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Influences of aspect ratio and wall boundary condition on laminar Rayleigh-Bénard convection of Bingham Fluids in rectangular enclosures

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ABSTRACT

The aspect ratio AR (ratio of enclosure height: length) dependence of steady-state Rayleigh-Bénard convection of Bingham fluids within rectangular enclosures has been investigated for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions. A nominal Rayleigh number range $10^3 \leq Ra \leq 10^5$ (Ra defined based on the height) for a single representative value of nominal Prandtl number (i.e. $Pr = 500$) has been considered for $1/4 \leq AR \leq 4$. It has been found that buoyancy driven flow strengthens with increasing nominal Rayleigh number Ra but the convective transport weakens with increasing Bingham number Bn due to additional flow resistance arising from yield stress in Bingham fluids. The relative contribution of thermal conduction (advection) to the total thermal transport strengthens (diminishes) with increasing AR for a given set of values of Ra and Pr for both Newtonian and Bingham fluids for both boundary conditions, and the thermal transport takes place purely due to conduction for tall enclosures. Although Ra , AR and Bn dependences of the mean Nusselt number $\overline{Nu}$ remains qualitatively similar for both CWT and CWHF boundary conditions, $\overline{Nu}$ assumes smaller values in the CWHF (CWT) configuration than in the case of CWT (CWHF) boundary condition for high (small) values of Ra . Correlations for the mean Nusselt number $\overline{Nu}$ have been proposed for both boundary conditions using scaling arguments and the correlations have been demonstrated to predict $\overline{Nu}$ obtained from simulation data for $1/4 \leq AR \leq 4$, $10^3 \leq Ra \leq 10^5$ and $Pr = 500$.

1. INTRODUCTION

Natural convection in rectangular enclosures is one of most widely analysed configurations in the heat transfer literature because of its wide relevance to a broad range of flows, such as, solar collectors, electronic cooling, and geo-physical flows. However, most analyses in this regard were carried out for Newtonian fluids (Catton *et al.*, 1974; Ostrach, 1988; Bodenschatz *et al.*, 2000; Quertatani *et al.*, 2008; Ganguli *et al.*, 2009). Recently, a number of analyses (Park and Ryu, 2001; Vola, 2003; Zhang *et al.*, 2006; Balmforth and Rust, 2009; Vikhansky, 2009,2010; Turan *et al.*, 2010; 2011a,b; 2012a,b; 2014; Hassan *et al.*, 2013; Karimfazli and Frigaard, 2013; Darbouli *et al.*, 2013; Kebiche *et al.*, 2014; Balmforth *et al.*, 2014; Yigit *et al.*, 2015a,b) concentrated on various aspects of natural convection of yield stress fluids, which have potential applications in cryogenic storage, nuclear cooling, chemical processing and food preparation and preservation, to name a few. Such yield stress fluids flow like fluids once a critical stress (i.e. yield stress) is surpassed but act like a rigid solid below this stress. Most magneto- and electro-rheological fluids exhibit yield stress behaviour where the yield stress can be manipulated by applying magnetic and electrical fields, respectively (Wereley, 2014, Stangroom, 1983). The analyses on natural convection of yield stress fluids in the existing literature can be categorised into two broad aspects. The first aspect deals with the critical temperature difference which determines the condition under which buoyancy force just overcomes the yield stress to induce flow within the enclosure, which has been extensively analysed using semi-analytical (Zhang *et al.*, 2006; Balmforth and Rust, 2009; Vikhansky, 2009,2010) and experimental (Darbouli *et al.*, 2013; Kebiche *et al.*, 2014) methods. The second aspect focuses on the effects of Rayleigh, Prandtl and Bingham (i.e. non-dimensional yield stress) numbers on the heat transfer characteristics under conditions which are far beyond the critical condition for the onset of fluid motion. This aspect is principally analysed based on semi-analytical (Karimfazli and Frigaard, 2013) and numerical (Vola, 2003; Turan *et al.*, 2010; 2011a,b; 2012a,b; 2014; Hassan *et al.*, 2013; Yigit *et al.*, 2015a,b) investigations. A number of investigations belonging to the second category specifically addressed the effects of aspect ratio AR (i.e. height to length ratio= $AR = H / L$) on natural convection of yield stress fluids in rectangular enclosures with different vertical wall temperatures (Turan *et al.*, 2012b,2014) and Rayleigh-Bénard (i.e. different horizontal wall temperatures with heated bottom

wall) (Yigit *et al.*, 2015a) configurations, and the present analysis concentrates on Rayleigh-Bénard configuration of yield stress fluids following a Bingham model (Barnes, 1999) in rectangular enclosures for different wall boundary conditions. It has been recently demonstrated by Turan *et al.* (2014) that the wall boundary condition significantly influences the aspect ratio AR dependence of heat transfer rate and the mean Nusselt number in the case of natural convection of Bingham fluids in the configuration with different vertical wall temperatures. It is worth noting that the Rayleigh-Bénard configuration is fundamentally different from the configuration with different vertical wall temperatures because flow initiates once a finite temperature difference is induced for the configuration with different vertical wall temperatures, whereas flow initiates only when a threshold Rayleigh number is surpassed in the Rayleigh-Bénard configuration. The present authors recently analysed the aspect ratio effects on laminar Rayleigh-Bénard configuration for the constant wall temperature boundary condition (Yigit *et al.*, 2015a). However, a number of previous analyses (Turan *et al.*, 2012c; 2014) indicated that aspect ratio effects on natural convection in rectangular enclosures for the constant wall heat flux (CWHF) boundary conditions can be significantly different from the corresponding results for the constant wall temperature (CWT) boundary condition for both Newtonian and Bingham fluids. Aspect ratio effects on heat transfer characteristics of yield stress fluids between CWT and CWHF boundary conditions for the laminar Rayleigh-Bénard configuration is yet to be analysed in detail in the open literature. In this respect, the main objectives of the current analysis are:

1. To analyse and explain the effects of wall boundary condition on the aspect ratio dependence of natural convection of yield stress fluids in the laminar Rayleigh-Bénard configuration for a range of different values of nominal Rayleigh and Bingham numbers;
2. To propose correlations for the mean Nusselt number for both CWT and CWHF boundary conditions in the aforementioned configuration based on scaling arguments and numerical simulation data;

In order to meet the above objectives numerical simulations of laminar two-dimensional natural convection have been carried out for aspect ratios AR ranging from $1/4$ to 4 within the nominal

Rayleigh number range $10^3 \leq Ra \leq 10^5$ for a single value of nominal Prandtl number (i.e. $Pr = 500$). This choice of nominal Prandtl number makes the current analysis consistent with previous analyses (Turan *et al.*, 2012a,b; Yigit *et al.*, 2015a,b) on aspect ratio effects. Furthermore, many yield stress fluids have a Prandtl number of the order of 100 (Turan *et al.*, 2010; 2011a; 2012a,b).

The remainder of the paper will be organised as follows. The mathematical background and numerical implementation pertaining to the current analysis are provided in the next two sections. Following this, results will be presented and discussed. The main findings will be summarised and conclusions will be drawn in the final section of this paper.

2. MATHEMATICAL BACKGROUND

Governing equations and non-dimensional numbers

The strain rate dependence of shear stress following the Bingham model (Barnes, 1999) can be expressed in tensorial form in the following manner:

$$\underline{\underline{\dot{\gamma}}} = 0 \quad \text{for} \quad \tau \leq \tau_y, \quad (1)$$

$$\underline{\underline{\tau}} = \left(\mu + \frac{\tau_y}{\dot{\gamma}} \right) \underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \tau > \tau_y, \quad (2)$$

where $\dot{\gamma}_{ij} = \partial u_i / \partial x_j + \partial u_j / \partial x_i$ are the components of the rate of strain tensor $\underline{\underline{\dot{\gamma}}}$, $\underline{\underline{\tau}}$ the stress tensor, τ and $\dot{\gamma}$ are evaluated based on the second invariants of the stress and the rate of strain tensors respectively (in a pure shear flow), which can be defined as:

$$\tau = \left[\frac{1}{2} \underline{\underline{\tau}} : \underline{\underline{\tau}} \right]^{1/2}, \quad (3)$$

$$\dot{\gamma} = \left[\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \right]^{1/2}. \quad (4)$$

O'Donovan and Tanner (1984) used a bi-viscosity model to model the Bingham model flow:

$$\underline{\underline{\tau}} = \mu_{yield} \dot{\underline{\underline{\gamma}}} \quad \text{for} \quad \dot{\underline{\underline{\gamma}}} \leq \frac{\tau_y}{\mu_{yield}} , \quad (5)$$

$$\underline{\underline{\tau}} = \tau_y \frac{\dot{\underline{\underline{\gamma}}}}{\dot{\underline{\underline{\gamma}}}} + \mu \dot{\underline{\underline{\gamma}}} \quad \text{for} \quad \dot{\underline{\underline{\gamma}}} > \frac{\tau_y}{\mu_{yield}} , \quad (6)$$

where μ_{yield} is the yield viscosity. O'Donovan and Tanner (1984) suggested that $\mu_{yield} = 10^3 \mu$ mimics the true Bingham model in a satisfactory way, a more conservative value of $\mu_{yield} = 10^4 \mu$ has been used for higher fidelity of the present simulations. Other regularisation methods are available and, to investigate the effect of this choice of regularisation, some limited simulations have been conducted using the Papanastasiou exponential model (Papanastasiou, 1987):

$$\underline{\underline{\tau}} = \tau_y (1 - e^{-m\dot{\underline{\underline{\gamma}}}}) + \mu \dot{\underline{\underline{\gamma}}} \quad (7)$$

where m is the stress growth exponent which has the dimensions of time. The differences between the predictions of the bi-viscosity and Papanastasiou regularisations have been found to be negligible (order of typical numerical uncertainty).

The spatial co-ordinates, velocity components, pressure and temperature are non-dimensionalised in the following manner:

$$x_i^+ = x_i / H , \quad u_i^+ = u_i / U_{ref} , \quad P^+ = P / \rho U_{ref}^2 \quad \text{and} \quad \Theta = (T - T_{ref}) / \Delta T_{ref} \quad (8)$$

where U_{ref} is the reference velocity scale and ΔT_{ref} is a reference temperature difference. For the CWT configuration ΔT_{ref} can be taken to be $\Delta T = (T_H - T_C)$ whereas ΔT_{ref} can be taken to be equal to qH / k (i.e. $\Delta T_{ref} = qH / k$) for the CWHF configuration. The reference temperature T_{ref} is taken to be the temperature at the centre of the domain T_{cen} for and CWHF boundary condition whereas T_{ref} is taken to be the cold wall temperature T_C for CWT boundary condition. One obtains the following non-dimensional forms of steady-state mass, momentum and energy conservation

equations if U_{ref} is taken to be equal to $\sqrt{g\beta\Delta T_{ref}H}$ (i.e. $U_{ref} = \sqrt{g\beta\Delta T_{ref}H}$) based on the equilibrium of inertial and buoyancy forces :

Non-dimensional mass conservation equation

$$\frac{\partial u_i^+}{\partial x_i^+} = 0 ; \quad (9)$$

Non-dimensional momentum conservation equations

$$u_j^+ \frac{\partial u_i^+}{\partial x_j^+} = -\frac{\partial P^+}{\partial x_i^+} + \delta_{i2} \Theta + \frac{Pr^{1/2}}{Ra^{1/2}} \frac{\partial \tau_{ij}^+}{\partial x_j^+} ; \quad (10)$$

Non-dimensional energy conservation equation

$$u_j^+ \frac{\partial \Theta}{\partial x_j^+} = \frac{1}{Pr^{1/2} Ra^{1/2}} \frac{\partial^2 \Theta}{\partial x_j^+ \partial x_j^+} . \quad (11)$$

In Eq. 10 τ_{ij}^+ is the non-dimensional stress tensor which is given by:

$$\tau_{ij}^+ = \frac{\tau_{ij} H}{\mu \sqrt{g\beta\Delta T_{ref}H}} . \quad (12)$$

The bi-viscosity model (O'Donovan and Tanner, 1984) takes the following form:

$$\underline{\dot{\gamma}}^+ = 0 \quad \text{for} \quad \tau^+ \leq Bn , \quad (13i)$$

$$\underline{\tau}^+ = \left(1 + \frac{Bn}{\dot{\gamma}^+} \right) \underline{\dot{\gamma}}^+ \quad \text{for} \quad \tau^+ > Bn \quad (13ii)$$

where $\underline{\dot{\gamma}}^+ = \dot{\gamma}H / \sqrt{g\beta\Delta T_{ref}H}$ is the non-dimensional strain rate tensor. The quantities Ra , Pr and Bn in Eqs. 10, 11 and 13 are “nominal” (Turan *et al.*, 2010; 2011a; 2012a,b) Rayleigh, Prandtl and Bingham numbers respectively, which can be defined for the CWT and CWHF boundary conditions as:

$$Ra_{CWT} = \frac{\rho^2 c_p g \beta \Delta T H^3}{\mu k} = Gr_{CWT} Pr ; Ra_{CWHF} = \frac{\rho^2 c_p g \beta q H^4}{\mu k^2} = Gr_{CWHF} Pr ; \quad (14i)$$

$$Bn_{CWT} = \frac{\tau_y}{\mu} \sqrt{\frac{H}{g\beta\Delta T}} \quad ; \quad Bn_{CWHF} = \frac{\tau_y}{\mu} \sqrt{\frac{k}{g\beta q}} \quad (14ii)$$

where Gr_{CWT} and Gr_{CWHF} are the Grashof number in the CWT and CWHF configurations respectively and Pr is the Prandtl number, which are defined as:

$$Gr_{CWT} = \frac{\rho^2 g \beta \Delta T H^3}{\mu^2} \quad ; \quad Gr_{CWHF} = \frac{\rho^2 g \beta q H^4}{\mu^2 k} \quad \text{and} \quad Pr = \frac{\mu c_p}{k}. \quad (15)$$

An effective viscosity which may be estimated as $\mu_{eff} = \tau_y / \dot{\gamma} + \mu$ may be more representative of the viscous stress within the flow than the constant plastic viscosity μ because the viscosity varies throughout with the local instantaneous shear rate in a Bingham fluid flow. Therefore the Rayleigh, Prandtl and Bingham numbers could have been defined differently if μ_{eff} was used instead of μ .

However $\dot{\gamma}$ will exhibit local variations in the flow domain so using a single characteristic value in the definitions of the non-dimensional numbers may not offer any additional benefit in comparison to the definitions given by Eqs. 14 and 15 (see detailed discussion in Ref. 10).

According to dimensional analysis, the Nusselt number $Nu = hL/k$ in the CWT (CWHF) configuration can be expressed as: $Nu = f(Ra_{CWT}, Pr, Bn_{CWT}, AR)$ (

$Nu = f(Ra_{CWHF}, Pr, Bn_{CWHF}, AR)$) where h is the heat transfer coefficient, which is defined as:

$$h = \left| -k \frac{\partial T}{\partial x_2} \right|_{wf} \times \frac{1}{(T_{wall} - T_{ref})} \quad (16)$$

where subscript 'wf' refers to the condition of the fluid in contact with the wall, T_{wall} is the wall temperature and T_{ref} is the appropriate reference temperature, which can be taken to be the temperature of the hot (cold) wall respectively.

Boundary conditions

The simulation domain is shown schematically in Fig. 1 where the two horizontal walls of a rectangular enclosure are subjected to either CWT or CWHF, whereas the other boundaries are considered to be adiabatic in nature. The velocity components (i.e. $u_1 = u$ and $u_2 = v$) are identically zero on each boundary because of the no-slip condition and impenetrability of rigid boundaries. For the CWHF configuration, the heat fluxes for the cold and hot vertical walls are specified (i.e. $-k(\partial T / \partial x_2)|_{x_2=0} = q$ and $-k(\partial T / \partial x_2)|_{x_2=H} = q$). In contrast, for the CWT configuration, the temperatures for the hot and cold vertical walls are specified (i.e. $T(x_2 = 0) = T_H$ and $T(x_2 = H) = T_C$). The temperature boundary conditions for the vertical insulated boundaries are given by: $\partial T / \partial x_1 = 0$ at $x_1 = 0$ and $x_1 = L$. The non-dimensional forms of the above boundary conditions are:

Bottom horizontal wall (i.e. $x_2^+ = 0$)

$u_1^+ = 0$ (no slip); $u_2^+ = 0$ (impenetrability); $\Theta = 1.0$ (for CWT); $-\partial \Theta / \partial x_2^+ = 1$ (for CWHF);

Top horizontal wall (i.e. $x_2^+ = 1$)

$u_1^+ = 0$ (no slip); $u_2^+ = 0$ (impenetrability); $\Theta = 0.0$ (for CWT); $-\partial \Theta / \partial x_2^+ = 1$ (for CWHF);

Vertical walls (i.e. $x_1^+ = 0$ and $x_1^+ = 1/AR$)

$u_1^+ = 0$ (impenetrability); $u_2^+ = 0$ (no slip); $\partial \Theta / \partial x_1^+ = 0$.

3. NUMERICAL IMPLEMENTATION

The coupled conservation equations given by eqs. 9-13 have been solved in a coupled manner using the finite-volume method where a second-order central differencing scheme is used for the diffusive terms and a second-order up-wind scheme for the convective terms. The well-known SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm (Patankar, 1980) is used for the coupling of the pressure and velocity and the convergence criteria were set to 10^{-6} for all the relative (scaled) residuals for the iterative solution method.

The grid independence of the numerical findings (i.e. maximum uncertainty in $\overline{Nu}$ and flow patterns for Newtonian and Bingham fluids) has been examined based on a careful analysis of a number of different non-uniform Cartesian meshes. It has been ensured that a change in the computational grid does not lead to a change in flow pattern within the enclosure during the course of the grid independency analysis for $AR < 1$ because the flow pattern changes (i.e. number of cells) significantly depending on Ra and AR prevalently for CWT configuration. Accordingly, the simulations have been carried out using Cartesian meshes with non-uniform grid spacing. For the $AR = 0.25$ case non-uniform meshes 100×100 , 180×220 and 200×240 have been used in order to establish grid independence. The maximum difference in $\overline{Nu}$ has been found to be smaller than 0.3% for the range of Ra (i.e. $10^3 \leq Ra \leq 10^5$) for Newtonian fluids (i.e. $Bn = 0$). Based on a compromise between computational cost and numerical accuracy the non-uniform mesh of 180×220 has been used for all of the $AR = 0.25$ simulations. Non-uniform meshes of 100×100 and 200×240 have been utilized for the simulations involving $AR = 0.5$ and the maximum uncertainty in $\overline{Nu}$ for Newtonian fluids was found to be smaller than 0.15%. Thus, the non-uniform mesh of 100×100 has been used for $AR = 0.5$ cases. Non-uniform meshes of 100×100 and 160×160 (100×150 and 120×240) have been used to check grid-dependency for $AR = 1.0$ ($AR = 2.0$) cases and the maximum uncertainty in $\overline{Nu}$ for Newtonian fluids was found to be smaller than 0.1%. Thus the non-uniform meshes of 100×100 and 100×150 have been used for $AR = 1.0$ and $AR = 2.0$ cases respectively. The non-uniform mesh of 100×200 has been used for $AR = 4.0$ simulations. The non-dimensional minimum grid spacing ($\Delta_{min,cell} / L$) and grid expansion ratio (r) values for the Cartesian meshes are 6.651×10^{-4} , 1.015 for $AR=0.25$ (180×220), 1.269×10^{-3} , 1.015 for $AR=0.5$ (100×100), 2.5373×10^{-3} , 1.036 for $AR=1.0$ (100×100), 2.5373×10^{-3} , 1.015 for $AR=2$ (100×150) and 2.5373×10^{-3} , 1.015 for $AR=4.0$ (100×200) respectively. Simulations involving Bingham fluid with $Bn = 0.01$ have also been carried out to perform a grid-convergence analysis and the maximum numerical uncertainty in $\overline{Nu}$ for Bingham fluid is found to be slightly greater than that in the case of Newtonian fluid. The maximum

numerical uncertainty in $\overline{Nu}$ for $Bn = 0.01$ was found to be less than 0.7% for both CWT and CWHF boundary conditions. Additionally, the mean Nusselt numbers $\overline{Nu}$ for laminar Rayleigh-Bénard convection of Newtonian fluids in square enclosures (i.e. $AR = 1$) for $10^3 \leq Ra \leq 10^6$ and $Pr = 0.71$ are compared to the benchmark data by Quertatani *et al.* (2008) in Table 1. It is evident from Table 1 that an excellent agreement has been achieved between the present results and the benchmark data reported by Quertatani *et al.* (2008). Furthermore, the Bingham fluid simulations have been benchmarked with respect to the results reported by Vola *et al.* (2003) for natural convection in square enclosures with vertical walls with different uniform temperatures and an excellent agreement (e.g. maximum deviation of mean Nusselt number $\overline{Nu}$ is of the order of 3%) has been obtained. Interested readers are referred to Turan *et al.* (2010) for further discussion in this respect (e.g. see Table 5 of Turan *et al.*, 2010).

It is worth noting that only pure-conduction results are obtained when a quiescent initial condition is used for Bingham fluid simulations in the current configuration because Bingham fluid flow is unconditionally linearly stable under quiescent initial conditions (Zhang *et al.*, 2006). However, convection of Bingham fluids can be obtained when an initial condition with an established flow is used. Here, the steady-state solution for Newtonian (i.e. $Bn = 0$) fluid is used as the initial condition for Bingham fluids by increasing Bn for a given set of values of nominal Rayleigh and Prandtl numbers. It has been confirmed that the same values of mean Nusselt number obtained by increasing Bn from Newtonian fluid (i.e. $Bn = 0$) flow can be obtained when Bn is reduced. It is possible to turn on or off the yield stress by applying electrical or magnetic fields in electro-rheological/magneto-rheological fluids so a Newtonian solution can be considered as a realistic initial condition with practical relevance. The Bingham fluid simulations for both CWT and CWHF boundary conditions have been run for Bingham numbers Bn ranging from 0 to $Bn_{\max}$ where $Bn_{\max}$ is the Bingham number at which $\overline{Nu}$ approaches to unity (i.e. $\overline{Nu} = 1.0$) and the solution essentially becomes the steady-state pure-conduction result.

4. RESULTS & DISCUSSION

4.1 Aspect ratio effects on flow and temperature distributions

The variations of non-dimensional vertical velocity component $V = u_2 L / \alpha$ with x_1 / L along the horizontal mid-plane for a Newtonian (i.e. $Bn = 0$) fluid and a representative Bingham fluid (i.e. $Bn = 0.025$) are shown in Fig. 2 for both CWT and CWHF boundary conditions at different values of nominal Rayleigh number Ra and aspect ratio AR . It is evident from Fig. 2 that the magnitude of V decreases with increasing aspect ratio for Newtonian and Bingham fluids for both CWT and CWHF boundary conditions. Equating the order of magnitudes of the inertial and buoyancy forces (i.e. $\rho u_2^2 / H \sim \rho g \beta \Delta T$) yields (Turan *et al.*, 2014; Yigit *et al.*, 2015a):

$$u_2 \sim \sqrt{g \beta \Delta T H} \text{ (for CWT); } u_2 \sim \sqrt{g \beta q \delta_{th} H / k} \text{ (for CWHF);} \quad (17i)$$

$$V = u_2 L / \alpha \sim AR^{-1} \sqrt{Ra Pr} \text{ (for CWT); } V = u_2 L / \alpha \sim AR^{-1} \sqrt{Ra Pr (\delta_{th} / H)} \text{ (for CWHF).} \quad (17ii)$$

Equation 17ii indicates that the magnitude of V is expected to decrease with increasing AR for both CWT and CWHF boundary conditions, which is consistent with the observations made from Fig. 2. It can further be seen from eq. 17ii that the magnitude of V is expected to increase with increasing Ra for a given set of values of AR and Pr , which is also confirmed in Fig. 2. The number of rolls within the enclosure changes with the change in aspect ratio for the CWT boundary condition, and thus the qualitative nature of the distribution of V , is different for different values of aspect ratio AR . However, the qualitative nature of the flow pattern remains unaltered with the change in AR in the case of CWHF boundary condition. This fundamental difference in flow structure can be illustrated by the contours of non-dimensional stream function $\Psi = \psi / \alpha$ shown in Fig. 3 for different values of AR at $Ra = 10^5$, $Pr = 500$ and $Bn = 0.025$.

The magnitude of V provides a measure of the strength of advective transport within the enclosure. This can be confirmed from Fig. 4 where the variations of non-dimensional temperature

$\theta_{CWT} = (T - T_{cen}) / (T_H - T_C)$ and $\theta_{CWHF} = (T - T_{cen})k / qH$ with x_2 / H along the vertical mid-plane (i.e. $x_1 / L = 0.5$) are provided for the cases shown in Fig. 2. A comparison between Figs. 2 and 4 reveals that the distributions of θ_{CWT} and θ_{CWHF} with x_2 / H become increasingly non-linear as the magnitude of V increases and for tall enclosures with $AR \gg 1$ (e.g. $AR = 4.0$) the non-dimensional vertical velocity component V assumes negligible values and the non-dimensional temperatures θ_{CWT} and θ_{CWHF} become almost a linear function of x_2 / H indicative of conduction-driven thermal transport. The strengthening of the relative contribution of conduction to overall thermal transport with increasing AR can be confirmed from the distributions of θ_{CWT} and θ_{CWHF} shown in Fig. 5 for different values of AR at $Ra = 10^5$, $Pr = 500$ and $Bn = 0.025$. It is evident from Fig. 5 that the contours of θ_{CWT} and θ_{CWHF} become increasingly curved due to convection with decreasing AR , whereas the isotherms remain parallel to the horizontal wall for tall enclosures with $AR \gg 1$ (e.g. $AR = 4.0$) due to conduction-driven thermal transport.

The weakening of advective transport with increasing aspect ratio AR is reflected in the increasing size of Apparently Unyielded Regions (AURs) (regions where $|\tau| \leq \tau_y$ (Mitsoulis and Zisis, 2001), which are shown in grey in Fig. 3. The weakening (strengthening) of advective (conductive) transport with increasing AR can be explained with the help of the energy flux integral at the vertical mid-plane, which is given by (Yigit *et al.*, 2015a):

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{diff} = \int_0^H \rho c_p u_1 dx_2 - \int_0^H k(\partial T / \partial x_1) dx_2 \quad (18i)$$

where

$$\dot{Q}_{conv} = \int_0^H \rho c_p u_1 dx_2 \approx \int_0^\delta \rho c_p u_1 dx_2 \quad \text{and} \quad \dot{Q}_{diff} = - \int_0^H k(\partial T / \partial x_1) dx_2 \quad (18ii)$$

where δ is the hydro-dynamic boundary-layer thickness on the horizontal walls. Using the continuity equation (i.e. $u_1 / L \sim u_2 / H$) it is possible to obtain:

$$u_1 \sim u_2 AR^{-1} \sim (\alpha / L) \sqrt{RaPr} AR^{-2} \text{ (for CWT);} \quad (19i)$$

$$u_1 \sim u_2 AR^{-1} \sim (\alpha / L) \sqrt{RaPr(\delta_{th} / H)} AR^{-2} \text{ (for CWHF).} \quad (19ii)$$

Equating the order of magnitudes of inertial and viscous resistance yields:

$$\rho u_1^2 / L \sim [\tau_y + \mu(u_1 / \delta)] / \delta. \quad (20)$$

Using eq.19 in eq. 20 leads to:

$$\delta / H \sim \sqrt{Pr / Ra} [(Bn \cdot AR) / 2 + 0.5 \sqrt{(Bn \cdot AR)^2 + 4(Ra / Pr)^{1/2}}] \text{ (for CWT)} \quad (21)$$

$$\frac{1}{f_1^{1/2}} \sqrt{\frac{Ra}{Pr}} \left(\frac{\delta}{H} \right)^{5/2} \sim (Bn \cdot AR) f_1^{1/2} \left(\frac{\delta}{H} \right)^{1/2} + 1 \text{ (for CWHF)} \quad (22)$$

where $f_1 = \delta / \delta_{th}$ and this quantity is a function of Ra, Pr, Bn and AR , which is expected to increase with increasing nominal Prandtl number. It is not possible to solve eq. 22 analytically but useful limiting conditions can be inferred from it. For $Bn = 0$ (i.e. Newtonian fluid) eq. 22 yields:

$\delta / H \sim f_1^{1/5} (Pr / Ra)^{1/5}$, whereas for large values of $(Bn \cdot AR)$ one gets:

$\delta / H \sim f_1^{1/2} (Bn \cdot AR)^{1/2} (Pr / Ra)^{1/4}$. Both eqs. 21 and 22 suggest that δ is expected to decrease

(increase) with increasing Ra (Bn and AR). Using eqs. 21 and 22 in eq. 18ii yields the following

scaling estimates for the maximum magnitudes of $\dot{Q}_{conv}$ and $\dot{Q}_{diff}$:

$$\dot{Q}_{conv} \sim \rho c_p u_1 \Delta T \delta \sim 0.5(k \Delta T) Pr [Bn + \sqrt{Bn^2 + 4(Ra / Pr)^{1/2} / AR^2}] \text{ (for CWT);} \quad (23i)$$

$$\dot{Q}_{diff} = - \int_0^H k (\partial T / \partial x_1) dx_2 \sim (k \Delta T) AR \text{ (for CWT);} \quad (23ii)$$

$$\dot{Q}_{conv} \sim \rho c_p u_1 q \delta_{th} \delta / k \sim q H \sqrt{RaPr} (\delta / H)^{5/2} f_1^{-3/2} AR^{-1} \text{ (for CWHF);} \quad (24i)$$

$$\dot{Q}_{diff} = - \int_0^H k (\partial T / \partial x_1) dx_2 \sim q H (\delta / H) f_1^{-1} AR \text{ (for CWHF).} \quad (24ii)$$

Equations 23 and 24 indicate that $\dot{Q}_{conv}$ ($\dot{Q}_{diff}$) weakens (strengthens) with increasing AR when

$k, \Delta T, Ra, q$ and Pr are held unaltered for both Newtonian and Bingham fluids, which is consistent

with the observations made from Figs. 2-5. As the relative contribution of the advective (diffusive)

thermal transport to overall heat transfer rate weakens (strengthens) with increasing AR , the mean Nusselt number $\overline{Nu}$ decreases monotonically with an increase in AR for both Newtonian (i.e. $Bn = 0$) and Bingham fluids (see Figs. 2 and 4) for all values of Ra except for Newtonian fluids at $Ra = 10^5$ in the case of the CWT boundary condition. For Newtonian fluids the number of convection cells within the enclosure changes between $Ra = 5 \times 10^4$ and $Ra = 10^5$ at $AR = 0.25$ for the CWT boundary condition but the flow pattern remains unchanged for the CWHF boundary condition, which is demonstrated in Fig. 6. This change in flow structure leads to a slightly smaller value of $\overline{Nu} = AR \int_0^L h dx_1 / k$ in the $AR = 0.25$ than in the $AR = 0.5$ case for $Ra = 10^5$ for CWT boundary condition, whereas a single cell flow pattern is observed for all the parameter range considered here for the CWHF boundary condition. Nevertheless, the mean Nusselt number $\overline{Nu}$ is found to be insignificantly smaller for $AR = 0.25$ and $Ra = 10^5$ than that is for $AR = 0.5$ and $Ra = 10^5$ for the CWHF boundary condition. Apart from that issue, the mean Nusselt number $\overline{Nu}$ decreases monotonically with increasing AR . The change in number of cells within the enclosure in response to the a change in AR in the case of Newtonian fluids for CWT boundary condition is qualitatively consistent with previous findings by Daniels (1984) and Gelfgat (1999) who showed that a decrease in aspect ratio AR of the enclosure leads to an increase in number of rolls in Rayleigh-Bénard convection of Newtonian fluids. For Bingham fluids this change in cell structure is not observed as a consequence of the nominal definition of Ra . For example, it is possible to estimate an effective viscosity within the horizontal boundary layer using eqs. 21 and 22 as:

$$\begin{aligned} \mu_{eff} &\sim \mu + \tau_y \delta / u_1 \sim \mu [1 + (Bn \cdot AR)(\delta / H)] \\ &\sim \mu [1 + 0.5 \cdot AR \cdot Bn \sqrt{Pr / Ra} \{ (Bn \cdot AR) + \sqrt{(Bn \cdot AR)^2 + 4(Ra / Pr)^{1/2}} \}] \end{aligned} \quad (\text{for CWT}); \quad (25i)$$

$$\mu_{eff} \sim \mu + \tau_y \delta / u_1 \sim \mu [1 + (Bn \cdot AR) f_1^{0.5} (\delta / H)^{1/2}] \quad (\text{for CWHF}); \quad (25ii)$$

which yields an expression for the effective Rayleigh number Ra_{eff} :

$$\begin{aligned} Ra_{eff} &= \rho g \beta (T_H - T_C) H^3 / \mu_{eff} \alpha \sim Ra / [1 + (Bn \cdot AR)(\delta / H)] \\ &\sim Ra / [1 + 0.5 \cdot AR \cdot Bn \sqrt{Pr / Ra} \{ (Bn \cdot AR) + \sqrt{(Bn \cdot AR)^2 + 4(Ra / Pr)^{1/2}} \}] \end{aligned} \quad (\text{for CWT}); \quad (26i)$$

$$Ra_{eff} = \rho g \beta q H^4 / k \mu_{eff} \alpha \sim Ra / [1 + (Bn \cdot AR) f_1^{0.5} (\delta/H)^{1/2}] \text{ (for CWHF)}. \quad (26ii)$$

It can be inferred from eq. 26 that Ra_{eff} assumes a smaller value than Ra for Bingham fluids for both CWT and CWHF boundary conditions. Thus at $Ra = 10^5$ and $Bn = 0.025$ the flow pattern within the enclosure remains qualitatively similar to Newtonian fluids at smaller values of Ra in the case of CWT boundary condition which can be confirmed from inspection of Fig. 6. As a result, the flow structure does not change for Bingham fluids between $Ra = 5 \times 10^4$ and $Ra = 10^5$ for the CWT boundary condition, which leads to a monotonic decrease in $\overline{Nu}$ with increasing AR for Bingham fluids (see Fig. 2 and 4).

A comparison between CWT and CWHF results in Figs. 2-4 reveals that the magnitude of V is smaller for the CWHF boundary condition than in the CWT boundary condition for a given set of values of Ra , AR and Pr . This difference can be explained from eqs. 17i and ii, which indicates that V is expected to be smaller in the CWHF case than in the CWT situation because $0 < \delta_{th}/H < 1$. Furthermore, it is evident from Fig. 4 that the temperature difference between the active walls decreases with increasing (decreasing) Ra (AR) for a given set of values of AR (Ra) and Pr . The non-dimensional temperature scales as:

$$\theta_{CWT} \sim O(1) \text{ (for CWT); } \theta_{CWHF} \sim O[(q\delta_{th}/k)/(qH/k)] \sim O[(\delta_{th}/H)] \text{ (for CWHF)}. \quad (27)$$

It has been discussed earlier (see the discussion related to eq. 22) that δ_{th}/H is expected to decrease (increase) with increasing Ra (AR) and thus the distribution of θ_{CWHF} for CWHF boundary condition is consistent with the observations made from Fig. 4. The smaller temperature difference between the active walls in the case of CWHF boundary condition induces a smaller magnitude of vertical velocity in the case of the CWHF boundary condition than in the case of the CWT boundary condition.

4.2 Bingham number effects

The variations of non-dimensional temperature (i.e. θ_{CWT} and θ_{CWHF}) and non-dimensional vertical velocity component V along the vertical and horizontal mid-planes are shown in Figs. 7 and 8 respectively for the CWT and CWHF boundary conditions for different values of Bn for $Ra = 5 \times 10^4$ and $AR = 0.25, 0.5, 1.0, 2.0$. Figures 7 and 8 reveal that an increase in Bn leads to a decrease in the magnitude of V and the extent of θ non-linearity along the mid-planes, which is indicative of the weakening of advective thermal transport. The additional resistance arising from yield stress leads to weakening of thermal advective transport in Bingham fluids in comparison to a Newtonian fluid, and this trend strengthens with increasing Bn . Equations 25 and 26 suggest that the effective viscosity μ_{eff} (effective Rayleigh number Ra_{eff}) increases (decreases), which indicates weakening of buoyancy effects in comparison to viscous forces, with increasing Bingham number. The weakening of convective transport with increasing Bn can be confirmed from Fig. 9 where the contours of θ_{CWT} , θ_{CWHF} and Ψ for different values of Bn for $Ra = 5 \times 10^4$, $Pr = 500$ and $AR = 0.5$ are shown. Figure 9 reveals that AURs expand in size and isotherms become increasingly parallel to the horizontal walls with increasing Bn . Equations 25 and 26 indicate that an increase in AR acts to increase (decrease) effective viscosity μ_{eff} (effective Rayleigh number Ra_{eff}) in case of Bingham fluids and it has indeed been found that the aspect ratio AR at which convection ceases to play a significant part in thermal transport decreases with increasing Bn .

4.3 Behaviour of mean Nusselt number $\overline{Nu}$

The wall heat flux q can be scaled as: $q = h\Delta T \sim k\Delta T / \delta_{th}$ which leads to $\overline{Nu} \sim qH / \Delta Tk \sim H / \delta_{th}$ which in turn leads to the following expression:

$$\overline{Nu} \sim \text{Max} \left[1.0, \left[\frac{(Ra / Pr)^{1/2}}{\left[\frac{Bn \cdot AR}{2} + \frac{1}{2} \sqrt{(Bn \cdot AR)^2 + 4 \left(\frac{Ra}{Pr} \right)^{1/2}} \right]} f_2(AR, Ra, Pr, Bn) \right] \right] \quad (\text{for CWT}); \quad (28)$$

where $f_2(Ra, Pr, AR, Bn)$ represents the ratio δ_{th} / δ for the CWT boundary condition. Equation 28 indicates that the mean Nusselt number is expected to decrease with increasing Bn and AR , whereas $\overline{Nu}$ increases with increasing Ra . Eq. 22 suggests that δ_{th} for the CWHF boundary condition is expected to increase with increasing Bn and AR , whereas δ_{th} decreases with increasing Ra . This suggests that Ra , Bn and AR dependences of $\overline{Nu}$ in the CWHF configuration are expected to be qualitatively similar to those for the CWT boundary condition.

The variations of the mean Nusselt number for Newtonian fluids (at a given Ra and Pr) Nu_0 with AR for Newtonian fluids are shown for both CWT and CWHF boundary conditions in Fig. 10 for $5 \times 10^3 \leq Ra \leq 10^5$. It can be seen from Fig. 10 that the mean Nusselt number values are comparable for both CWT and CWHF boundary conditions for small values of Ra but $\overline{Nu}$ is larger for the CWT boundary condition than in the case of the CWHF boundary condition for high values of Ra . This behaviour can be explained by the scaling estimate of Nu_0 for the CWT boundary condition which can be obtained by putting $Bn = 0$ in eq. 28, which yields $Nu_0 \sim (Ra / Pr)^{1/4} f_2$ for the CWT boundary condition. Similarly, putting $Bn = 0$ in eq. 22 gives rise to $Nu_0 \sim (Ra / Pr)^{1/5} f_1^{4/5}$ for the CWHF boundary condition. A comparison between $Nu_0 \sim (Ra / Pr)^{1/4} f_2$ (for CWT) and $Nu_0 \sim (Ra / Pr)^{1/5} f_1^{4/5}$ (for CWHF) reveals that $(Ra / Pr)^{1/5}$ is expected to be smaller than $(Ra / Pr)^{1/4}$ for large values of nominal Rayleigh number for a given value of nominal Prandtl number. This, in turn, gives rise to a greater magnitude of Nu_0 for the CWT boundary condition than in the case of the CWHF boundary condition. However, $Pr^{1/4} > Pr^{1/5}$ may overcome $Ra^{1/4} > Ra^{1/5}$ for small magnitudes of Ra and lead to a greater magnitude of Nu_0 for the CWHF boundary condition than in the case of the CWT boundary condition.

Here the variation of $\overline{Nu}$ for Newtonian fluids is parameterised as:

$$Nu_0 = \left[\{m_0 / [1 + \exp\{(AR - x_0) / n_0\}]\} - y_0 \right] (AR - 1)^{0.01} + Nu_0(AR = 1) \quad (29i)$$

where x_0, y_0, m_0 and n_0 are given by:

$$\begin{aligned} x_0 &= 0.55 + 29.25 \exp[-0.5\{(\ln Ra - 11.12) / 0.306\}^2]; \quad y_0 = 0.455 + 4097 / [1 + \exp\{(46754 - Ra) / 4258\}]; \\ m_0 &= 0.765 + 4097 / [1 + \exp\{(46743 - Ra) / 4270\}] \quad \text{and} \quad n_0 = 0.022 + 3.31 \exp[-0.5\{(\ln Ra - 11.10) / 0.301\}^2] \quad (\text{for CWT}); \end{aligned} \quad (29ii)$$

and

$$\begin{aligned} x_0 &= 0.613 + 2.21 / [1 + \exp\{(10.43 - \ln Ra) / 0.42\}]; \quad y_0 = -33.6 + 40.95 / [1 + \exp\{(-\ln Ra - 4.658) / 8.03\}]; \\ m_0 &= 0.767 + 1.917 / [1 + \exp\{(9.194 - \ln Ra) / 0.715\}] \quad \text{and} \quad n_0 = 0.04 + 0.69 / \left\{ 1 + [(\ln Ra - 11) / 0.644]^2 \right\} \quad (\text{for CWHF}). \end{aligned} \quad (29iii)$$

The correlations proposed by Turan *et al.* (2012a,b) are used here for the evaluation of $Nu_0(AR = 1)$:

$$Nu_0(AR = 1) = 0.178 Ra^{0.269} \left(\frac{Pr}{1 + Pr} \right)^{0.02} \quad \text{for (CWT);} \quad (29iv)$$

$$Nu_0(AR = 1) = 0.289 Ra^{0.214} \left(\frac{Pr}{1 + Pr} \right)^{0.017} \quad \text{for (CWHF).} \quad (29v)$$

Equations 29iv and 29v are consistent with the scaling estimates of Nu_0 . The prediction of eq. 29 is shown in Fig. 10 where it can be seen to satisfactorily predict the variations of Nu_0 with AR for Newtonian fluids except for $AR=0.25$ at $Ra=10^5$ for the CWT configuration where the overprediction of Nu_0 by eq. 29 originates due to the change in flow pattern, which is not captured by eq. 29 and leads to about a 5% overprediction of Nu_0 at this single point ($AR=0.25$ and $Ra=10^5$ for CWT).

Equation 28 and Figs. 7-9 indicate that $\overline{Nu}$ is expected to decrease with increasing Bn , which can be confirmed from Fig. 11 where the variations of $\overline{Nu}$ with Bn are shown for different values of Ra and AR for both CWT and CWHF boundary conditions. Equations 25 and 26 suggest that an increase in Bn for large values of Bingham number leads to a rapid increase (decrease) in effective viscosity μ_{eff} (effective Rayleigh number Ra_{eff}). In Rayleigh-Bénard convection fluid motion stops once the effective Rayleigh number Ra_{eff} drops below the critical Rayleigh number Ra_{crit} for the onset of convection. An abrupt cessation of convection is obtained once the effective Rayleigh number Ra_{eff} drops below Ra_{crit} (i.e. $Ra_{eff} < Ra_{crit}$), which leads to a sudden drop of the mean Nusselt number $\overline{Nu}$ to a unity value indicating pure conduction-driven transport in the absence of thermal convection. Figure 11 suggests that $\overline{Nu}$ attains a value equal to unity once a certain Bingham number is exceeded, which is defined here as Bn_{max} (i.e. $\overline{Nu} > 1$ for $Bn < Bn_{max}$ and $\overline{Nu} = 1$ for $Bn \geq Bn_{max}$). The value of Bn_{max} increases with increasing (decreasing) Ra (AR) as yield stress effects are surpassed by stronger buoyancy effects up to high values of Bn for high (small) values of Ra (AR). This can further be substantiated by inserting $\overline{Nu} = 1.0$ in eq. 28, which leads to:

$$Bn_{max} \sim AR^{-1} [f_2 \sqrt{Ra / Pr} - 1 / f_2] \quad (\text{for CWT}) \quad (30)$$

which suggests that Bn_{max} is expected to rise with increasing (decreasing) Ra (AR). Although it is not possible to obtain a relation similar to eq. 30 based on eq. 22, Bn_{max} for the CWHF configuration is expected to exhibit the same qualitative behaviour as in the case of the CWT boundary condition.

Similarly, the mean Nusselt number $\overline{Nu}$ for Newtonian fluids (i.e. $Bn = 0$) can be obtained by putting $Bn = 0$ in eq. 28:

$$\overline{Nu} \sim \text{Max}[1.0, (Ra / Pr)^{1/4} f_2(AR, Ra, Pr, 0)] \quad (\text{for CWT}). \quad (31i)$$

For laminar convective regime where $\overline{Nu} \gg 1$ one can rewrite eqs. 28 and 31i as:

$$\overline{Nu} - 1 \sim \frac{2(Ra / Pr)^{1/2}}{[Bn \cdot AR + \sqrt{(Bn \cdot AR)^2 + 4(Ra / Pr)^{1/2}}]} f_2(AR, Ra, Pr, Bn); \quad Nu_0 - 1 \sim (Ra / Pr)^{1/4} f_2(AR, Ra, Pr, 0) \quad (31ii)$$

Using eq. (31ii) one can write:

$$\frac{\overline{Nu} - 1}{Nu_0 - 1} \sim \frac{2(Ra / Pr)^{1/4}}{[Bn \cdot AR + \sqrt{(Bn \cdot AR)^2 + 4(Ra / Pr)^{1/2}}]} \frac{f_2(AR, Ra, Pr, Bn)}{f_2(AR, Ra, Pr, 0)} \quad (31iii)$$

Yigit *et al.*, (2015a) utilised eq. 31iii to propose correlations of $\overline{Nu}$ for natural convection of Bingham fluids in rectangular enclosures for CWT boundary condition in the ranges of $1/4 \leq AR \leq 4$ and $0 \leq Bn \leq Bn_{max}$:

$$\frac{\overline{Nu} - 1}{Nu_0 - 1} = \frac{2[1 - (Bn^* / Bn_{max}^*)^b]^c}{Bn^* + \sqrt{Bn^{*2} + 4}} \quad \text{when } \overline{Nu}_{Bn=0} > 1 \quad \text{and} \quad \overline{Nu} = 1 \quad \text{when } \overline{Nu}_{Bn=0} = 1 \quad (32)$$

where $Bn^* = Bn \cdot AR \cdot (Ra / Pr)^{-1/4}$ and $Bn_{max}^* = Bn_{max} \cdot AR \cdot (Ra / Pr)^{-1/4}$. Accordingly, the eq. 32 is used for $1/4 \leq AR \leq 4$ when b and c are appropriately parameterised. Although it is not possible to obtain a closed form scaling estimate for $\overline{Nu}$ in the case of CWHF boundary condition, eq. 32 can still be used for the parameterisation of mean Nusselt number following previous analyses (Turan *et al.*, 2011a,b; 2014). The parameters b and c are proposed in the following manner:

$$b = 0.6 \quad \text{and} \quad c = 0.025 Ra^{0.171} Pr^{0.095} - m_1 \exp[-0.5\{(AR - 0.366)/n_1\}^2] \cdot |AR - 1| \quad \text{for (CWT);} \quad (33i)$$

$$b = 0.75 \text{ and } c = 0.0818 Ra^{0.1019} Pr^{0.0540} - m_2 \exp[-0.5\{(AR - 0.366)/0.024\}^2] \cdot |AR - 1| \text{ for (CWHF)} \quad (33ii)$$

where

$$m_1 = 0.03 + 0.0334/[1 + \exp\{(9.48 - \ln Ra)/0.06\}] \text{ and } n_1 = 0.09 + 0.032/[1 + \exp\{(\ln Ra - 9.13)/0.083\}] \quad (33iii)$$

$$\text{and } m_2 = -2091 + 28070/[1 + \exp\{(9.419 - \ln Ra)/0.583\}]. \quad (33iv)$$

Finally $Bn_{\max}$ is estimated by identifying the Bingham number for which the mean Nusselt number deviates from $\overline{Nu} = 1.0$ in the second decimal place. Here $Bn_{\max}$ obtained from numerical simulation data is parameterised as:

$$Bn_{\max}(AR) = \left[\left\{ m_3 / (1 + \exp\{(AR - x_1)/n_2\}) \right\} - y_1 \right] |AR - 1|^{0.01} + Bn_{\max(AR=1)} \quad (\text{for CWT}); \quad (34i)$$

where $Bn_{\max(AR=1)}$ and the parameters y_1 , x_1 , n_2 and m_3 for the CWT configuration are given by:

$$Bn_{\max(AR=1)} = \left[\left\{ (0.00048Pr + 0.132)/(Pr + 18.15) \right\} Ra^{0.525} \right] - [(0.0389Pr + 7.687)/(Pr + 13.56)] \quad \text{for } 10 \leq Pr \leq 500; \quad (34ii)$$

$$y_1 = -0.033 + 1.105 / [1 + \exp\{(13.04 - \ln Ra)/1.417\}]; \quad x_1 = 1.686 - 1.154 \exp[-0.5\{(\ln Ra - 9.04)/0.841\}^2];$$

$$n_2 = 0.354 - 0.43 \exp[-0.5\{(\ln Ra - 9.01)/0.282\}^2]; \quad m_3 = -0.01 + 0.804 / [1 + \exp\{(12.07 - \ln Ra)/1.158\}]. \quad (34iii)$$

The quantity $Bn_{\max(AR=1)}$ and the parameters y_1 , x_1 , n_2 and m_3 for the CWHF configuration are given by:

$$Bn_{\max(AR=1)} = \left[\left\{ (Pr + 236)/(2707Pr + 40280) \right\} Ra^{0.525} \right] - [(Pr + 128)/(44.6Pr + 271)] \quad \text{for } 10 \leq Pr \leq 500; \quad (34iv)$$

$$y_1 = -0.0337 + 7.967 / [1 + \exp\{(18.52 - \ln Ra)/1.986\}]; \quad m_3 = 0.0014 + 1.034 / [1 + \exp\{(12.7 - \ln Ra)/1.315\}];$$

$$x_1 = -34.35 + 35.72 / [1 + \exp\{(0.819 - \ln Ra)/1.733\}]; \quad n_2 = 0.214 + 0.266 / [1 + \exp\{(10.2 - \ln Ra)/0.398\}]. \quad (34v)$$

It is worth noting that the correlation given by eqs. 34ii and 34iv are different from simpler correlations for the CWT and CWHF boundary conditions proposed earlier by Turan *et al.* (2012a,b):

$$Bn_{\max (AR=1)} = [0.0019 \ln(Ra_{CWT}) - 0.0128] Ra_{CWT}^{0.55} Pr^{-0.50} \text{ for (CWT);} \quad (35i)$$

$$Bn_{\max (AR=1)} = [0.0412 \ln(Ra_{CWHF}) - 0.3201] Ra_{CWHF}^{0.287} Pr^{-0.50} \text{ for (CWHF).} \quad (35ii)$$

The predictions of eq. 34 are shown in Fig. 12, which shows that this correlation satisfactorily captures the variation of $Bn_{\max}$ for the range of Ra and AR explored here for both CWT and CWHF boundary conditions. In previous analyses (Turan *et al.*, 2012a,b) a less precise definition of $Bn_{\max}$ was used and also less refined meshes in comparison to those used here. Furthermore, the correlations by Turan *et al.* (2012a,b) were proposed for square enclosures for $0.1 \leq Pr \leq 10^2$, whereas the correlation given by eq. 34 not only captures the current simulation data but also satisfactorily predicts the simulation data by Turan *et al.* (2012a,b) and is valid for $0.1 \leq Pr \leq 500$.

A comparison between Figs. 10-12 reveals convection can be sustained up to high values of $Bn_{\max}$ for the cases with high values of Nu_0 . Thus, $Bn_{\max}$ assumes greater values for the CWT boundary condition than in the CWHF configuration for the combination of Ra and Pr where Nu_0 assumes greater values in the CWT boundary condition than in the case of CWHF boundary condition.

The prediction of eq. 32 is shown in Fig. 11 where eqs. 29 and 34 are used for parameterising $Bn_{\max}$ and Nu_0 . It can be seen from Fig. 11 that eq. 32 satisfactorily captures the variation of $\overline{Nu}$ with Bn for the range of Ra and AR explored here. The R^2 value of eq. 32 is greater than 0.99 for both $\overline{Nu}$ and $Bn_{\max}$ correlations. The maximum percentage error for these correlations with respect to the simulation data is found to be about 5% for both CWT and CWHF boundary conditions. Also, it is worth noting that the predictions of eq. 34 are shown in Fig. 12 for $Pr = 500$ but the parameterisation of $Bn_{\max}$ has been calibrated for a large range of Pr (i.e. $10 \leq Pr \leq 500$). Thus, it can be expected that the correlation of $\overline{Nu}$ given by eq. 34 is likely to be valid for a range of different Prandtl

numbers. Indeed, a limited number of additional simulations have been carried out for $Pr = 100$, which showed that eq. 34 is able to predict $\overline{Nu}$ satisfactorily also for these cases.

5. CONCLUSIONS

The influences of wall boundary condition on the effects of aspect ratio AR ($= H/L$ where the H is the enclosure height and L is length of the enclosure) on steady-state laminar Rayleigh-Bénard convection of Bingham fluids with a nominal Prandtl number $Pr = 500$ in rectangular enclosures for $1/4 \leq AR \leq 4$ and $10^3 \leq Ra \leq 10^5$ have been studied. It has been found that advection strengthens with increasing Ra for both Newtonian and Bingham fluids regardless of boundary conditions. The temperature difference between the horizontal walls for the CWHF boundary condition assumes smaller values than in the CWT case for high values of nominal Rayleigh number Ra , which induces weaker convection in the case of CWHF boundary condition than in the case of the CWT boundary condition. This is reflected in the higher values of mean Nusselt number $\overline{Nu}$ for the CWT boundary condition than in the case of CWHF boundary condition for high values of nominal Rayleigh number Ra . However, the mean Nusselt number $\overline{Nu}$ and the temperature difference between the horizontal walls assume comparable values for both CWT and CWHF boundary conditions for small values of nominal Rayleigh number Ra . The mean Nusselt number $\overline{Nu}$ is found to decrease with increasing Bingham number for both CWT and CWHF boundary conditions due to additional flow resistance arising from the yield stress in Bingham fluids. Furthermore, it has been demonstrated that advective (conductive) transport weakens (strengthens) with increasing AR for both Newtonian and Bingham fluids irrespective of the precise boundary condition. Thus, conductive thermal transport becomes dominant for tall enclosures which is reflected in the unity value of the mean Nusselt number (i.e. $\overline{Nu} = 1.0$) for these cases. However, the number of convection cells changes with changing AR for both Newtonian and Bingham fluids in the case of CWT boundary condition, whereas only a single cell flow pattern is observed for the parameter range considered here for the CWHF boundary condition. Although Ra , AR and Bn dependences of $\overline{Nu}$ remain qualitatively similar for both the

CWT and CWHF boundary conditions, smaller $Bn_{\max}$ (where $\overline{Nu}$ assumes a value equal to unity) values are obtained in the CWHF (CWT) configuration than in the case of CWT (CWHF) boundary condition for the high (small) values of Ra . A detailed scaling analysis has been carried out to explain the observed effects of AR dependences on steady-state natural convection of Bingham fluids in rectangular enclosures for both CWT and CWHF boundary conditions. Correlations for the mean Nusselt number $\overline{Nu}$ have been proposed for both CWT and CWHF boundary conditions, the functional forms of which are based on scaling arguments. It has been shown that $\overline{Nu}$ obtained from simulation data for $1/4 \leq AR \leq 4$ and $10^3 \leq Ra \leq 10^5$ in the case of both Newtonian and Bingham fluids can be satisfactorily predicted by the proposed correlations for both CWT and CWHF boundary conditions. The physical insights gained from this analysis and newly proposed correlations will help in optimising the heat transfer rate in natural convection of in rectangular enclosures in practical applications (e.g. cryogenic storage, nuclear cooling, chemical processing and food preparation and preservation) involving yield stress fluids including magneto-rheological and electro-rheological fluids.

REFERENCES

- Balmforth N. J. and Rust A.C. (2009), “Weakly nonlinear viscoplastic convection”, J. Non-Newt. Fluid Mech., Vol. 158, pp. 36-45.
- Balmforth N., Frigaard I., Ovarlez G. (2014), “Yielding to stress: Recent developments in viscoplastic fluid mechanics”, Ann. Rev. Fluid Mech., Vol. 46, 2014, pp. 121-146.
- Barnes H. A. (1999), “The yield stress—a review or ‘ $\pi\alpha\nu\tau\alpha \rho\epsilon\iota$ ’—everything flows?”, J. Non-Newt. Fluid Mech., Vol. 81, pp. 133-178.
- Bodenschatz E., Pesch W. and Ahlers G. (2000), “Recent Developments in Rayleigh-Bénard convection”, Ann. Rev. Fluid Mech., Vol. 32, pp. 709-778.
- Catton I., Ayyaswamy P.S., and Clever R.M. (1974), “Natural convection flow in a finite, rectangular slot arbitrarily oriented with respect to the gravity vector”, Int. J. Heat Mass Transfer, Vol. 17, pp. 173-184.
- Daniels, P.G.(1984), “Roll-pattern evolution in finite-amplitude Rayleigh-Benard convection in a two-dimensional fluid layer bounded by distant sidewalls” J. Fluid Mech., Vol. 143, pp. 125-152.
- Darbouli M., Métivier C., Piau J.M., Magnin A. and Abdelali A. (2013), “Rayleigh-Bénard convection for viscoplastic fluids”, Physics of Fluids, Vol. 25, pp. 023101.
- Ganguli A.A., Pandit A.B., Joshi J.B. (2009), “CFD simulation of heat transfer in two-dimensional vertical enclosure”, Chem. Eng. Res. Design, Vol. 87, pp. 711-727
- Gelfgat A. Y., (1999), “Different modes of Rayleigh-Bénard instability in two and three dimensional rectangular enclosures”, J. Computational Physics, Vol. 156, pp. 300-324.
- Hassan M.A., Pathak M., Khan M.K. (2013), “Natural convection of viscoplastic fluids in a square enclosure”, J. Heat Transfer Transactions of ASME, Vol. 135, pp. 122501.
- Karimfazli I., Frigaard I.A. (2013), “Natural convection flows of a Bingham fluid in a long vertical channel”, J. non-Newt. Fluid Mech., Vol. 201, pp. 39-55.
- Kebiche Z., Castelain C., Burghellea T. (2014), “Experimental investigation of the Rayleigh-Bénard convection in a yield stress fluid”, J. Non-Newt. Fluid Mech., Vol. 203, pp. 9-23.

- Mitsoulis E. and Zisis T. (2001), "Flow of Bingham plastics in a lid-driven square cavity", *J. Non-Newt. Fluid Mech.*, Vol. 101, pp. 173-180.
- O'Donovan E. J., Tanner R. I. (1984), "Numerical study of the Bingham squeeze film problem", *J. Non-Newt. Fluid Mech.*, Vol. 15, pp. 75-83.
- Ostrach S. (1988), "Natural convection in enclosure", *J. Heat Trans.*, Vol. 110, pp. 1175-1190.
- Papanastasiou T. C. (1987), "Flow of materials with yield," *J. Rheol.*, Vol. 31, pp. 385.
- Park H.M. and Ryu D.H. (2001), "Rayleigh-Bénard convection of viscoelastic fluids in finite domains", *J. Non-Newt. Fluid Mech.*, Vol. 98, pp. 169-184.
- Patankar S. V. (1980), "Numerical Heat Transfer and Fluid Flow", Hemisphere, Washington, D.C.
- Quertatani N., Cheikh N. B., Beya B. B., Lili T. (2008), "Numerical simulation of two dimensional Rayleigh- Bénard convection in an enclosure", *C.R. Mecanique*, Vol. 336, pp. 464-470.
- Stangroom J.E. (1983), "Electrorheological Fluids". *Phys. Technol.* 14 (6): 290–296.
- Turan O., Chakraborty N., Poole R.J. (2010), "Laminar natural convection of Bingham fluids in a square enclosure with differentially heated side walls", *J. Non-Newt. Fluid Mech.*, Vol. 165, pp.901-913.
- Turan O., Sachdeva A., Poole R.J., Chakraborty N. (2011a), "Laminar natural convection of Bingham fluids in a square enclosure with vertical walls subjected to constant heat flux", *Num. Heat Trans. A*, Vol. 60, pp. 381-409.
- Turan O., Chakraborty N., Poole R.J. (2011b), "Aspect ratio effects in laminar natural convection of Bingham fluids in rectangular enclosures with differentially heated side walls", *J. Non-Newt. Fluid Mech.*, Vol. 166, pp. 208-230.
- Turan O., Sachdeva A., Poole R.J., Chakraborty N. (2012a), "Laminar Rayleigh-Bénard convection of yield stress fluids in a square enclosure", *J. Non-Newt. Fluid Mech.*, Vol. 172, pp. 83-96.
- Turan O., Poole R.J., Chakraborty N. (2012b), "Boundary condition effects on natural convection of Bingham fluids in a square enclosure with differentially heated horizontal walls", *J. Comput. Therm. Sci.*, Vol. 4, pp. 77-97.

- Turan O., Poole R.J., Chakraborty N. (2012c), “Influences of boundary conditions on the aspect ratio effects in laminar natural convection in rectangular enclosures with differentially heated side walls”, *Int. J. Heat Fluid Flow*, Vol. 33, pp. 131-146.
- Turan O., Poole R.J., Chakraborty N. (2014), “Influences of boundary conditions on laminar natural convection of Bingham fluids in rectangular enclosures with differentially heated side walls”, *Heat Trans. Eng.*, Vol. 35, pp. 822-849.
- Vikhansky A. (2009), “Thermal convection of a viscoplastic liquid with high Rayleigh and Bingham numbers”, *Phys. Fluids*, Vol. 21, pp. 103103.
- Vikhansky A. (2010), “On the onset of natural convection of Bingham liquid in rectangular enclosures”, *J. Non-Newt. Fluid Mech.*, Vol. 165, pp. 1713-1716.
- Vola D., Boscardin L., Latché J.C. (2003), “Laminar unsteady flows of Bingham fluids: a numerical strategy and some benchmark results,” *J. Comput. Phys.*, Vol. 187, 2003, pp. 441-456.
- Wereley N.M. (2014) “*Magnetorheology : Advances and Applications*”, Royal Society of Chemistry, RSC Smart materials, Cambridge, UK.
- Yigit S., Poole R. J., Chakraborty N. (2015a), “Effects of aspect ratio on natural convection of Bingham Fluids in rectangular enclosures with differentially heated horizontal walls heated from below”, *Int. J. Heat Mass Transfer*, Vol. 80, pp. 727-736.
- Yigit S., Poole R. J., Chakraborty N. (2015b), “Laminar natural convection of Bingham fluids in inclined differentially heated square enclosures subjected to uniform wall temperatures”, *J. Heat Transfer Transactions of ASME*, Vol. 137, 052504.
- Zhang J., Vola D., Frigaard I. A. (2006), “Yield stress effects on Rayleigh-Bénard convection,” *J. Fluid Mech.*, Vol. 566, pp. 389-419.

Table 1. Comparison of the mean Nusselt number $\overline{Nu}$ for Newtonian fluid with the results by Quertatani *et al.*, (2008) for square enclosure ($AR = 1$) at $Pr = 0.71$.

Ra	Present study	Quertatani <i>et al.</i> , (2008)
1×10^3	1.000	1.000
1×10^4	2.154	2.158
1×10^5	3.907	3.910
1×10^6	6.363	6.309

FIGURE CAPTIONS

Figure 1: Schematic diagram of simulation domain for a) CWT, b) CWHF configurations.

Figure 2: Variations of non-dimensional vertical velocity component $V = u_2 L / \alpha$ along the vertical mid-plane x_1 / L for Newtonian ($Bn=0$) and Bingham ($Bn=0.025$) fluid cases for different AR and Ra a) 10^4 , b) 5×10^4 , c) 10^5 at $Pr=500$. The aspect ratio at which the maximum $\overline{Nu}$ is obtained for a given set of Ra and Pr is highlighted with an asterix * in the figure legend.

Figure 3: Contours of non-dimensional stream function Ψ with AURs (shown in grey) for different values of AR for $Ra = 10^5$, $Bn=0.025$ and $Pr = 500$, a) CWT, b) CWHF. The dimensions of the configurations are scaled in such a manner that the widths of the enclosures look the same merely for ease of plotting.

Figure 4: Variations of non-dimensional temperature $\theta_{CWT} = (T - T_{cen}) / (T_H - T_C)$ and $\theta_{CWHF} = (T - T_{cen}) k / qH$ along the vertical mid-plane x_2 / H for Newtonian ($Bn=0$) and Bingham ($Bn=0.025$) fluid cases for different AR and Ra a) 10^4 , b) 5×10^4 , c) 10^5 at $Pr=500$. The aspect ratio at which the maximum $\overline{Nu}$ is obtained for a given set of Ra and Pr is highlighted with an asterix * in the figure legend.

Figure 5: Contours of non-dimensional temperature θ_{CWT} and θ_{CWHF} for different values of AR for $Ra = 10^5$, $Bn=0.025$ and $Pr = 500$, a) CWT, b) CWHF. The dimensions of the configurations are scaled in such a manner that the widths of the enclosures look the same merely for ease of plotting.

Figure 6: Contours of non-dimensional temperature θ_{CWT} and θ_{CWHF} and non-dimensional stream functions Ψ with AURs (shown in grey) for a) Newtonian ($Bn=0$), b) Bingham ($Bn=0.025$) fluid cases for different Ra for $AR = 0.25$ at $Pr=500$.

Figure 7: Variations of non-dimensional temperature θ_{CWT} and θ_{CWHF} along the vertical mid-plane for different values of Bn and AR for $Ra = 5 \times 10^4$.

Figure 8: Variations of non-dimensional vertical velocity component V along the horizontal mid-plane for different values of Bn and AR for $Ra = 5 \times 10^4$.

Figure 9: Contours of non-dimensional temperature θ_{CWT} and θ_{CWHF} and non-dimensional stream function Ψ with AURs (grey region) for different values of Bn at $AR=0.5$, $Ra = 5 \times 10^4$ and $Pr=500$.

Figure 10: Variation of $\overline{Nu}$ with AR for Newtonian fluids along with the prediction of Eq. 29i, a) CWT, b) CWHF.

Figure 11: Variation of $\overline{Nu}$ with Bn for different values of AR and $Ra =$ (a) 10^4 , (b) 5×10^4 , (c) 10^5 obtained from simulations for CWT ($\circ$) and CWHF ($\bullet$) along with the prediction of Eq. 32 for CWT (solid line) and for CWHF (dashed line).

Fig. 12: Variation of $Bn_{\max}$ with different AR for values of Ra at $Pr=500$ along with the prediction of Eq. 34.

Figure 1: Schematic diagram of simulation domain for a) CWT, b) CWHF configurations.

Figure 2: Variations of non-dimensional vertical velocity component $V = u_2 L / \alpha$ along the vertical mid-plane x_1 / L for Newtonian ($Bn=0$) and Bingham ($Bn=0.025$) fluid cases for different AR and Ra a) 10^4 , b) 5×10^4 , c) 10^5 at $Pr=500$. The aspect ratio at which the maximum $\overline{Nu}$ is obtained for a given set of Ra and Pr is highlighted with an asterisk $*$ in the figure legend.

a)

b)

Figure 3: Contours of non-dimensional stream functions Ψ with AURs (shown in grey) for different values of AR for $Ra = 10^5$, $Bn = 0.025$ and $Pr = 500$, a) CWT, b) CWHF. The dimensions of the configurations are scaled in such manner that the widths of the enclosures look the same merely for ease of plotting.

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a)

b)

$AR = 0.25$

$AR = 0.5$

$AR = 1$

$AR = 2$

$AR = 4$

Figure 5: Contours of non-dimensional temperature θ_{CWT} and θ_{CWHF} for different values of AR for $Ra = 10^5$, $Bn = 0.025$ and $Pr = 500$, a) CWT, b) CWHF. The dimensions of the configurations are scaled in such a manner that the widths of the enclosures look the same merely for ease of plotting.

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Fig. 12: Variation of Bn_{max} with different AR for values of Ra at $Pr=500$ along with the prediction of Eq. 34.