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Laminar natural convection of Bingham fluids in square cross-sectioned cylindrical annular cavity with differentially heated vertical walls subjected to constant heat fluxes

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ABSTRACT

Steady-state numerical simulations have been conducted to investigate natural convection of yield stress fluids obeying Bingham model in square cross-sectioned axisymmetric cylindrical annular enclosure with vertical walls subjected to constant heat fluxes for nominal Rayleigh number range of 10^3 to 10^6 , nominal Prandtl number of 10 to 10^3 for different values of internal cylinder radius. It is found that the mean Nusselt number on the inner periphery increases (decreases) with increasing nominal Rayleigh (Bingham) number due to strengthening (weakening) of thermal advection. However, the values of the mean Nusselt number on the inner periphery obtained for Bingham fluids are smaller than that obtained for Newtonian fluids for the same set of nominal Rayleigh and Prandtl numbers. The mean Nusselt number normalised by the corresponding value obtained for pure conductive transport increases with increasing internal radius before asymptotically approaching the mean Nusselt number for a square enclosure. This suggests that the ratio of the convective to the conductive transport strengthens with increasing cylinder radius in the cylindrical annular cavity. Detailed physical explanations have been provided for the effects of the aforementioned parameters on the mean Nusselt number on the inner periphery and correlations have been proposed for the mean Nusselt number on the inner periphery for both Newtonian and Bingham fluids.

INTRODUCTION

A number of recent semi-analytical [1-4], numerical [5-13] and experimental [14,15] studies analysed natural convection of yield stress fluids (i.e. fluids exhibit solid-like properties below the yield stress and flows like fluids on surpassing the yield stress) following the Bingham model in rectangular enclosures. Natural convection of yield stress fluids has applications in preservation of canned food, solar energy collectors and cryogenic fluid storages but the annular cylindrical geometry (e.g. annular space around a cylindrical tank) is more relevant to these applications than the rectangular enclosures. However, natural convection of yield stress fluids in cylindrical annular space is yet to be analysed in detail. This deficit is addressed here by numerical investigation of natural convection of yield stress fluids in square cross-sectioned cylindrical annular enclosure with differentially heated vertical walls subjected to constant heat fluxes. Previous analyses [1-15] demonstrated that an increase in Bingham number Bn (i.e. ratio of yield stress to plastic viscous stress) for a given nominal value of Rayleigh number leads to a reduction in heat transfer rate, and convection entirely stops for large enough values of Bingham number. A similar trend is expected for cylindrical annular enclosures but the yield stress effects on heat transfer rate are likely to be quantitatively different to those in rectangular enclosures. Natural convection in vertical cylindrical annular enclosures has been studied both numerically [16,17] and experimentally [18,19] for Newtonian fluids and the augmentation of the heat transfer rate has been reported with increasing internal radius. Thus, the curvature of cylindrical enclosures is also expected to play a key role in heat and momentum transport in natural convection of yield stress fluids. Here steady state two-dimensional axisymmetric numerical simulations of laminar natural convection of yield stress fluids obeying Bingham model in square cross-sectioned cylindrical annular space with differentially heated vertical walls subjected to constant heat fluxes have been carried out for a range of different values of nominal Rayleigh (e.g. $Ra = 10^3 - 10^6$) and Prandtl (e.g. $Pr = 10 - 10^3$)

numbers for different values of normalised cylinder radius (e.g. $r_i/L = 1/8 - 16$ where r_i and L are the inner radius and the cylinder height respectively). In this respect, the main objectives of this analysis are as follows:

1. To demonstrate the effects of Ra , Pr and Bn on natural convection of Bingham fluids in cylindrical annular space with differentially heated vertical walls for different values of r_i/L .
2. To provide physical explanations for the above effects and propose correlations for the mean Nusselt number.

The mathematical background and numerical implementation pertaining to the current analysis are provided in the next two sections. Following this results are presented and subsequently discussed. The main findings are summarised and conclusions are drawn in the final section of this paper.

MATHEMATICAL BACKGROUND

The strain rate dependence of viscous stresses for Bingham fluids can be expressed as:

$$\underline{\dot{\gamma}} = 0 \quad \text{for} \quad \tau \leq \tau_y \quad (1)$$

$$\underline{\underline{\tau}} = (\mu + \tau_y/\dot{\gamma})\underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \tau > \tau_y \quad (2)$$

where $\dot{\gamma}_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ are the components of the rate of strain tensor $\underline{\underline{\dot{\gamma}}}$, $\underline{\underline{\tau}}$ is the stress tensor, τ_y is the yield stress, μ is the plastic viscosity, τ and $\dot{\gamma}$ are second invariants of the stress and the rate of strain tensor respectively, which are expressed as:

$$\tau = \left[\frac{1}{2} \underline{\underline{\tau}} : \underline{\underline{\tau}} \right]^{1/2} \quad (3)$$

$$\dot{\gamma} = \left[\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \right]^{1/2} \quad (4)$$

For the current analysis the bi-viscosity regularisation proposed by O'Donovan and Tanner [20] has been used to mimic the stress-shear rate characteristics for a Bingham fluid:

$$\underline{\underline{\tau}} = \mu_{yield} \underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \dot{\gamma} \leq \tau_y / \mu_{yield} \quad (5a)$$

$$\underline{\underline{\tau}} = \tau_y (\underline{\underline{\dot{\gamma}}} / \dot{\gamma}) + \mu \underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \dot{\gamma} > \tau_y / \mu_{yield} \quad (5b)$$

where μ_{yield} is the yield viscosity, and μ is the plastic viscosity. O'Donovan and Tanner [20] indicated that a value of μ_{yield} equal to 1000μ mimics the true Bingham model in a satisfactory manner but here $\mu_{yield}/\mu = 10^4$ is taken to ensure higher fidelity of the simulations. Here, a limited number of simulations have also been carried out in square enclosure with differentially heated vertical walls subjected to constant wall temperature at $Ra = 10^6$ and $Pr = 10$ based on the regularization proposed by Papanastasiou [21] in order to assess the sensitivity of the simulations on the choice of regularization. Papanastasiou's regularisation [21] takes the following form:

$$\underline{\underline{\tau}} = \underline{\underline{\tau_y}} (1 - \exp(-m\dot{\gamma})) + \mu \underline{\underline{\dot{\gamma}}} \quad (6)$$

where m is the stress growth exponent which has the dimension of time. The stress growth exponent must be chosen $m \gg 10L^2/\alpha$ in order to mimic true Bingham model satisfactorily and it has been chosen equal to $m \gg 10^5 L^2/\alpha$ in the current analysis. Both Eqs. (5) and (6) transform the “unyielded” region to a zone of high viscosity. The maximum difference between the mean Nusselt numbers for square enclosure (i.e. $\overline{Nu}_{sq}$) obtained from these two regularizations remains less than 2% where the details can be seen in Table 1.

It is worth noting that according to the Bingham model (i.e. Eqs. (1) and (2)) the effective viscosity for Bingham fluids can be expressed as $\mu_{eff} = (\mu + \tau_y/\dot{\gamma})$ and thus μ_{eff} exhibits singularities in the unyielded region where the shear rate is zero. Similarly, a singularity can

be expected for a pure conduction state due to zero velocities. To avoid these problems bi-viscosity [20] and Papanastasiou [21] regularizations have been proposed for numerical simulations of Bingham fluid flows. These regularizations replace the unyielded regions with high values of viscosity so that the numerical solutions predict vanishingly small magnitudes of velocity in the unyielded regions, and under this condition heat transfer takes place principally due to conduction, and fluid flow does not influence the thermal transport.

A schematic diagram of the configuration analysed here is presented in Fig. 1. Using Buckingham's Π theorem it is possible to show that the Nusselt number based on the inner periphery $Nu_i = h_i L / k$ can be expressed as: $Nu_i = f_1(Ra, Pr, Bn, r_i/L)$ ¹ where r_i/L represents the slenderness ratio of the square cross-sectioned cylindrical enclosures and the nominal Rayleigh, Bingham and Prandtl numbers are given by:

$$Ra = \frac{\rho^2 c_p g \beta q_i L^4}{\mu k^2} = Gr Pr \quad (7)$$

$$Bn = \frac{\tau_y}{\mu \sqrt{g \beta q_i / k}} \quad \text{and} \quad Pr = \frac{\mu c_p}{k} \quad (8)$$

where h_i is the heat transfer coefficient on the internal surface which is defined as:

$$h_i = \left| -k \left(\frac{\partial T}{\partial r} \right)_{r=r_i} \times 1 / (T_{r=r_i} - T_{r=r_i+L}) \right| \quad (9)$$

It is possible to obtain the mean heat transfer coefficient $\bar{h}_i$ and the mean Nusselt number $\overline{Nu}_i$ at the internal radius as:

$$\bar{h}_i = \frac{1}{L} \int_0^L h_i dz \quad \text{and} \quad \overline{Nu}_i = \frac{\bar{h}_i L}{k} \quad (10)$$

¹ It is also possible to express the mean Nusselt number as a function of Grashof, Prandtl and Bingham numbers, and r_i/L . In natural convection literature (e.g. [1,3,4,13-15]), most criteria are specified in terms of Ra . Furthermore, the correlations for the mean Nusselt number are often proposed in terms of Ra and Pr [5-12]. Thus, Ra and Pr is changed independently of each other so the range of Gr is different for each value of Pr . The importance of this issue is of limited importance for non-Newtonian fluids because Ra and Pr are only nominal values because of spatial variation of viscosity.

Based on steady state one obtains:

$$\frac{q_o}{q_i} = \frac{r_i}{r_o} \quad (11)$$

where q_o is the heat flux magnitudes at the external circumference, which is defined as:

$$q_i = \frac{1}{L} \int_0^L \left| k \frac{\partial T}{\partial r} \right|_{r=r_i} dz \text{ and } q_o = \frac{1}{L} \int_0^L \left| k \frac{\partial T}{\partial r} \right|_{r=r_o} dz \quad (12)$$

This leads to:

$$\bar{h}_o = \bar{h}_i \frac{r_i}{r_o} \text{ and } \overline{Nu}_o = \overline{Nu}_i \frac{r_i}{r_o} \quad (13)$$

In the context of this discussion it is worth mentioning that the heat flux due to pure conduction on the internal surface is given by:

$$(q_i)_{cond} = \frac{k |T_{r=r_i} - T_{r=r_o}|}{r_i \ln(1+L/r_i)} \quad (14)$$

Equation (14) leads to the Nusselt number value on the internal surface due to pure conduction:

$$(\overline{Nu}_i)_{cond} = \frac{(q_i)_{cond} L}{k(T_H - T_C)} = \frac{(L/r_i)}{\ln(1+L/r_i)} \quad (15)$$

For the purpose of generalization the current analysis is done in non-dimensional form, where the spatial co-ordinates, velocity components, pressure and temperature can be non-dimensionalised in the following manner:

$$r_i^+ = r_i/L; z^+ = z/L; u_i^+ = u_i/U_{ref}; P^+ = P/\rho U_{ref}^2 \text{ and } \theta = (T - T_{ref})k/q_i L \quad (16)$$

where the reference velocity scale U_{ref} is taken to be equal to $\sqrt{g\beta\Delta T_{ref}L}$ (i.e. $U_{ref} = \sqrt{g\beta q_i L^2/k}$) based on the equilibrium of inertial and buoyancy forces [10,12,13] and temperature of the center of the domain is taken to be the reference temperature T_{ref} . One obtains the following non-dimensional forms of steady-state mass, momentum and energy conservation equations with constant temperature-independent thermo-physical properties:

Non-dimensional mass conservation equation

$$\frac{1}{r^+} \frac{\partial(r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0 \quad (17)$$

Non-dimensional momentum conservation equations

Radial direction

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = -\frac{\partial P^+}{\partial r^+} + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rr}^+)}{\partial r^+} - \frac{\tau_{\phi\phi}^+}{r^+} + \frac{\partial(\tau_{rz}^+)}{\partial z^+} \right] \quad (18a)$$

Vertical direction

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = -\frac{\partial P^+}{\partial z^+} + \theta + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rz}^+)}{\partial r^+} + \frac{\partial(\tau_{zz}^+)}{\partial z^+} \right] \quad (18b)$$

Non-dimensional energy conservation equation

$$u^+ \frac{\partial \theta}{\partial r^+} + w^+ \frac{\partial \theta}{\partial z^+} = \frac{1}{Pr^{1/2} Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial}{\partial r^+} \left(r^+ \frac{\partial \theta}{\partial r^+} \right) + \frac{\partial^2 \theta}{\partial z^+ \partial z^+} \right] \quad (19)$$

In Eq. (18) τ_{ij}^+ is the non-dimensional stress tensor which is given by:

$$\tau_{ij}^+ = \frac{\tau_{ij}}{\mu \sqrt{g\beta q_i/k}} \quad (20)$$

For the current analysis axisymmetric nature of the flow field is utilised to carry out steady state two-dimensional simulations where the radial coordinate is r and z axis is taken to align with the vertical direction, whereas the flow is assumed to be independent of the azimuthal direction ϕ . However, $\dot{\gamma}_{\phi\phi}$ takes the form $\dot{\gamma}_{\phi\phi} = u/r$ in axisymmetric configuration and thus $\tau_{\phi\phi}$ appears in Eq. (18a). The components of viscous stress tensor (i.e. τ_{rr} , $\tau_{\phi\phi}$, τ_{rz} and τ_{zz}) are expressed according to Eq. (5). Eqs. (17-20) have been solved in a coupled manner in conjunction with the following boundary conditions. The two horizontal walls are kept under adiabatic conditions (i.e. $\partial\theta/\partial z^+ = 0$ at $z^+ = 0.0$ and $z^+ = 1.0$), whereas constant wall heat flux boundary condition is imposed for the vertical walls (i.e. $-\partial\theta/\partial r^+ = 1$ and $-\partial\theta/\partial r^+ = 1/(1 + L/r_i)$ at $r^+ = r_i/L$ and $r^+ = r_i/L + 1$ respectively). Both velocity components (i.e. u^+ and w^+) are identically zero on each boundary because of the no-slip condition and impenetrability of rigid walls.

NUMERICAL IMPLEMENTATION

The numerical simulations have been carried out using three different non-uniform meshes M1 (60×60), M2 (100×100), and M3 (140×140). The details of these meshes are provided in

Table 2. For some representative simulations (e.g. Newtonian (i.e. $Bn = 0$) and Bingham ($Bn = 0.1$) fluid simulations for $Ra = 10^6$ and $Pr = 10^2$), the numerical uncertainty levels for the mean Nusselt number $\overline{Nu}_i$ for $r_i/L = 1.0$ and 4.0 remain smaller than 1% for these three different meshes. The uncertainty levels are typically higher for the Bingham fluid than for the Newtonian fluid for the same set of nominal values of Ra and Pr . Moreover, the uncertainty level decreases with decreasing Ra . Here, the simulations have been carried out using mesh M2 for the sake of high accuracy and computational efficiency. It has been demonstrated elsewhere (see Table 3 of Ref. [5]) that the results obtained for natural convection of Newtonian fluids in square enclosures with differentially heated vertical constant temperature walls are in excellent agreement with pioneering work of de Vahl Davis [22]. Moreover, the variation of mean Nusselt number for Bingham fluids in square enclosures with differentially heated vertical walls subjected to constant wall temperature boundary condition has also been compared to the corresponding results presented by Vola *et al.* [23] in Ref. [5] and an excellent agreement was reported.

The ANSYS-FLUENT software has been used for the current simulations. This commercial package has been used successfully for simulating non-Newtonian fluid flows in several previous analyses [5-13]. The coupled governing equations of mass, momentum and energy have been solved in the context of finite volume methodology. A second-order central differencing scheme is used for the diffusive terms and a second-order up-wind scheme is considered for the convective terms. The well-known SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm [24] is used for coupling of the pressure and velocity. The convergence criteria were set to 10^{-6} for all the relative (scaled) residuals.

SCALING ANALYSIS

A scaling analysis is conducted to predict the anticipated influences of Ra , Pr , r_i/L and Bn on the Nusselt number for Bingham fluids. The wall heat flux on the inner periphery q_i scaled as: $q_i = h_i \Delta T \sim k \Delta T / \delta_{th}$ where δ_{th} is the thermal boundary layer on the vertical walls. This gives rise to the scaling estimates of the mean Nusselt number $\overline{Nu}_i \sim h_i k / L \sim L / \delta_{th}$ or $\overline{Nu}_i \sim (L/\delta) f_2(Ra, Pr, Bn, r_i/L)$ for the regime of convection where thermal transport principally takes place within thermal boundary layer with $r_i \gg \delta$ with the function $f_2(Ra, Pr, Bn, r_i/L)$ depicting the ratio of hydro-dynamic and thermal boundary layer thicknesses (i.e. δ/δ_{th}) on the vertical wall.

Based on the continuity equation for axisymmetric geometry one obtains:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left(\frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{L} \quad (21)$$

which leads to the following relation based on scaling estimate of w :

$$u \sim \frac{wr}{(r+L)} \sim \frac{w}{(1+L/r_i)} \sim \frac{\alpha}{L} \frac{\sqrt{RaPr}}{(1+L/r_i)} \sqrt{\frac{\delta_{th}}{L}} \quad (22)$$

Eq. (22) suggests that the magnitude of w increases with increasing r_i/L and $u \sim w$ is obtained for large values of r_i/L . Equating the order of magnitudes of inertial and viscous terms in the radial direction yields:

$$\rho \frac{u^2}{L} \sim \frac{1}{\delta_1} \left(\tau_y + \mu \frac{u}{\delta_1} \right) \quad (23a)$$

$$\delta_1 \sim L \left[\sqrt{\left(\frac{Bn}{2} \right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2}} + \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2} \right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \quad (23b)$$

where δ_1 is the hydrodynamic boundary layer thickness on the horizontal wall. The hydro-dynamic boundary layer thickness on the vertical wall δ can be scaled in the following manner by equating the magnitudes of inertial and viscous terms in the vertical direction (i.e. $\rho w^2 / L \sim (\tau_y + \mu w / \delta) / \delta$):

$$\delta \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \quad (23c)$$

Based on the scaling estimates in Eq. (23) it is possible to estimate the effective viscosity in both vertical and horizontal boundary layers in the following manner:

$$\mu_{eff}^V \sim \mu + \tau_y \delta / w \quad (24a)$$

$$\mu_{eff}^H \sim \mu + \tau_y \delta_1 / u \quad (24b)$$

As Eqs. (23b) and (23c) indicate that $\delta_1 > \delta$ and $u < w$ (i.e. Eq. (18)), μ_{eff}^H is expected to be greater than μ_{eff}^V in cylindrical annular space and these two quantities approach each other for $r_i/L \rightarrow \infty$ (i.e. in the limit of square enclosures). It is worth noting that the velocity scales in both vertical and horizontal directions remain of the same order for a two-dimensional square enclosure, which also leads to $\delta_1 \sim \delta$ and $\mu_{eff}^V \sim \mu_{eff}^H$ for square enclosures.

RESULTS & DISCUSSION

Effects of nominal Rayleigh number Ra

The variations of non-dimensional temperature ($\theta = k(T - T_{cen})/q_i L$) and vertical ($W = wL/\alpha$) velocity component (radial ($U = uL/\alpha$) velocity component) in the radial (axial) at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) direction are shown in Fig. 2 for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids for $Ra = 10^3 - 10^6$ at $r_i/L = 1.0$ and $Pr = 10^2$. The extent of the deviation of θ distribution from the corresponding variation obtained from pure conduction solution (where fluid flow does not affect thermal transport) provides the measure of the strength of convective thermal transport. Thus, it can be noted from Fig. 2 that the distribution of the non-dimensional temperature θ deviates increasingly from the conduction solution with increasing nominal Rayleigh number for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids due to relative strengthening of buoyancy force with an increase in Ra . The strengthening of convective transport with increasing Ra can also be confirmed from Fig. 2, which shows

that the magnitudes of W and U increase with increasing Ra for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids. However, the magnitudes of W and U for the Bingham fluid cases are found to be smaller than the values which are obtained for the Newtonian fluid case for the same set of values of Ra , Pr and r_i/L . The additional flow resistance due to yield stress in Bingham fluids leads to weaker convective transport than in Newtonian fluids and this is reflected in the smaller magnitudes of W and U in Bingham fluid cases. It can be seen from Fig. 2 that the non-dimensional temperature θ distribution deviates slightly from the pure conduction solution for the Newtonian ($Bn = 0$) fluid case at $Ra = 10^3$. By contrast, the non-dimensional temperature θ distribution coincides with the pure conduction solution (shown with the dashed line in Fig. 2) for Bingham fluid case ($Bn = 0.1$) at $Ra = 10^3$ due to stronger flow resistance in Bingham fluids. A deviation of θ from the pure conduction solution at $Ra = 10^3$ for the Newtonian fluid case suggests that the convective transport plays a non-negligible role in thermal transport which is reflected in the non-zero magnitudes of U and W in Fig. 2. However, the magnitudes of U and W remain negligible for $Bn = 0.1$ at $Ra = 10^3$, which is consistent with the non-dimensional temperature distribution being coincident with the pure conduction solution (i.e. thermal transport is principally governed by conduction).

The contours of non-dimensional temperature θ and non-dimensional stream function $\Psi = \psi/\alpha$ are shown in Fig. 3 for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for different values of Ra at $r_i/L = 1.0$ and $Pr = 10^2$. A flow structure consisting of a large convection cell can be seen from Fig. 3 which is qualitatively consistent with previous findings for natural convection of both rectangular [5,6,9,10] and cylindrical [16,17] enclosures. Figure 3 shows that the magnitude of Ψ increases and the isotherms become more curved with increasing Ra indicating strengthening of convection-driven thermal transport with increasing Ra for both Newtonian (i.e. $Bn = 0$) and Bingham (e.g. $Bn = 0.1$) fluid flows. The grey regions

on the streamline plot in Fig. 3 indicate the Apparently Unyielded Regions (AURs) (regions where $|\tau| \leq \tau_y$ [25]). The bi-viscosity regularisation assumes that AURs as the regions of slow moving fluids because of the high viscosity ($\mu_{yield}/\mu \geq 10^3$). The shapes and sizes of AURs are dependent on the choice of μ_{yield} [5-14, 25] but the qualitative and quantitative distributions of stream function and isotherms remain independent of the value of μ_{yield} (at least for $\mu_{yield}/\mu \geq 10^3$). Therefore, the precise shape and size of AURs do not affect the $\overline{Nu}_i$ in the present study. Figure 3 further shows the isotherms for Bingham fluid case ($Bn = 0.1$) for $Ra = 10^3$ remains parallel to the active walls (i.e. hot and cold) since thermal transport takes place purely due to conduction as a result of the unyielded state of the fluid. By contrast, the isotherms are more curved in the case of Newtonian fluid ($Bn = 0$) for $Ra = 10^3$ than in the corresponding Bingham fluid case. As the viscous resistance is stronger in Bingham fluids, the convective transport is weaker in Bingham fluids than in Newtonian fluids for the range of r_i/L , Ra and Pr is taken into consideration in this analysis.

Effects of Bingham number Bn

The variations of non-dimensional temperature (θ) and vertical (W) velocity component (radial (U) velocity component) in the radial (axial) direction at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) are shown in Fig. 4 for different values of Bn for $r_i/L = 0.125$ and 16 at $Ra = 10^6$ and $Pr = 10^2$. As can be seen from Fig. 4, the non-dimensional temperature distribution tends to approach the conduction solution with increasing Bn due to weakening (strengthening) of convective transport (viscous flow resistance). This can further be substantiated from the decreasing magnitudes of W and U with increasing Bn . Further insight into the effects of Bn can be obtained from the contours of non-dimensional temperature θ and non-dimensional stream function Ψ for different values of Bn , as shown in Fig. 5 for $r_i/L = 0.125$ and 16 at $Ra = 10^6$ and $Pr = 10^2$. It is clear from Fig. 5 that the magnitude of Ψ decreases and isotherms tend to

be parallel to the sidewalls with increasing Bn due to weakening of convective transport. Moreover, the size of the AURs increases as the value of Bn increases and thermal transport becomes conduction dominated for large values of Bn .

Effects of nominal Prandtl number Pr

The variations of the mean Nusselt number on the inner periphery normalised by the corresponding conductive Nusselt number $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with different Pr are shown in Fig. 6 for different values of Ra and r_i/L for Newtonian fluids ($Bn = 0$). The mean Nusselt number normalised by the corresponding value obtained for pure conductive transport ($\overline{Nu}_i/(\overline{Nu}_i)_{cond}$) provides the measure of the ratio of convective to conductive transport since $(\overline{Nu}_i)_{cond} = (L/r_i)/\ln(1 + L/r_i)$ decreases with increasing r_i/L . It can be seen from Fig. 6 that Pr does not have any significant influence on $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for Newtonian fluids, which is also consistent with several previous findings [5-9]. For $Pr \gg 1$ Newtonian fluids, the thermal boundary layer is much thinner than the hydro-dynamic boundary layer thickness so a change in Prandtl number modifies the relative balance between the buoyancy and viscous forces and does not affect the thermal transport within the thermal boundary layer [26]. Thus, the nominal Prandtl number does not have any major influence on $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for all values of Ra , r_i/L for Newtonian fluids.

The variation $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different Pr and r_i/L at $Ra = 10^6$ are shown in Fig. 7 for the purpose of demonstrating the effects of Pr on the mean Nusselt number behaviour for Bingham fluids. It can be seen from Fig. 7 that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Pr for large values of Bn but an opposite behaviour has been observed for small values of Bn . Moreover, Fig. 7 demonstrates that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Bn , indicating diminishing strength of convection with an increase in Bingham number. It is clear from Fig.

7 that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ settles to unity for either equal to or greater than a threshold value of Bn which is referred to as Bn_{max} in this analysis. It is also worth noting that Bn_{max} is defined based on the Bingham number where $\overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1.01$ in the present analysis. According to bi-viscosity regularisation fluid flow always exists in the cavity but this flow does not affect thermal transport for $Bn > Bn_{max}$. Figure 7 further shows that the value of Bn_{max} decreases with increasing Pr . The same qualitative behaviour is also observed for other values of Ra and thus is not shown here for the sake of conciseness.

It can be seen from Eqs. (23b) and (23c) that the effective viscosity μ_{eff} increases with increasing Bn . Thus, both the effective Rayleigh and Grashof numbers (i.e. $Ra_{eff} = \rho g \beta \Delta T_{ref} L^3 / \mu_{eff} \alpha$ and $Gr_{eff} = Ra_{eff} / Pr = \rho^2 g \beta \Delta T_{ref} L^3 / \mu_{eff}^2$) decrease with increasing Pr for large values of Bn . An increase in Pr for a given value of Ra gives rise to a thinning of thermal boundary layer relative to the hydro-dynamic boundary layer for small values of Bn , which leads to an increase in $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with increasing Pr . A similar non-monotonic nominal Prandtl number dependence of the mean Nusselt number has been observed earlier in the context of rectangular enclosures [5-9].

Effects of r_i/L

The variations of θ and $W(U)$ in the radial (axial) direction at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) for $Bn = 0$ and $Bn = 0.5$ cases are shown in Fig. 8 for $Ra = 10^6$ and $Pr = 10^2$ in order to demonstrate the effects of r_i/L . It is evident from Fig. 8 that the temperature difference between the vertical walls increases with increasing r_i/L for both $Bn = 0$ and $Bn = 0.5$ cases. It is also important to understand r_i/L dependence of the temperature difference between vertical walls ΔT when the thermal transport occurs purely by thermal conduction in order to understand the wall curvature effects on the temperature difference between vertical walls. The

thermal resistance due to thermal conduction in axisymmetric cylindrical geometry is given by $\ln(r_o/r_i)/2\pi kL$, which suggests that the steady state heat flow rate for pure conductive transport is given by: $\dot{Q} = \Delta T_{cond}/\{\ln(r_o/r_i)/2\pi kL\}$. Furthermore, $\dot{Q}$ can be expressed in terms of heat flux on the inner periphery as: $\dot{Q} = q_i 2\pi r_i L$, which leads to:

$$\Delta T_{cond} = \frac{q_i(r_i/L)}{k} \ln\left(1 + \frac{L}{r_i}\right) \quad (25)$$

From Eq. (25), it is clear that the temperature difference between vertical walls when the thermal transport occurs purely by thermal conduction (i.e. ΔT_{cond}) is dependent on internal radius of the cylindrical annular cavity. Accordingly, Eq. (25) indicates that ΔT_{cond} increases with decreasing values of r_i/L . For the transport regime, where convection plays significant role on the thermal transport, the temperature difference between the vertical walls remains smaller than that in the case of purely conductive transport ($\therefore h_i = q_i/\Delta T > q_i/\Delta T_{cond}$) because heat transfer coefficient $h_i = q_i/\Delta T$ for convective transport is expected to be greater than that for pure conductive transport $(h_i)_{cond} = q_i/\Delta T_{cond}$. It is also worth noting that pure conduction regime can be obtained in this configuration for either small values of Rayleigh number or large values of Bingham number. In the context of bi-viscosity regularisation fluid flow remains present within the domain under all condition but it does not influence to the thermal transport for either small values of Rayleigh number or large values of Bingham number.

Furthermore, Fig.8 shows that u increases while the axial velocity component w decreases with increasing r_i/L for Newtonian fluids ($Bn = 0$). According to Eq. (22) (i.e. $u \sim w(1 + L/r_i)$) the magnitude of the radial velocity component u increases with increasing r_i/L , which is consistent with numerical findings for both Newtonian and Bingham fluid cases (see Fig. 8). It can also be seen from Fig. 8 that the magnitudes of W and U for Bingham fluids are smaller than the corresponding Newtonian fluid values. In order to understand this behaviour the

contours of non-dimensional temperature θ and non-dimensional stream function Ψ for $Bn = 0$ and $Bn = 0.5$ are shown in Fig. 9 for different values of r_i/L at $Ra = 10^6$ and $Pr = 10^2$. It is clear from Fig. 9 that the magnitude of Ψ increases with increasing r_i/L for same Ra for both $Bn = 0$ and $Bn = 0.5$, which indicates that the convective transport strengthens with increasing r_i/L . However, the distributions of Ψ in Fig. 9 suggest that the flow structure in $Bn = 0.5$ case is considerably different from the Newtonian (i.e. $Bn = 0$) case. Although the nominal Rayleigh number is the same for both $Bn = 0$ and $Bn = 0.5$ cases shown in Fig. 9, the effective Rayleigh number Ra_{eff} in the $Bn = 0.5$ case remains smaller than the $Bn = 0$ case, which justifies the difference in flow structure between these cases. It can be seen from Eq. (23b) that the hydro-dynamic boundary layer thickness in the radial direction decreases with increasing r_i/L , which suggests that μ_{eff}^H is expected to decrease with an increase in r_i/L . This indicates the effective Rayleigh number in the vertical direction $Ra_{eff}^V = \rho g \beta q_i L^4 / \mu_{eff}^V \alpha$ is greater than that in the horizontal direction $Ra_{eff}^H = \rho g \beta q_i L^4 / \mu_{eff}^H \alpha$ for cylindrical annular enclosures and the effective Rayleigh numbers Ra_{eff}^V and Ra_{eff}^H approach each other in the limit of square enclosures (i.e. $r_i/L \rightarrow \infty$). Thus, the convective transport is expected to strengthen with increasing r_i/L , which can be verified from Fig. 9 by noting the increases in Ψ magnitudes and the decreases of AUR sizes with an increase in r_i/L .

As the variations W and U with r_i/L exhibit opposite behaviour in the case of Newtonian fluid (i.e. $Bn = 0$), the relative contributions of thermal transport due to convection and conduction for different values of r_i/L can be understood clearly from the variations of $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ with r_i/L . Therefore, the variations of $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ with different r_i/L for different values of Ra at $Pr = 10^2$ are shown in Fig. 10 for Newtonian fluids (i.e. $Bn = 0$). It can be seen from Fig. 10 that $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ increases with increasing r_i/L for all values of Ra and

approaches the value of mean Nusselt number for square enclosures (i.e. $\overline{Nu}_{sq}$) in the limit $r_i/L \rightarrow \infty$. This indicates that the relative contribution of convection to overall thermal transport increases with increasing r_i/L , which is consistent with previous experimental [18,19] and numerical [16,17] findings in the context of Newtonian fluids. The strengthening of convective transport with increasing r_i/L for Bingham fluids can also be substantiated from the increasing trend of $(Bn_{max})_i$ with increasing r_i/L in Fig. 11, which suggests that convective transport can be sustained up to high values of Bingham number for large values of r_i/L , and $(Bn_{max})_i$ approaches the corresponding value for square enclosures (i.e. $(Bn_{max})_{sq}$) for large values of r_i/L . In the light of these findings, it can be concluded that the effects of convection on thermal transport are weaker in cylindrical annular enclosures than in rectangular enclosures but natural convection in annular enclosures approaches that in rectangular enclosures for large values of internal radius.

Behaviour of mean Nusselt number $\overline{Nu}_i$

The wall heat flux on the inner periphery q_i scaled as: $q_i = h_i \Delta T \sim k \Delta T / \delta_{th}$ where δ_{th} is the thermal boundary layer on the vertical walls for two dimensional square enclosures [5-13]. It is not possible to obtain an analytical relation for δ_{th} from Eqs. (23a) and (23b) but Turan *et al.* [6] proposed a correlation for mean Nusselt number $\overline{Nu}_{sq}$ for natural convection of Bingham fluids in square enclosures in the range $0 \leq Bn \leq Bn_{max}$:

$$\frac{\overline{Nu}_{sq}-1}{(\overline{Nu}_{sq})_{Bn=0}-1} = \frac{2[1-(Bn^*/Bn_{max}^*)^b]^c}{Bn^* + \sqrt{Bn^{*2}+4}} \text{ when } (\overline{Nu}_{sq})_{Bn=0} > 1 \quad (26a)$$

$$\overline{Nu}_{sq} = 1 \text{ when } (\overline{Nu}_{sq})_{Bn=0} = 1 \quad (26b)$$

where $Bn_{max}^* = Bn_{max}(Ra/Pr)^{-1/4}$, b and c are the correlation parameters. Turan *et al.* [6] proposed a correlation for natural convection of Newtonian fluids ($Bn = 0$) in square enclosures for $10^3 \leq Ra \leq 10^6$ and $0.1 \leq Pr \leq 10^2$:

$$(\overline{Nu}_{sq})_{Bn=0} = \left[0.209 Ra^{0.249} \left(\frac{Pr}{1+Pr} \right)^{0.037} \right] \quad 0.1 \leq Pr \leq 100 \quad (27)$$

This correlation (i.e. Eq. (27)) can be utilized for $r_i/L \gg 1$ as $\overline{Nu}_i$ approaches $\overline{Nu}_{sq}$ and $(\overline{Nu}_i)_{cond}$ approaches unity (i.e. $\lim_{r_i/L \rightarrow \infty} [(L/r_i)/\ln(1 + L/r_i)] = 1.0$) for $r_i/L \rightarrow \infty$. Here the variation of $(\overline{Nu}_i)_{Bn=0}$ for Newtonian fluids in the cylindrical annular cavity is parameterised using Eq. (27) as:

$$(\overline{Nu}_i)_{Bn=0} = (\overline{Nu}_{sq})_{Bn=0} \frac{[m_0(L/r_i)]^d}{\ln[1 + (m_0(L/r_i))^d]} \quad (28)$$

where,

$$m_0 = \{0.914/[1 + \exp((9.25 - \ln Ra)/2.15)]\}^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \text{ and } d = 0.927^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \quad (29)$$

Eq. (28) is not only valid for $0.125 \leq r_i/L \leq 16$ but also valid for square cavity as $(\overline{Nu}_i)_{Bn=0}$ will be equal to $(\overline{Nu}_{sq})_{Bn=0}$ in the limit of $r_i/L \rightarrow \infty$. In addition to this, $(\overline{Nu}_i)_{Bn=0}$ will be equal to $(\overline{Nu}_i)_{cond} = [(L/r_i)/\ln(1 + L/r_i)]$ for $(\overline{Nu}_{sq})_{Bn=0} = 1.0$. The predictions of Eq. (28) are shown in Fig. 10 which shows the correlation satisfactorily predicts ($R^2 = 0.99$) $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for Newtonian fluids in square cross-sectioned cylindrical annular cavity for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

Turan *et al.* [6] correlated $(Bn_{max})_{sq}$ for square enclosures in the range of $10^3 \leq Ra \leq 10^6$ and $0.125 \leq r_i/L \leq 16$ as:

$$(Bn_{max})_{sq} = 0.019 Ra^{0.56} Pr^{-0.46} \quad 0.1 \leq Pr \leq 100 \quad (30)$$

The variation of $(Bn_{max})_i$ for Bingham fluids in the cylindrical annular enclosure is parameterised using the correlation for $(Bn_{max})_{sq}$ as:

$$(Bn_{max})_i = (Bn_{max})_{sq}/[1 + a_o(L/r_i)^{b_o}] \text{ for } r_i/L \leq 1 \quad (31a)$$

where,

$$a_o = \left[\frac{50.8+0.97Pr}{77.6+Pr} \right] Ra^{0.048} \text{ and } b_o = \left[\frac{1.145+0.117Pr}{1+0.117Pr} \right] - 0.02 \ln Ra \quad (31b)$$

$$(Bn_{max})_i = (Bn_{max})_{sq} - [a_o(L/r_i)^{b_o}] \quad \text{for } r_i/L > 1 \quad (31c)$$

where,

$$a_o = (0.077 - 0.01 \ln Pr) + \left[\frac{1+0.0028Pr}{311.4+11Pr} \right] Ra^{0.585} \text{ and } b_o = \left[\frac{1+0.153Pr}{0.064+0.372Pr} \right] - \left[\frac{1-0.226Pr}{63.76-9.63Pr} \right] \ln Ra \quad (31d)$$

It can be seen from Fig 11 that Eq. (31) satisfactorily captures (i.e. $R^2 = 0.99$) the variation of the $(Bn_{max})_i$ for different values of r_i/L and Pr for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

Finally, Eq. (26a) has been extended to propose a correlation for $\overline{Nu}_i$ for natural convection of Bingham fluids in square-cross sectioned cylindrical annular cavity in the range $0 \leq Bn \leq Bn_{max}$:

$$\frac{[\overline{Nu}_i/(\overline{Nu}_i)_{cond}] - 1}{[(\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond}] - 1} = \frac{2[1 - (Bn^*/Bn_{max}^*)^{b_i}]^{c_i}}{Bn^* + \sqrt{Bn^{*2} + 4}} \text{ when } (\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond} > 1 \quad (32a)$$

$$\text{and } \overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1 \text{ when } (\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond} = 1 \quad (32b)$$

where $Bn_{max}^* = Bn_{max}(Ra/Pr)^{-1/4}$, b_i and c_i are the correlation parameters. Turan *et al.* [6] proposed $b_{sq} = 0.643$ and $c_{sq} = 0.143Ra^{0.226}Pr^{0.062}$ for square enclosures. Accordingly, in the current study b_i and c_i have been parameterised as:

$$b_i = b_{sq} \left[\left(\frac{\ln Ra - 15.48}{0.512 \ln Ra - 9.76} \right) (L/r_i)^{0.01} \right] \text{ and } c_i = c_{sq} - [k_o(L/r_i)^n] \quad (32c)$$

where,

$$k_o = \left[\frac{0.015}{1-0.05 \ln Pr} \right] Ra^{0.37} - \left[\frac{0.733}{1+0.267 \ln Pr} \right] ; n = \left[\frac{1-0.072 \ln Ra}{\ln Ra - 7.643} \right] + \left[\frac{1-0.06 \ln Ra}{0.22 \ln Ra - 1.98} \right] \exp(-n_o \ln Pr)$$

$$\text{and } n_o = \frac{1-0.064\ln Ra}{0.148\ln Ra-0.994} \quad (32d)$$

The predictions of Eq. (32) are shown in Fig. 12, which shows that the correlation satisfactorily predicts ($R^2 = 0.99$) $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for Bingham fluids in square cross-sectioned cylindrical annular cavity for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$. It is also worth noting that the variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for cylindrical annular enclosures with square cross-section is qualitatively similar to the variation of mean Nusselt number with Bingham number for square enclosures [6] but there are important quantitative differences due to the influences of r_i/L . The variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for $r_i/L = 16.0$ is quantitatively similar to the Bingham number dependence of the mean Nusselt number for square enclosures.

CONCLUSIONS

Laminar steady-state natural convection of yield stress fluids obeying Bingham model in axisymmetric square cross-sectioned cylindrical annular enclosures with differentially heated vertical walls subjected to constant heat flux boundary condition has been numerically investigated for nominal Rayleigh number range $10^3 \leq Ra \leq 10^6$, nominal Prandtl number range $10 \leq Pr \leq 10^3$ and different internal cylinder radius normalized by its height $0.125 \leq r_i/L \leq 16$. It has been found that the mean Nusselt number based on the inner periphery $\overline{Nu}_i$ increases (decreases) with increasing nominal Rayleigh number (Bingham number) due to strengthening (weakening) of thermal convection. However, $\overline{Nu}_i$ values obtained for Bingham fluids are smaller than that is obtained for Newtonian fluids for the same set of nominal Rayleigh and Prandtl number values because of the augmented flow resistance in Bingham fluids due to the yield stress. It has been found that the mean Nusselt number normalised by its value for pure conductive transport ($\overline{Nu}_i/(\overline{Nu}_i)_{cond}$) increases with increasing r_i/L and eventually approaches to the corresponding mean Nusselt number value for a square enclosure $\overline{Nu}_{sq}$. Moreover, $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ has been found to be insensitive to change in

Prandtl number for Newtonian fluids, whereas $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Pr for large values of Bn in contrast to Newtonian fluids. Furthermore, $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Bn and settles unity value (i.e. $\overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1.0$) for large values of Bn as the heat transfer occurs purely by thermal conduction for unyielded fluids. Detailed physical explanations have been offered for the observed r_i/L , Rayleigh, Prandtl and Bingham number dependence of the mean Nusselt number based on a detailed scaling analysis. The numerical findings and the results of the scaling analysis have been utilised to propose a new correlation for the mean Nusselt number $\overline{Nu}_i$ for natural convection of both Newtonian and Bingham fluids in square cross-sectioned cylindrical annular enclosures, which satisfactorily captures the mean Nusselt number obtained from these steady state simulations.

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NOMENCLATURE

$AURs$	Apparently Unyielded Regions
a_o, a_1	Correlation parameters, dimensionless
b, b_o, b_1	Correlation parameters, dimensionless
Bn	Bingham number, dimensionless
Bn_{max}	Threshold value of Bingham number above which $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ settles to unity, dimensionless
Bn_{max}^*	Scaled threshold value of Bingham number, dimensionless
c	Correlation parameter, dimensionless
c_p	Specific heat at constant pressure, J/kgK
d	Correlation parameter, dimensionless
e_{rel}	Relative error, dimensionless
f_1, f_2	Functions, dimensionless
g	Gravitational acceleration, m/s ²
Gr	Grashof number, dimensionless

h	Heat transfer coefficient, W/m ² K
$\bar{h}_i$	Mean heat transfer coefficient, W/m ² K
k	Thermal conductivity, W/mK
k_o	Correlation parameter, dimensionless
L	height of the enclosure and difference between inner and outer radius, m
m	Stress growth exponent, s
m_o	Correlation parameter, dimensionless
n, n_o	Correlation parameter, dimensionless
Nu	Nusselt number, dimensionless
$\overline{Nu}$	Mean Nusselt number, dimensionless
$\overline{Nu}_i$	Mean Nusselt number on the inner periphery, dimensionless
P	Pressure, Pa
Pr	Prandtl number, dimensionless
$\dot{Q}$	Heat flow rate, W
q	Heat flux, W/m ²
r	Radius, m
r_e	Grid expansion ratio, dimensionless
r_i	Inner radius, m
r_o	Outer radius, m
R^2	Coefficient of determination, dimensionless
Ra	Rayleigh number, dimensionless
T	Temperature, K
T_H	Hot wall temperature, K
T_C	Cold wall temperature, K
u, w	Radial, vertical velocity component, m/s
U, W	Dimensionless radial ($U = u L / \alpha$) and vertical velocity ($W = w L / \alpha$), dimensionless
U_{ref}	Reference velocity scale, m/s
z	Coordinate in vertical direction, m

Greek Symbols

ϑ	Characteristic velocity, m/s
α	Thermal diffusivity, m ² /s
β	Coefficient of thermal expansion, 1/K
δ_{th}	Thermal boundary-layer thickness, m

δ	Hydro-dynamic boundary layer thickness for vertical walls, m
δ_l	Hydro-dynamic boundary layer thickness for horizontal walls, m
θ	Dimensionless temperature, dimensionless
μ	Dynamic viscosity, Ns/m ²
ν	Kinematic viscosity, m ² /s
ρ	Density, kg/m ³
$\dot{\gamma}$	Second invariant of strain rate tensor, s ⁻¹
$\underline{\underline{\dot{\gamma}}}$	Strain rate tensor, s ⁻¹
τ	Second invariant of viscous stress tensor, Pa
τ_{ij}	Components of viscous stress tensor, Pa
τ_y	Yield stress, Pa
$\underline{\underline{\tau_y}}$	Yield stress tensor, Pa
$\underline{\underline{\tau}}$	Viscous stress tensor, Pa
ϕ	Azimuthal co-ordinate, dimensionless
ψ	Stream function, m ² /s
Ψ	Dimensionless stream function, dimensionless
ΔT	Temperature difference, K
$\Delta_{min,cell}$	Minimum cell distance, m

Subscripts

<i>cen</i>	Geometrical centre of the domain
<i>cond</i>	Conduction
<i>eff</i>	Effective value
<i>max</i>	Maximum value
<i>nom</i>	Nominal value
<i>ref</i>	Reference value
<i>sq</i>	Square
<i>i</i>	Inner periphery wall
<i>o</i>	Outer periphery wall

Superscripts

—	Mean value
+	Non-dimensionalised value
<i>V</i>	Representative value in the vertical boundary layer
<i>H</i>	Representative value in the horizontal boundary layer

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Table 1 Variations of $\overline{Nu}_{sq}$ for different values of Bn for both bi-viscosity [20] and Papanastasiou [21] regularisations for $Ra = 10^6$ and $Pr = 10$.

Bn	$\overline{Nu}_{sq}$	
	Bi-viscosity Reg. ($\mu_{yield} = 10^4 \mu$)	Papanastasiou's Reg.
0	9.224	9.256
1	7.224	7.252
3	4.259	4.316
5	2.581	2.573
7	1.693	1.722
10	1.000	1.000

Table 2 Non-dimensional minimum cell distance ($\Delta_{min,cell}/L$) and grid expansion ratio (r_e) values and the numerical uncertainty for mean Nusselt number on the inner periphery $\overline{Nu}_i$ at $Ra = 10^6$ and $Pr = 10^2$ for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids for $r_i/L = 1.0$ and $r_i/L = 4.0$.

Grid	M1 60 x 60	M2 100 x 100	M3 140 x 140
$\Delta_{min,cell}/L$	6.658×10^{-3}	4.006×10^{-3}	2.865×10^{-3}
r_e	1.027	1.016	1.011

		$r_i/L = 1.0$			$r_i/L = 4.0$		
		M1	M2	M3	M1	M2	M3
Newtonian Fluid ($Bn = 0$)	$\overline{Nu}_i$	7.965	7.973	7.979	6.905	6.906	6.908
	$e_{rel} (\%)$	0.1	0.075		0.014	0.028	
Bingham Fluid ($Bn = 0.1$)	$\overline{Nu}_i$	5.857	5.864	5.891	5.211	5.215	5.228
	$e_{rel} (\%)$	0.12	0.45		0.07	0.25	

FIGURE CAPTIONS

Fig. 1: Schematic diagram of the simulation domain.

Fig. 2: Variations of non-dimensional temperature θ and non-dimensional axial (radial) $W = wL/\alpha$ ($U = uL/\alpha$) velocity components along the horizontal (vertical) mid-plane for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for different values of Ra for $r_i/L = 1$ at $Pr = 10^2$. The pure conduction solution is shown by the dashed line.

Fig. 3: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for $Ra =$ a) 10^3 , b) 10^4 , c) 10^5 , d) 10^6 at $r_i/L = 1$ and $Pr = 10^2$.

Fig. 4: Variations of non-dimensional temperature θ and non-dimensional axial (radial) W (U) velocity components along the horizontal (vertical) mid-plane for different Bn values for $r_i/L = 0.125$ (left column) and $r_i/L = 16$ (right column) at $Ra = 10^6$ and $Pr = 10^2$. The pure conduction solution is given by the dashed line.

Fig. 5: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for different Bn values for $r_i/L = 0.125$ (left column) and $r_i/L = 16$ (right column) at $Ra = 10^6$ and $Pr = 10^2$.

Fig. 6: Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Pr for different values of Ra and $r_i/L =$ a) 0.125, b) 1, c) 16 for Newtonian fluids ($Bn = 0$).

Fig. 7: Variation of the $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different values of Pr in the case of $r_i/L =$ a) 0.125, b) 1, c) 16 at $Ra = 10^6$ for Bingham fluids.

Fig. 8: Variations of non-dimensional temperature θ and non-dimensional axial (radial) W (U) velocity components along the horizontal (vertical) mid-plane for different values of r_i/L at $Ra = 10^6$ and $Pr = 10^2$ for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.5$) fluid cases. The corresponding pure conduction solution for θ is shown by dashed lines.

Fig. 9: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.5$) fluid cases for $r_i/L =$ a) 0.125 , b) 1 , c) 4 , d) 16 at $Ra = 10^6$ and $Pr = 10^2$.

Fig. 10 : Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with r_i/L for different values of Ra at $Pr = 10^2$ for Newtonian Fluids ($Bn = 0$) along with the prediction of (Eq. (28)). The mean Nusselt number $\overline{Nu}_{sq}$ is also shown with the dashed line (Eq. (27)).

Fig. 11 : Variation of the $(Bn_{max})_i$ with r_i/L for different values of Pr and $Ra =$ (a) 10^4 , (b) 5×10^5 , (c) 10^6 along with the predictions of the correlation (Eq. (31)). The corresponding $(Bn_{max})_{sq}$ values are shown by dashed lines for each Pr .

Fig. 12: Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different values of r_i/L and Ra at $Pr = 10^2$ for both along with the prediction of (Eq. (32)).

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