving a conjecture stated *e.g.* in Refs. [8, 30].

The outline of this paper is as follows: In Section 2, we review and present further details on the diagrammatic expansion of the correlation functions from Stochastic Inflation that has been introduced in Refs. [35, 49]. Turning to the Field Theory approach, the elementary propagators as building blocks for Amphichronous Feynman diagrams in the Keldysh representation are presented in Section 3. The Feynman diagrams are then evaluated and brought to a form that can be compared with their stochastic counterparts. The necessary approximations to leading IR order are carefully justified, and this central part of the present work is presented in Section 4. Having demonstrated the perturbative equivalence between QFT and the stochastic approach in the IR, we then proceed in Section 5 to discuss the non-perturbative late-time limit of the stochastic dynamics where the field reaches an equilibrium state. We thus conclude that the non-perturbative equilibrium distribution that is attained by the field corresponds to the correct late-time resummation of the QFT series. This resummation also works for very light fields for which the QFT expansion fails completely at late times. We conclude in Section 6 by summarizing our results and touching upon the issue of de Sitter invariance in the stochastic formalism. The notations and conventions used here are in line with Refs. [34, 35].

2 Diagrammatic Expansion of Correlators in the Stochastic Approach

We consider a real scalar field in four spacetime dimensions within the expanding half of de Sitter space, also known as the Poincaré patch, see *e.g.* Ref. [50] for a comprehensive discussion of the properties and parameterizations of this spacetime. The metric is therefore fully parameterized through the value of the Hubble rate H . Moreover, in comoving coordinates, it is manifest that the spatial sections are flat such that we may seek spatially homogeneous solutions. In terms of this coordinate choice, the stochastic approach consists of separating long-wavelength modes from short-wavelength ones. Due to their IR enhancement, the long modes can be treated as classical random variables that can effectively be described by Langevin dynamics [1, 2], *i.e.* they obey the Starobinsky Equation

$$\dot{\phi} + \frac{\partial_{\phi} V}{3H} = \xi(t), \quad (1)$$

where $\xi(t)$ is a stochastic noise term that originates from integrating out the short-wavelength modes, and where a dot denotes a derivative with respect to the comoving time t . The noise is Gaussian and white, such that it is fully determined by its two point

correlation function, which reads

$$\langle \xi(t)\xi(t') \rangle = \frac{H^3}{4\pi^2} \delta(t - t'), \quad (2)$$

where the expectation value $\langle \dots \rangle$ denotes the average over noise realizations. The field ϕ in Eq. (1) should be understood as the average over a patch of physical size $\sim H^{-1}$ with the stochastic noise acting on each of these patches with practically zero correlation between different patches.

In order to cast the stochastic approach into a form that can be readily connected with Field Theoretic elements we employ functional (path integral) techniques, making use of the formalism and the diagrammatic representation developed in Refs. [35, 49]. In the functional formulation of the Starobinsky Equation (1), expectation values of any function $\mathcal{O}[\phi]$ w.r.t. the different realizations of the stochastic force ξ can be obtained from the following path integral

$$\langle \mathcal{O}[\phi] \rangle = \int D[\xi] e^{-\frac{1}{2} \int dt \xi^2 \frac{4\pi^2}{H^3}} \int D[\phi] \mathcal{O}[\phi] \delta(\dot{\phi} + \partial_\phi V/3H - \xi) \mathcal{J}[\phi], \quad (3)$$

where $\mathcal{J}[\phi]$ is the Jacobian of the argument of the delta function with respect to the “integration variable” ϕ : $\mathcal{J}[\phi] = \left| \text{Det} \left[\frac{\delta}{\delta\phi} \left(\dot{\phi} + \partial_\phi V/3H - \xi \right) \right] \right|$. The integration over the noise ξ reflects the assumption that the latter is Gaussian. To compute the determinant, we discretize the time interval in N time steps of extent Δt such that $\phi_i = \phi(t_i)$ and $\xi_i = \xi(t_i)$ with $i = 0, \dots, N$. Then the determinant can be written $\mathcal{J} = |\text{Det} \mathcal{J}_{ij}|$, where

$$\mathcal{J}_{ij} = \frac{\partial}{\partial \phi_j} \left(\frac{\phi_i - \phi_{i-1}}{\Delta t} + \frac{\partial_\phi V(\phi_{i-1})}{3H} - \xi_{i-1} \right). \quad (4)$$

It is important to note that we have chosen a retarded regularization for the operator, *i.e.* all functions of ϕ and the stochastic force ξ are calculated at the start of each timestep. With this choice, $\mathcal{J}_{ij}$ is $\frac{1}{\Delta t}$ on the diagonal and the only other non-zero elements are the $\mathcal{J}_{i,i-1}$ entries. Absorbing the $\frac{1}{\Delta t}$ factors in the measure, we see that $\mathcal{J}[\phi] = 1$. Note that, with the appropriate normalization of the measure $D[\xi]$, the integral is a realization of the identity: $\langle \mathbb{I} \rangle = 1$. Nevertheless, it is convenient to make use of the partition function

$$\mathcal{Z} \equiv \int D[\xi] e^{-\frac{1}{2} \int dt \xi^2 \frac{4\pi^2}{H^3}} \int D[\phi] \delta(\dot{\phi} + \partial_\phi V/3H - \xi) = 1, \quad (5)$$

then to introduce an auxiliary field ψ in order to replace the δ function with a functional Fourier integral and eventually to integrate out the noise ξ . The partition function then reads¹

$$\mathcal{Z} = \int D[\phi] D[\psi] e^{-\int dt \left\{ \frac{i}{H^2} \psi \left(\dot{\phi} + \frac{\partial_\phi V}{3H} \right) + \frac{1}{8\pi^2 H} \psi^2 \right\}}. \quad (6)$$

¹Here, the auxiliary field ψ has a different mass dimension than in previous work [35, 49], such that the propagators $G^{R,A}$ and F all have mass dimension *two*.

Adding couplings $-i \int dt J_\psi \psi$ and $-i \int dt J_\phi \phi$ with currents in the exponent defines $\mathcal{Z}[J_\psi, J_\phi]$ from which n -point functions can be computed in the usual way by taking derivatives with respect to $-iJ_\psi$ and $-iJ_\phi$. Note that, unlike the standard QFT partition function, $\mathcal{Z}[0, 0] = 1$, and expectation values can be obtained directly from $\mathcal{Z}$ without receiving multiplicative contributions from vacuum bubbles.

In order to prepare for a perturbative expansion, we replace $\dot{\phi}\psi \rightarrow \frac{1}{2}(\dot{\phi}\psi - \phi\dot{\psi})$ under the integral, and we decompose the potential as $V = \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi)$. This yields

$$\begin{aligned} \mathcal{Z} &= \int D[\phi] D[\psi] e^{-i \int dt \left\{ \frac{1}{2}(\dot{\phi}, \dot{\psi}) \begin{pmatrix} 0 & \frac{1}{H^2}(-\partial_t + \frac{m^2}{3H}) \\ \frac{1}{H^2}(\partial_t + \frac{m^2}{3H}) & -\frac{i}{4\pi^2 H} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \frac{\partial_\phi V_{\text{int}}}{3H^3} \psi \right\}} \quad (7) \\ &\equiv \int D[\phi] D[\psi] e^{-i \int dt \left\{ \frac{1}{2}(\dot{\phi}, \dot{\psi}) \mathbb{G}_0^{-1} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \frac{\partial_\phi V_{\text{int}}}{3H^3} \psi \right\}}, \end{aligned}$$

where the last equality defines $\mathbb{G}_0^{-1}$. The latter is the functional and matrix inverse of the free propagator $\mathbb{G}_0$, that is given by

$$\mathbb{G}_0(t, t') = \begin{pmatrix} \langle \phi(t) \phi(t') \rangle & \langle \phi(t) \psi(t') \rangle \\ \langle \psi(t) \phi(t') \rangle & \langle \psi(t) \psi(t') \rangle \end{pmatrix} \equiv \begin{pmatrix} F(t, t') & -iG^R(t, t') \\ -iG^A(t, t') & 0 \end{pmatrix}, \quad (8)$$

where the above equality defines the free propagators $G^{R,A}$ and F . It should be emphasized that the null entry in $\mathbb{G}_0$ is a direct consequence of the definition of $\mathbb{G}_0^{-1}$. It occurs due to the fact that ψ is an auxiliary field and therefore not dynamical. Using Eqs. (7,8) and the relation $\mathbb{G}_0 \star \mathbb{G}_0^{-1}(t, t') = \mathbb{I}_{2 \times 2} \delta(t, t')$, we observe that $G^{R,A}$ are the retarded and advanced propagators for the operator $\frac{1}{H^2}(\partial_t + \frac{m^2}{3H})$, while the statistical correlator is the two-point function of the original field $F(t, t') = \langle \phi(t) \phi(t') \rangle$. These Green functions can be easily found, and they read

$$G^R(t, t') = G^A(t', t) = H^2 e^{-\frac{m^2}{3H}(t-t')} \Theta(t - t'), \quad (9a)$$

$$F(t, t') = \frac{3H^4}{8\pi m^2} \left(e^{-\frac{m^2}{3H}|t-t'|} - e^{-\frac{m^2}{3H}(t+t')} \right), \quad (9b)$$

where we have imposed the initial condition $F(0, 0) = 0$. Before going on, let us remark that $F(t, t) \approx \frac{H^3}{4\pi} t$ for $m \rightarrow 0$. This famous secular behaviour [51–54] signals the breakdown of the perturbative expansion for large enough times and masses that are small (*e.g.* $m^2 \ll \sqrt{\lambda} H^2$ for the quartic interaction assumed below) or vanishing. In order to maintain $F(t, t)$ regular also for infinitely late times, we first restrict to the case where the squared mass is strictly positive. For $\min(Ht, Ht') \gg (3H^2)/2m^2$, *i.e.* assuming that the stochastic process (equivalently inflation) has started early enough, the growth of the equal-time correlations saturates, such that $F(t, t')$ only depends on

Figure 1: The elements out of which stochastic diagrams are constructed. The choice of vertex factor implies that the assembled diagrams should be divided by their symmetry factor.

the time separation $|t - t'|$:

$$F(t, t') = \frac{3H^4}{8\pi^2 m^2} e^{-\frac{m^2}{3H}|t-t'|}. \quad (10)$$

In Section 5, we comment on the important case $m^2 = 0$.

The partition function in its form (7) readily leads to a diagrammatic expansion for the correlation functions, using the free propagators (9) as internal lines. The vertices derive from the interaction potential V_{int} , for which we take a quartic coupling $V_{\text{int}} = \frac{\lambda}{4!}\phi^4$, such that the mixing term in Eq. (7) becomes

$$\frac{\partial_\phi V_{\text{int}}}{3H^3} \psi = \frac{\lambda}{3!} \frac{\psi \phi^3}{3H^3}. \quad (11)$$

Putting these elements together, we can derive a set of Feynman rules that is presented in Figure 1 (*cf.* Ref. [35]).

Note that the “symmetric” interaction term $\psi^3\phi$ is absent in the stochastic description. This is in contrast to the corresponding QFT Feynman rules in the Keldysh representation which contain both $\psi^3\phi$ and $\psi\phi^3$ vertices - *cf.* Eq. (22) below. However, the $\psi^3\phi$ vertex is irrelevant for long wavelengths due to the IR enhancement of the field correlations: While the statistical propagator (9b) is enhanced in the IR, the retarded and advanced propagators (9a) remain regular in the limit $m^2/H^2 \rightarrow 0$. QFT diagrams with vertices connecting with three retarded or advanced propagators are therefore suppressed in powers of m^2/H^2 compared to diagrams where only one of these propagators connects to each vertex. The irrelevance of the $\psi^3\phi$ vertex can be understood as a consequence of the fact that the long modes behave classically. In the stochastic approach this vertex is therefore absent by construction.

3 Field Theoretical Approach

The agreement of the Field Theoretical and stochastic calculations for the 2-point function was previously demonstrated in Ref. [35] up to order λ^2 , a result relevant in the regime where $H^2 \gg m^2 \gg \sqrt{\lambda}H^2$, such that a perturbative truncation at some order in λ is meaningful. In that work, the QFT calculation has been performed in the basis defined by the $\pm$ branches of the CTP. We refer to this basis as the $\pm$ basis or the Wightman basis because the Wightman propagators appear there explicitly as the Green functions connecting the $+$ with the $-$ branch of the Amphichronous time evolution or vice versa. For the present purposes, it is however advantageous to work in the Keldysh basis [20] instead, where the retarded, advanced and statistical Green functions appear as the elementary propagators. The reason is twofold: a) It can readily be noted from the calculation in Ref. [35] that the expressions for the loop integrals are best arranged in terms of the Keldysh propagators. b) The structure of the free stochastic Green function (8) from Section 2 emerges naturally within the Keldysh basis, making it a preferred choice when comparing between the stochastic and Field Theoretic formalisms. In the following, to facilitate a comparison with Ref. [35], we review how to transform between the Wightman and the Keldysh bases.

In the Wightman basis, the field is divided into ϕ_+ and ϕ_- , corresponding to the forward and backward branches of the Amphichronous time-evolution. The free propagator $i\Delta$ satisfies the Klein-Gordon equation

$$a^4 (\nabla_x^2 - m^2) i\Delta^{(0)fg}(x; x') = fg\delta^{fg} i\delta^4(x - x'), \quad (12)$$

where $f, g = \pm$ are CTP indices and $(\nabla_x)_\mu$ is the covariant derivative with respect to x . The transformation from the Wightman to the Keldysh basis is performed with the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (13)$$

that acts on the field components as follows:

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = U \cdot \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix}. \quad (14)$$

For the free propagators, this implies the relation

$$U \cdot \begin{pmatrix} i\Delta^T & i\Delta^< \\ i\Delta^> & i\Delta^{\bar{T}} \end{pmatrix} \cdot U^\dagger = \begin{pmatrix} i\Delta^< + i\Delta^> & i\Delta^T - i\Delta^< \\ i\Delta^T - i\Delta^> & 0 \end{pmatrix} \quad (15)$$

$$\equiv \begin{pmatrix} F(x, x') & -iG^R(x, x') \\ -iG^A(x, x') & 0 \end{pmatrix}, \quad (16)$$

where the last equality defines the statistical propagator $F(x, x')$ as well as the causal ones $G^{R,A}(x, x')$. For the first equality, we have used that

$$i\Delta^T + i\Delta^{\bar{T}} = i\Delta^< + i\Delta^>. \quad (17)$$

We employ here the same symbol for the QFT propagators on de Sitter space as for their counterparts in the stochastic formalism, defined in Section 2. Eq. (16) should then be compared with Eq. (8), which both share a similar structure. It should be clear however that these quantities are intrinsically different. It is particularly important to note that

- in the stochastic approach, ϕ is a classically stochastic random variable while in the Field Theoretical approach, it refers to a field operator,
- the field ψ is here fully dynamical, in contrast to its stochastic counterpart, which is an auxiliary field.

However, as we will see eventually, the stochastic and QFT fields can be identified in the IR regime.

The solutions to Eq. (12) are well known and can be expressed exactly in terms of hypergeometric functions. In Ref. [35], these are then expanded to leading order in IR-enhancement H^2/m^2 and in large separations $|y|$, where the distance function is given by [30, 55]

$$y(x; x') = \frac{(\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2}{\eta\eta'}, \quad (18)$$

which we express in terms of the conformal time η that is related to comoving time as $\eta = -\frac{1}{a_0 H}e^{-Ht}$, where a_0 is a constant. When transformed into the Keldysh basis, these results take the form

$$-iG^{R,A}(x, x') = i\Delta^{(0)R,A}(x, x') = \frac{H^2}{4\pi^2} \left(-\frac{i}{2} \right) \arg y^{R,A} |y|^{-\frac{m^2}{3H^2}}, \quad (19a)$$

$$F(x, x') = i\Delta^{(0)>}(x, x') + \Delta^{(0)<}(x, x') = \frac{3H^4}{4\pi^2 m^2} |y|^{-\frac{m^2}{3H^2}}, \quad (19b)$$

where the argument of y can be expressed as

$$\arg y^R(x, x') = \arg y^A(x', x) = 2\pi\vartheta(\eta - \eta')\vartheta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2), \quad (20)$$

which follows from the appropriate boundary conditions for the Green functions. The divergence in the statistical propagator $F(x, x')$ for $m \rightarrow 0$ is due to the fact that for a minimally coupled and massless free scalar field, there is no de Sitter invariant quantum state [56, 57].

This form for the propagators is valid only for a light scalar field ($m^2 \ll H^2$). According to Eqs. (19a) and (19b), the propagators decay for large separations², *i.e.* for

²This can be interpreted as the same physical effect that leads to a blue-tilted power-spectrum of inflationary perturbations from a massive scalar field due to the η term.

Figure 2: The elements out of which the QFT Feynman diagrams are constructed. The choice of vertex factor implies that the assembled diagrams should be divided by their symmetry factor. Note the similarity with the stochastic diagrams of Section 2 as well as the obvious differences: Integrations extend over all of spacetime and the extra $\psi^3\phi$ vertex appears.

large values of $|y|$. However, as the mass $m \rightarrow 0$, *i.e.* for $m^2 \ll \sqrt{\lambda}H^2$, the decay of the IR fluctuations can become slow enough for the perturbation expansion to break down. Note also that the statistical propagator has a factor H^2/m^2 of IR enhancement. In contrast, the retarded and advanced propagators that encompass the spectral information for the field excitations remain regular for $m^2/H^2 \rightarrow 0$. This is in accordance with the corresponding stochastic two-point functions given in Eq. (9).

For the diagrammatic expansion we need the Feynman rules for the vertices that are connected by the propagators discussed above. To obtain them, we start from the Amphichronous effective action

$$S[\phi_+, \phi_-] = \int d^4x [\mathcal{L}(\phi_+) - \mathcal{L}(\phi_-)], \quad (21)$$

which reads in the Keldysh basis

$$S[\phi_+, \phi_-] = \int d^4x \sqrt{-g} \left[\frac{1}{2} (\psi \hat{\mathcal{O}} \phi + \phi \hat{\mathcal{O}} \psi) - \frac{1}{2} \frac{\lambda}{3!} (\phi^3 \psi + \phi \psi^3) \right], \quad (22)$$

with the kinetic operator

$$\hat{\mathcal{O}} = \nabla^2 + m^2. \quad (23)$$

Notice that there are two types of vertices: $\phi\psi^3$ connects with at least three retarded or advanced propagators, and $\phi^3\psi$ connects with at least one retarded or advanced propagator. It can therefore readily be seen that, due to the smaller degree of IR enhancement, diagrams containing a $\phi\psi^3$ vertex can be neglected at leading order in the IR enhancement. This is clearly reflected in the stochastic diagrammatic expansion, where the $\phi\psi^3$ is absent in first place, *cf.* the discussion at the end of Section 2 and also in Ref. [49].

We show the Feynman rules for QFT in the Keldysh basis in Figure 2. The graphical similarities with the diagrammatic expansion of the stochastic theory are evident, as

are the obvious differences: There is an extra vertex and integrations extend over the whole spacetime instead of temporal integrations only. With these rules one can formally express correlation functions of ϕ to all orders in the loop expansion, but of course, an exact evaluation of all convolution integrals appears not to be practicable. In the next Section, we show however that to leading IR approximation, the spatial integrals can be performed, establishing an agreement with the stochastic diagrams of Section 2.

4 Equivalence between Field Theoretical and Stochastic Diagrams at Leading IR Order

In this Section, we explicitly demonstrate that the QFT diagrams calculated from the Feynman rules of Figure 2 evaluate to the same results at leading IR order as the corresponding stochastic diagrams constructed from Figure 1. Since the extra $\psi^3\phi$ vertex is irrelevant in the IR, the main difference is that the stochastic propagators (9) are purely time-dependent, which is a consequence of the separation into short and long modes, that abandons manifest de Sitter invariance. In contrast, the QFT propagators depend also on the spatial coordinates, and they are de Sitter invariant, which also holds true for their approximate forms (19a) and (19b).

In the following, we therefore establish the agreement between the diagrams by performing the spatial integrals to leading IR approximation, thus abandoning manifest de Sitter invariance as well. For this purpose, we first show how the agreement is achieved when treating the factors of the propagators that depend on the spatial separations as approximately constant to leading IR order. Then, we justify that the contributions from the integration regions where the latter approximation is not valid are exponentially small (in the parameter H^2/m^2) and therefore negligible.

Reduction of QFT Diagrams to the Stochastic Form. – Since the stochastic propagators are purely time-dependent, we first separate the space dependence from the QFT propagators within an explicit factor. The causal and the statistical propagators (19a) and (19b) share the same non-trivial dependence on the de Sitter invariant length $y(x, x')$, which is raised to a power that determines the leading IR behavior at large space- or time-like separations. We rewrite these terms as

$$|y|^{-\frac{m^2}{3H^2}} = \left| \frac{\eta\eta'}{(\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2} \right|^{\frac{m^2}{3H^2}} = \left(\frac{\eta\eta'}{(\eta - \eta')^2} \right)^{\frac{m^2}{3H^2}} \left| \frac{1}{1 - \delta^2} \right|^{\frac{m^2}{3H^2}}, \quad (24)$$

where $\delta^2 = \frac{(\mathbf{x} - \mathbf{x}')^2}{(\eta - \eta')^2}$. As it is shown in Ref. [35], the integration over regions with large time separations (*i.e.* $|t - t'| \gg 1/H$) accumulates IR-enhancement factors $\sim \frac{H^2}{m^2}$ in addition to the explicit factors present in the statistical propagators. In these integration regions, based on the approximation

$$\frac{\eta\eta'}{(\eta - \eta')^2} = \frac{e^{-H|t-t'|}}{1 + e^{-2H|t-t'|} - 2e^{-H(t+t')}} \approx e^{-H|t-t'|}, \quad (25)$$

we can replace the time-dependent factor in Eq. (24) with $e^{-\frac{m^2}{3H}|t-t'|}$. Moreover, since $m^2 \ll H^2$, we can expand

$$\left| \frac{1}{1-\delta^2} \right|^{\frac{m^2}{3H^2}} = 1 + \mathcal{O}\left(\frac{m^2}{H^2}\right), \quad (26)$$

for separations satisfying

$$1 - \delta^2 \gg \exp\left(-\frac{3H^2}{m^2}\right) \quad \text{and} \quad \exp\left(-\frac{3H^2}{m^2}\right) \ll \delta^2 - 1 \ll \exp\left(\frac{3H^2}{m^2}\right). \quad (27)$$

Since $\delta^2 \in [0, +\infty]$ for any physical separation, one observes that the larger the IR enhancement, *i.e.* the closer m is to zero, the wider is the window of validity (27) for the approximation (26). Then, for separations satisfying the above conditions, the power term (24) is approximated by

$$|y|^{-\frac{m^2}{3H^2}} = e^{-\frac{m^2}{3H}|t-t'|} \times \left(1 + \mathcal{O}\left(\frac{m^2}{H^2}\right)\right), \quad (28)$$

such that we can reexpress the Field Theoretical propagators as

$$F(x, x') = \frac{3H^4}{4\pi^2 m^2} e^{-\frac{m^2}{3H}|t-t'|}, \quad (29)$$

and

$$G^R(x, x') = G^A(x', x) = \frac{H^2}{4\pi} \theta(t-t') \theta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2) e^{-\frac{m^2}{3H}(t-t')}. \quad (30)$$

We note that up to a factor of 2, Eq. (29) is of the same form as the statistical propagator $F(t, t')$ in the stochastic approach (10); in particular it is space independent. Moreover, in diagrams contributing to leading IR order, each of the vertex integrals involves only one retarded or advanced propagator. These contribute a factor $\frac{1}{4\pi} \theta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2)$ that we absorb into the vertex integral,

$$-i \frac{\lambda}{2} \int d^4 x' a^4(x') \frac{1}{4\pi} \theta((\eta - \eta')^2 - (\mathbf{x} - \mathbf{x}')^2) = -i \frac{\lambda}{6} \int d\eta' \frac{(\eta - \eta')^3}{H^4 \eta'^4} \approx -i \frac{\lambda}{6H^3} \int dt'. \quad (31)$$

The accuracy of the latter approximation is to be understood in the sense that when integrating over a function $\sim \vartheta(t-t') \exp(-\alpha \frac{m^2}{3H}|t-t'|)$, there is a correction $\mathcal{O}(\alpha \frac{m^2}{3H})$, which is negligible to leading IR order.

The above integration can be done for each individual vertex in a Feynman diagram contributing to leading IR order, *i.e.* a diagram where each vertex connects to precisely one causal propagator. Hence, we can effectively replace the retarded and advanced propagators with

$$G^R(x, x'), G^A(x', x) \rightarrow \theta(t-t') H^2 e^{-\frac{m^2}{3H}(t-t')}, \quad (32)$$

such that the effective vertex contribution is now given by $-i\frac{\lambda}{6H^3} \int dt'$. We stress that both the effective vertex and the effective causal propagators are only valid when evaluated under the convolution integrals that arise from Feynman diagrams at leading IR order. Their form is almost identical to the corresponding quantities in the stochastic formulation, *cf.* Eqs. (9a) and (11). The only remaining differences compared with the stochastic calculation are a factor of 1/2 in the vertex coefficients as well as a factor 2 in the statistical correlations (29) and (10). We see that these discrepancies compensate when rescaling the QFT fields in the Keldysh basis $\phi \rightarrow \phi/\sqrt{2}$ and $\psi \rightarrow \sqrt{2}\psi$. Then, ignoring the $\psi^3\phi$ vertex, ϕ becomes equivalent to the average field between the forward and backward time contours $\phi \rightarrow (\phi_+ + \phi_-)/2$ while ψ now corresponds to the auxiliary field of the stochastic formalism. G^R and G^A remain unaffected while F and the vertex $\psi\phi^3$ now coincide with the expressions from the stochastic approach.

Contributions Close to the Light Cone and at Large Spatial Separations. –

We have demonstrated above that the truncation of the series (26) at leading order readily leads to QFT Feynman diagrams that are identical to the stochastic ones. Next, we estimate the contributions from those regions of integration where this expansion breaks down and show that these are negligible to leading IR order. This is necessary because in any generic Feynman diagram the convolution integral runs over the entire Poincaré patch and hence also receives contributions from separations where the approximation (26) is not valid. From the criteria (27), we can categorize these separations into a) regions in the vicinity of the light cone:

$$1 - \alpha \exp\left(-\frac{3H^2}{m^2}\right) < \delta^2 < 1 + \alpha \exp\left(-\frac{3H^2}{m^2}\right) \quad (33)$$

and b) large space-like separations, *i.e.*, large δ^2 :

$$\delta^2 > 1 + \frac{1}{\alpha} \exp\left(\frac{3H^2}{m^2}\right), \quad (34)$$

where $1 \ll \alpha \ll \exp(\frac{3H^2}{m^2})$ is a constant.

We aim to show that the regions specified above are exponentially small and therefore only lead to contributions to the Feynman diagrams that are negligible at leading IR order. While it is somewhat obvious that region a) corresponds to an exponentially small restriction of the integration volume in the directions perpendicular to the light cone, we nonetheless work this out explicitly in the following. The identification of region b) turns out to be slightly less straightforward because we need to make use of the causal structure of the vertex integrals in a Feynman diagram.

In a Feynman diagram contributing at leading IR order (*i.e.* all vertices are of the $\phi^3\psi$ type), the vertex integrations are of the general form

$$\begin{aligned} I(\{x_i\}) &= -i\frac{\lambda}{2} \int d^4x' a^4(x') \frac{H^2}{4\pi} \vartheta(\eta' - \eta_1) \vartheta((\eta' - \eta_1)^2 - (\mathbf{x}' - \mathbf{x}_1)^2) \\ &\times |y(x', x_1)|^{-\frac{m^2}{3H^2}} |y(x', x_2)|^{-\frac{m^2}{3H^2}} |y(x', x_3)|^{-\frac{m^2}{3H^2}} |y(x', x_4)|^{-\frac{m^2}{3H^2}}. \end{aligned} \quad (35)$$

Here the vertex at x' connects with another vertex or an external point x_1 through a causal propagator and with three additional points through statistical propagators. While it does not appear possible to exactly evaluate this integral analytically, it is straightforward to estimate (*cf.* Ref. [35])

$$I(\{x_i\}) \sim \frac{\lambda}{m^2} \prod_{i=2,3,4} \left(\frac{\eta_i}{\eta_1} \right)^{-\epsilon}, \quad (36)$$

where $\epsilon = \frac{m^2}{3H^2}$ and the IR enhancement is reflected by the divergence for $m \rightarrow 0$. For large ratios η_i/η_1 there is an extra suppression because of the decay of correlations with very early times.

The power terms can be rewritten in the form of Eq. (24) by introducing $\delta_i^2 = \frac{(\mathbf{x}_i - \mathbf{x}')^2}{(\eta_i - \eta')^2}$ where $i = 1, 2, 3, 4$. The spatial integration in Eq. (35) then takes the form

$$\int d^3\mathbf{x}' \vartheta((\eta' - \eta_1)^2 - (\mathbf{x}' - \mathbf{x}_1)^2) \prod_{i=1}^4 \left| \frac{1}{\tilde{\delta}_i^2} \right|^\epsilon = 4\pi(\eta_1 - \eta')^3 \int_0^1 d|\tilde{\delta}_1| \delta_1^2 \prod_{i=1}^4 \left| \frac{1}{\tilde{\delta}_i^2} \right|^\epsilon, \quad (37)$$

where $\tilde{\delta}_i^2 = 1 - \delta_i^2$. When all separations $x' - x_i$ comply with the condition (27), we approximately take $|1/\tilde{\delta}_i^2|^\epsilon \approx 1$, and the term $\frac{4\pi}{3}(\eta_1 - \eta')^3$ in Eq. (37) can be passed to the temporal integration according to Eq. (31) – see the discussion above.

Now consider an integration region where δ_j^2 lies in the vicinity of the light cone, *i.e.* within the region (33), while the remaining δ_i^2 satisfy the relation (27) and can therefore be approximated by $\tilde{\delta}_i^2 \approx 1$ for $i \neq j$. Then, using $|\tilde{\delta}_j|$ as a variable of integration, we obtain from the region defined by relation (33) the contribution

$$2 \int_0^{\frac{1}{2}\alpha e^{-1/\epsilon}} d|\tilde{\delta}_j| \delta_j^2 \left| \frac{1}{\tilde{\delta}_j} \right|^{2\epsilon} = \frac{2\tilde{\delta}_j^{1-2\epsilon}}{1-2\epsilon} \Big|_0^{\frac{1}{2}\alpha e^{-1/\epsilon}}, \quad (38)$$

where we have used that $\delta_j \approx 1$ close to the light cone. Therefore, compared to the contribution to the integral from the regime (27) [or, more directly, the corresponding factor in Eq. (36), which is of order one], the piece from the vicinity of the light cone is exponentially suppressed by a factor $\alpha \exp(-\frac{3H^2}{m^2})$. It should be clear that similar arguments apply when more than one of the separations in the integral (35) are close to the light cone simultaneously because the regions where this may occur are only exponentially small and only contain integrable singularities for $\epsilon \ll 1$. We should also recall that while the region (33) is exponentially narrow, the leading inaccuracy of our approximations is of order m^2/H^2 as stated through Eq. (26).

Now we turn to large space-like separations defined by relation (34). Since the long-wavelength fluctuations vary by definition only very slowly, these observables are obtained by evaluating the coincident correlation function $\langle \phi(x_o)\phi(x_o) \rangle$ and subtracting the short-wavelength contributions that are independent from the background expansion. We choose for simplicity $x_o = (\eta_o, \mathbf{0})$. Now, due to causality, all points that

Figure 3: For a correlation function $\langle \phi(x_o) \phi(x_o) \rangle$, the support of the spacetime integration is given by the past light cone of x_o because of causality. The diagram illustrates that this implies a maximal spatial distance for any two points with given η_a and η_b .

contribute to the Feynman diagrams must lie within the past light cone of x_o , a fact we illustrate in Figure 3. This implies that all two-point separations that occur in the generic integral (35) satisfy the bound (we identify $x' = x_a$ and $x_i = x_b$ in Figure 3)

$$\delta^2(x', x_i) = \frac{(\mathbf{x}' - \mathbf{x}_i)^2}{(\eta' - \eta_i)^2} \leq 1 + \frac{4\eta'\eta_i + 4\eta_o^2 - 4\eta_o(\eta' + \eta_i)}{(\eta' - \eta_i)^2}. \quad (39)$$

Combining this with relation (34) in the form $\exp(1/\epsilon)/\alpha \leq \delta^2$, we find the following strip for the range of the integration variable η' that is allowed by causality but where at the same time the approximation (26) is invalid due to a large separation between $\mathbf{x}'$ and $\mathbf{x}_i$:

$$\eta_i - 2\sqrt{\alpha}e^{-\frac{1}{2\epsilon}}(\eta_o - \eta_i) \leq \eta' \leq \eta_i + 2\sqrt{\alpha}e^{-\frac{1}{2\epsilon}}(\eta_o - \eta_i). \quad (40)$$

We denote the restriction of the integral (35) to the strip (40) by I_{strip} and aim to determine an upper bound on its absolute value. Note that while relation (40) defines an exponentially narrow strip around η_i , it becomes wider as η_i takes large negative values. We therefore need to show in particular that $|I_{\text{strip}}|$ remains exponentially small even when $\eta_i \rightarrow -\infty$. To estimate the integrand, we note that in the region where the inequality (34) holds, the term (24) appearing in the propagators satisfies the following relation:

$$|y(x', x_i)|^{-\epsilon} \leq \alpha^\epsilon e^{-1} \left(\frac{(\eta'^2 - \eta_i^2)}{\eta' \eta_i} \right)^{-\epsilon}. \quad (41)$$

Now, as illustrated in Figure 4, the integration (35) over x' is confined to the past light cone of x_1 because of the causal θ functions. This implies that the condition (34) is never fulfilled for $\delta(x', x_1)$ and that we can replace $\delta^2(x', x_1)$ with the approximation (26). In order to estimate contributions from the regions where the expansion (26) breaks down,

Figure 4: The vertex integration in Eq. (35) is confined to the past light cone of x_1 . The large δ^2 region for $y(x', x_2)$ is the narrow strip between the red dashed lines.

we therefore consider the distance from x' to one of the remaining points, say x_2 , where $\eta_2 < \eta_1$. Besides, we see from the relation (40) that the strip becomes wider and we obtain a looser estimate on I_{strip} when taking $\eta_o \rightarrow 0^-$, such that we set $\eta_o = 0$ in order to obtain an upper bound in the following. In Figure 4, the shaded area between the two dashed lines then indicates the region where the inequality (40) is saturated because of a large value of $\delta(x', x_2)$. Assuming that x_3 and x_4 lie outside of the strip between the dashed lines, we can obtain an upper bound on $|I_{\text{strip}}|$ by replacing the power term involving $y(x', x_2)$ with the bound (41), while we make use of Eq. (24) and the approximation (26) for the remaining power terms. We thus find:

$$\begin{aligned}
|I_{\text{strip}}| &\leq \frac{\lambda}{2H^2} \int_{\eta_2 - 2\eta_2 \sqrt{\alpha} e^{-\frac{1}{2\epsilon}}}^{\eta_2 + 2\eta_2 \sqrt{\alpha} e^{-\frac{1}{2\epsilon}}} d\eta' \frac{1}{\eta'^4} \int_0^{\eta_1 - \eta'} dr 4\pi r^2 \alpha^\epsilon e^{-1} \prod_{i=1}^4 \left(\frac{(\eta_i - \eta')^2}{\eta_i \eta'} \right)^{-\epsilon} \\
&\approx \frac{2\pi\lambda}{3H^2} e^{-1} \frac{(\eta_1 - \eta_2)^3}{\eta_2^4} \prod_{i=3,4} \left(\frac{(\eta_i - \eta_2)^2}{\eta_i \eta_2} \right)^{-\epsilon} \int_{\eta_2 - 2\eta_2 \sqrt{\alpha} e^{-\frac{1}{2\epsilon}}}^{\eta_2 + 2\eta_2 \sqrt{\alpha} e^{-\frac{1}{2\epsilon}}} d\eta' \left(\frac{(\eta_2 - \eta')^2}{\eta_2^2} \right)^{-\epsilon} \\
&\approx \frac{4\pi\lambda}{3H^2} \sqrt{\alpha} e^{-\frac{1}{\epsilon}} \frac{(\eta_1 - \eta_2)^3}{\eta_2^3} \prod_{i=3,4} \left(\frac{(\eta_i - \eta_2)^2}{\eta_i \eta_2} \right)^{-\epsilon}, \tag{42}
\end{aligned}$$

which is exponentially small when compared with the estimate (36) for the leading IR contribution. Note in particular that this bound remains exponentially restrictive also for $\eta_2 \rightarrow -\infty$.

Provided the individual $\eta_{2,3,4}$ are separated far enough, the above argument can be successively applied as the η' integration sweeps over the disjoint strips (40). We should eventually comment on situations where the individual strips defined by relation (40) overlap or where these strips intersect with the light cones of the individual x_i . Since, as we have shown here, the problematic regions are exponentially small and the contained singularities integrable as long as $\epsilon \ll 1$, which is amply fulfilled here by assumption,

the contributions from intersections of the regions (33) and (34) are also exponentially small compared to the leading IR terms.

In conclusion, the approximation (28) can safely be used for the evaluation of Feynman diagrams to leading IR order and in the late-time limit. This implies in turn that in the same limit, the elements (propagators and vertices) of Feynman diagrams of the Amphichronous QFT are equivalent to their stochastic counterparts up to terms of order m^2/H^2 .

5 Resumming the QFT in the Late-Time Limit

In the previous Sections, we have demonstrated that the perturbative computations in QFT and in Starobinsky's stochastic approach agree at the leading IR order. In particular, there is a one-to-one correspondence between the Feynman diagram expansion of two-point functions derived from the stochastic partition function (7) and from Amphichronous QFT in the Keldysh basis. However, the stochastic approach offers the possibility to resum the perturbation series in terms of taking expectation values of a classical probability distribution function (*i.e.* effectively in terms of a one-dimensional integration), and consequently this may also serve as a resummation for all QFT diagrams to leading IR order. In fact, the stochastic resummation yields well defined results at late times even in those cases when the perturbative expansion fails *i.e.* for ultra-light and massless fields: $0 \leq m^2 \ll \sqrt{\lambda}H^2$. The result of this resummation was first obtained in the seminal work of Starobinsky and Yokoyama [2]. In the following, we show that this resummation procedure can be applied as well to the Feynman diagram expansions presented in the present work. For this purpose, we demonstrate that in the late-time limit, the stochastic partition function (7) can effectively be evaluated with the same result as the probability distribution found by Starobinsky and Yokoyama.

We start with the partition function $\mathcal{Z}$ as defined in Eq. (7) and integrate over the auxiliary field ψ . This yields

$$\mathcal{Z} = \int D[\phi] e^{-\frac{2\pi^2}{H^3} \int_0^T du \left(\dot{\phi} + \frac{\partial_{\phi} V}{3H} \right)^2} = \int D[\phi] e^{-\frac{2\pi^2}{H^3} \int_0^T dt \left(\dot{\phi}^2 + 2\dot{\phi} \frac{\partial_{\phi} V}{3H} + \left(\frac{\partial_{\phi} V}{3H} \right)^2 \right)}, \quad (43)$$

where we have explicitly specified the boundaries of the integral in the exponential term. Note that in the above equation the potential is the full interacting one $V = \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi)$; we have also absorbed a constant in the integration measure. The partition function can then be written as

$$\mathcal{Z} = \int_{-\infty}^{+\infty} d\phi_T e^{-\frac{4\pi^2}{3H^4} V(\phi_T)} \int D[\phi_{t < T}] e^{-\frac{2\pi^2}{H^3} \mathcal{S}[\phi]} \quad (44)$$

with

$$\mathcal{S}[\phi] = \int_0^T dt \left[\dot{\phi}^2 + \left(\frac{\partial_{\phi} V}{3H} \right)^2 \right], \quad (45)$$

where $\phi_T \equiv \phi(T)$, and we have assumed that $\phi(0) = 0$, $V(0) = 0$. Moreover, we have decomposed the measure in Eq. (44) as $\int D[\phi] = \int D[\phi_{t < T}] \int_{-\infty}^{+\infty} d\phi_T$. The choice of initial condition implies that the IR sector of the field ϕ at $t = 0$ does not exhibit significant fluctuations, *i.e.* there is no infrared enhancement yet. This could be, for example, due to inflation beginning at $t = 0$. It is useful to note that $\mathcal{S}[\phi]$ corresponds to the action of a particle moving in the one dimensional potential $-[\partial_\phi V/(3H)]^2$ along the trajectory $\phi(t)$.

Our goal is to resum the QFT series and we are interested in the late time limit when this series will possibly break down. According to our previous arguments, the late-time limit should be given by evaluating the partition function (44) non-perturbatively. This can be done by the steepest descent method. At late times, the integral is dominated by the path $\phi_0(t)$ that extremizes the pseudo action $\mathcal{S}$, *i.e.* $\left. \frac{\delta \mathcal{S}}{\delta \phi} \right|_{\phi=\phi_0} = 0$. The latter condition is equivalent to

$$\ddot{\phi}_0 - \frac{1}{9H^2} \partial_\phi V \partial_\phi^2 V \Big|_{\phi=\phi_0} = 0. \quad (46)$$

Moreover the solution has to satisfy the following boundary conditions: $\phi_0(0) = 0$ and $\phi_0(t) = \phi_t$. Integrating the above equation yields

$$\dot{\phi}_0^2 - \frac{1}{9H^2} \left(\partial_\phi V|_{\phi=\phi_0} \right)^2 = E, \quad (47)$$

where E is an integration constant that should be chosen in order to meet the boundary condition $\phi_0(T) = \phi_T$. For example, in the quartic potential $V(\phi) = \frac{\lambda}{4!} \phi^4$, we have $\partial_\phi V(\phi)|_{\phi=\phi_0(0)=0} = 0$, and consequently, we have to take $E \rightarrow 0$ for $T \rightarrow \infty$, corresponding to the fact that at $t = 0$, we have to choose the kinetic energy of ϕ at the top of the unbounded upside-down potential to be infinitesimally small for $\phi(T)$ to remain finite when $T \rightarrow \infty$. More precisely, since $\phi(0) = 0$, Eq. (47) implies that $\phi(t) \geq \sqrt{Et}$. With the boundary condition $\phi(T) = \phi_T$, this implies that $ET \leq \phi_T/T$ and therefore $ET \rightarrow 0$ in the late-time limit $T \rightarrow \infty$. Therefore, in this limit, the pseudo action (45) can be written as

$$\mathcal{S}[\phi_0] = 2 \int_0^T dt \dot{\phi}_0^2 = \frac{2}{3H} \int_0^T dt \dot{\phi}_0 \partial_\phi V|_{\phi=\phi_0} = \frac{2}{3H} V(\phi_T), \quad (48)$$

where we have used the boundary conditions $\phi_0(0) = 0$, $\phi_0(T) = \phi_T$, and we have dropped terms of order ET . Using Eq. (48) in Eq. (45) and relabeling $\phi_T = \varphi$, we find for the late time partition function

$$\mathcal{Z} = \int_{-\infty}^{+\infty} d\varphi e^{-\frac{8\pi^2}{3H^4} V(\varphi)}. \quad (49)$$

This coincides exactly with the result by Starobinsky and Yokoyama [2]. As mentioned above, the partition function (49) is of course valid even for light fields with $0 \leq m^2 \ll \sqrt{\lambda} H^2$, showing that their fluctuations remain well defined at late times.

Now one may note that the QFT calculations in this work assume $m > 0$ through the form of the free propagators (19). Since the stochastically resummed result is continuous in the limit $m \rightarrow 0$ and on the other hand, the stochastic and the QFT diagrams agree, we may therefore conclude that the QFT result for the two-point correlation coincides with the stochastic answer also for $m = 0$. This can be also seen by taking a different approach, *i.e.* by evaluating only diagrams that descend from the two-particle irreducible effective action for the theory with $m = 0$ and making a dynamical mass ansatz for the propagators. In Ref. [13], it is demonstrated that these diagrams can be resummed to leading IR order in Euclidean de Sitter space. This leads to Schwinger-Dyson equations that can be solved for the dynamical mass, which is then found to be in agreement with the result from Stochastic Inflation. Since at leading IR order, the QFT partition function in Euclidean de Sitter space coincides with the stochastic partition function, which we have shown here in the same approximation also to be consistent with QFT in Lorentzian de Sitter space, the resummation found in Ref. [13] can be directly applied to the Feynman diagram expansions developed in the present work.

6 Discussion

In this paper we have demonstrated that for a light scalar field with quartic self-interaction, the Quantum Field Theory in Lorentzian de Sitter space and the stochastic theory of Starobinsky are in one-to-one correspondence at the level of Feynman diagrams in the IR limit, *i.e.* for spatial distances or time separations $\gtrsim 1/H$. Corrections to this agreement will appear at the relative order m^2/H^2 for a light massive scalar field ($\sqrt{\lambda}H^2 \ll m^2 \ll H^2$) and at order $\sqrt{\lambda}$ for a massless or ultra-light field ($m^2 \ll \sqrt{\lambda}H^2$). Hence the stochastic formalism is perturbatively equivalent to the full QFT to leading IR order. In addition to truncating the Field Theoretical computations in a meaningful way, the stochastic approach offers the possibility of resumming the perturbation series in terms of late-time probability distribution functions. This is of particular relevance for the ultralight or massless regime, where the perturbative expansion breaks down. The equivalence at the level of Feynman diagrams implies that the stochastic resummation can also be applied to QFT calculations to leading IR order [2].

A number of comments on the relation between Field Theoretically and stochastically computed correlations can be found in the earlier literature. It has indeed been conjectured and emphasized *e.g.* in Refs. [8, 30] that the stochastic probability distribution function resums the ‘leading logarithms’ (*i.e.* the IR enhanced powers of H^2/m^2 in our nomenclature) that occur in the Feynman diagrams of the Amphichronous Field Theoretical formulation. Our analysis is the first that fully reproduces the stochastic correlation functions found by Starobinsky and Yokoyama from a QFT calculation truncated at leading IR order, and it therefore proves the above conjecture.

While the main results of this paper as well as of many other works concern equal-time correlations, it is also interesting to consider their behaviour at unequal times. In the massive case, the free theory propagators (9) and (19) exhibit an exponen-

tial decay on scales larger than $\sim 3H/m^2$. From the fact that a dynamical mass ansatz for the Schwinger-Dyson equations derived from the two-particle irreducible effective action in Euclidean de Sitter space reproduces the stochastic answers [13], one may anticipate that this decay of correlations at large separations should also appear in the massless theory on a Lorentzian metric. In particular, when replacing $m^2 \rightarrow m_{\text{dyn}}^2 = \Gamma(1/4)\sqrt{\lambda}H^2/(8\pi\Gamma(3/4))$, the relevant scale for the decay should be $3H/m_{\text{dyn}}^2 \sim H^{-1}/\sqrt{\lambda}$. This may be compared with Ref. [58], where it is reported that this scale should be $\sim H^{-1}(\lambda N)^{-1/3}$, where N is the number of e-folds. Moreover, it is found there that the decay of the correlations in momentum space is rather abrupt toward large scales, *i.e.* $\sim k^3$ where k is the momentum. This should be compared to a decay $\sim k^{m_{\text{dyn}}^2/(3H^2)}$ indicated by the dynamical mass ansatz, corresponding to a small and constant blue spectral tilt. The answer for the decay of correlations in Lorentzian space should be in principle attainable from the stochastic functional (7), a calculation which we leave in detail to future work. Here, we note that a simple estimate appears to support the decay behaviour suggested by the dynamical mass approach: We take Eq. (1) evaluated at the time t , multiply with $\phi(0)$ and take the expectation value such that we get

$$\langle \frac{d}{dt}\phi(t)\phi(0) \rangle + \lambda \langle \phi^3(t)\phi(0) \rangle / (18H) \approx 0, \quad (50)$$

where we assume that t is large enough for the field and the noise to be uncorrelated, $\langle \xi(t)\phi(0) \rangle \approx 0$. Next, we Wick expand the correlation of four fields to obtain

$$\frac{d}{dt}\langle \phi(t)\phi(0) \rangle \sim \frac{\lambda}{H} \langle \phi^2(t) \rangle \langle \phi(t)\phi(0) \rangle. \quad (51)$$

Since the Wick expansion is valid only for approximately Gaussian correlations, above relation should be understood as an estimate of order one accuracy for the massless field in the quartic potential. When noting that the late-time limit of the equal-time correlator follows from Eq. (49) to be [2]

$$\langle \phi^2(t) \rangle = \frac{\Gamma(\frac{7}{4})H^2}{\pi\Gamma(\frac{5}{4})\sqrt{\lambda}}, \quad (52)$$

we find from relation (51) that

$$\langle \phi(t)\phi(0) \rangle \sim e^{-A\sqrt{\lambda}Ht}. \quad (53)$$

The results from Ref. [13] for Euclidean de Sitter space suggest that $A\sqrt{\lambda}H = m_{\text{dyn}}^2/(3H)$. It would be interesting to explicitly verify this conjecture from the stochastic partition function (43) that should in principle allow for an evaluation of unequal time correlators.

Future research may progress into directions on Euclidean de Sitter space, *e.g.* to aim for resummations of the massless scalar theory beyond the leading IR order or for identifying the ground state of quantized Gravity, at least to leading IR approximation. As for the developments on Lorentzian de Sitter space performed in this work, these open opportunities to address some of the following questions:

- When going beyond the leading IR approximation, we expect also ultraviolet divergences, that should be renormalized. One should investigate, whether the known counterterms from the theory in Minkowski background are sufficient, or if new operators that couple ϕ to scalar curvature invariants appear (*cf.* Ref. [40] for a discussion of such matters in the Hartree approximation). This question is of importance for understanding the properties of scalar potentials when receiving radiative corrections in the curved background.
- While the Starobinsky Equation (1) in conjunction with the stochastic noise (2) can be readily solved in order to obtain the scalar field correlations at all times [2], this has not yet been achieved in the QFT framework. In the perturbative calculations [30–33], early time correlations are addressed, while here, we show that also the asymptotically time-independent correlations at late times can be obtained using QFT methods, including situations where the perturbation expansion breaks down due to the strong IR enhancement. It would be interesting to confirm the full evolution of the correlations from early-time growth to late-time saturation for the massless, self-interacting scalar theory within the QFT framework.
- Based on the conjecture that the stochastic approach sums all leading IR order diagrams, it has been proposed to generalize its application beyond the self-interacting scalar theory to *e.g.* Gravitation and Electrodynamics [4, 8]. It should be interesting to use the methods developed here to establish the link between Field Theoretical and stochastic formulations also for these well-motivated proposals.
- The stochastic probability distributions in Ref. [2] as well as the stochastic partition function (7) make only predictions for the time-dependence of the correlations. It would therefore be beneficial to develop a formulation of the stochastic approach that can also make predictions for the spatial dependencies. Ideally, this formulation should be manifestly de Sitter invariant, just as the underlying Field Theory.

While the stochastic approach is intuitive and compellingly simple, the Amphichronous QFT formulation allows to perform systematic calculations using controlled approximations. To this end, the agreement found to leading IR order should be of practical use for validating stochastic results by QFT calculations. In addition, the links established here may be a starting point to further develop calculational methods for Quantum Theory on de Sitter space that open paths to new results by combining the advantages of both approaches.

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