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Numerical investigation of laminar Rayleigh-Bénard convection of Bingham fluids in square cross-sectioned cylindrical enclosures

Sahin Yigit ^{1,✉}, Siqi Chen ¹, Paul Quinn ¹ and Nilanjan Chakraborty ¹

¹School of Mechanical and Systems Engineering, Newcastle University, Newcastle-Upon-Tyne, NE1 7RU, UK

✉Corresponding author.

s.yigit1@ncl.ac.uk

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ABSTRACT

Numerical simulations have been conducted under the assumption of axisymmetry to analyse steady-state laminar natural convection of yield stress fluids obeying Bingham model in square cross-sectioned cylindrical annular enclosures heated from below (i.e. Rayleigh-Bénard configuration). The simulations have been carried out at a representative value of nominal Prandtl number (i.e. $Pr = 500$) for different internal cylinder radius ($0 \leq r_i/L \leq 16$ where r_i and L are the inner radius and the cylinder height respectively) for a nominal Rayleigh number range $10^3 \leq Ra \leq 10^5$. Both constant wall temperature and constant wall heat flux boundary conditions have been imposed for differentially heated horizontal walls to analyse the effects of wall boundary condition. Although the buoyancy-driven transport strengthens with increasing Ra , the mean Nusselt number $\overline{Nu}_{cy}$ does not show a monotonic increase with increasing Ra for small values of r_i/L because of the change in flow pattern (i.e. number of convection rolls/cells). By contrast, $\overline{Nu}_{cy}$ monotonically increases with increasing Ra , and only one cell flow pattern is obtained for large values of r_i/L . Furthermore, $\overline{Nu}_{cy}$ has been found to increase with increasing r_i/L but asymptotically approaches the corresponding value obtained for square enclosures ($r_i \rightarrow \infty$) for both CWT and CWHF boundary conditions for large values of r_i/L . It has also been found that both the flow pattern and the mean Nusselt number $\overline{Nu}_{cy}$ are dependent on the initial conditions for Bingham fluid cases since hysteresis is evident for small values of r_i/L for both CWT and CWHF boundary conditions. It has been found that convection could be sustained up to a higher value of Bingham number due to stronger convection arising from higher temperature difference between horizontal walls in the case of CWT boundary condition than in the corresponding CWHF configuration. Finally, the numerical findings have been used to propose a correlation for $\overline{Nu}_{cy}$ in the range of $2 \leq r_i/L \leq 16$ ($0.25 \leq r_i/L \leq 16$) and $10^3 \leq Ra \leq 10^5$ for the CWT (CWHF) boundary condition.

NOMENCLATURE

a_o	[-]	Correlation parameter
b, b_o	[-]	Correlation parameter
Bn	[-]	Bingham number
Bn_{max}	[-]	Bingham number at which or above the mean Nusselt number attains a value of unity
c	[-]	Correlation parameter
c_p	[J/kgK]	Specific heat at constant pressure
e_{rel}	[-]	Relative error
g	[m/s ²]	Gravitational acceleration
h	[W/m ² K]	Heat transfer coefficient
k	[W/mK]	Thermal conductivity
L	[m]	height of the enclosure and difference between inner and outer radius
m	[-]	Stress growth exponent
m_o	[-]	Correlation parameter
n_o	[-]	Correlation parameter
$\overline{Nu}$	[-]	Mean Nusselt number
$\overline{Nu}_{cy}$	[-]	Mean Nusselt number for cylindrical annular enclosure
P	[Pa]	Pressure
Pr	[-]	Prandtl number
q	[W/m ²]	Heat flux
r	[m]	Radius
r_g	[-]	Grid expansion ratio
r_i	[m]	Inner radius
r_o	[m]	Outer radius
Ra	[-]	Rayleigh number
T	[K]	Temperature
u	[m/s]	Radial velocity component
u_i	[m/s]	i^{th} velocity component
U	[-]	Dimensionless radial ($U = u L / \alpha$) velocity component
U_{ref}	[m/s]	Reference velocity scale
w	[m/s]	Vertical velocity component
W	[-]	Dimensionless vertical ($W = w L / \alpha$) velocity component
z	[m]	Coordinate in vertical direction

$\mathcal{G}$	[m/s]	Characteristic velocity
x_i	[m]	Coordinate in i^{th} direction
α	[m ² /s]	Thermal diffusivity
β	[1/K]	Coefficient of thermal expansion
δ, δ_{th}	[m]	Velocity and thermal boundary-layer thickness
θ	[-]	Dimensionless temperature
μ	[Ns/m ²]	Dynamic viscosity
ν	[m ² /s]	Kinematic viscosity
ρ	[kg/m ³]	Density
$\tau_{ij} (\tau)$	[Pa]	Stress tensor (stress)
ϕ	[-]	Azimuthal co-ordinate
Ψ	[-]	Dimensionless stream function

Subscripts

C	Cold wall
cen	Geometrical centre
cy	Cylindrical Annular Enclosure
eff	Effective value
H	Hot wall
nom	Nominal value
ref	Reference value
sq	Square

Special characters

ΔT	[K]	Temperature difference
$\Delta_{min, cell}$	[m]	Minimum cell distance

1. INTRODUCTION

Natural convection of Newtonian fluids in rectangular enclosures has been extensively analysed (see Refs.[1,2] and references therein) but relatively limited attention has been given to natural convection of non-Newtonian fluids despite their wide relevance in many applications such as geophysics, power systems, cooling of electronics. Yield stress fluid is a special type of non-Newtonian fluid, which acts as a solid and does not flow until a threshold yield stress is surpassed. Materials such as mud-slurries in oil drilling, toothpaste, molten chocolate and anti-drip paints are common examples of yield fluids. Aqueous solutions of Carbopol is often used as a model yield stress fluid in laboratory experiments [3-5]. Some magneto- and electro-rheological fluids exhibit yield stress behaviour, and it is possible to modify the yield stress by applying electrical or magnetic fields [6]. This may be useful in reducing heat losses or designing new adaptive thermal systems to control convective thermal transfer in enclosed spaces (i.e. preservations of canned food, chemical processing, storage of cryogenics, solar and nuclear power systems). Several recent semi-analytical [7-10], numerical [3,11-19] experimental studies [3-5] have analysed natural convection of yield stress fluids in rectangular enclosures. These analyses indicated that the buoyancy-driven transport of yield stress fluids weakens with increasing Bingham number (i.e. non-dimensional yield stress) due to additional flow resistance arising from yield stress. This leads to a reduction in convective heat transfer rate, and thermal transport takes place purely due to conduction for large values of Bingham numbers since fluid flow practically stops under such a condition. A qualitatively similar behaviour can be expected for cylindrical annular enclosures but the heat transfer rate for a given set of nominal values of Rayleigh, Prandtl and Bingham numbers might be quantitatively different in comparison to that for rectangular enclosures since curvature of the curved walls in cylindrical enclosures can potentially influence the heat and momentum transport. Although cylindrical annular space is more relevant to real engineering applications than rectangular enclosures (i.e.

cylindrical cryogenic storages), laminar natural convection of yield stress fluids in cylindrical annular enclosures has not been analysed in detail in existing literature. This gap in existing literature has been addressed here by numerically analysing natural convection of Bingham fluids in square cross-sectioned cylindrical annular enclosures heated from below for the range of nominal Rayleigh number $10^3 \leq Ra \leq 10^5$ at a representative value of nominal Prandtl number (i.e. $Pr = \mu c_p / k = 500$) for $0 \leq r_i / L \leq 16$ where r_i and L are the internal radius and the cylinder height respectively. The main objectives of this study are:

- i) To demonstrate the effects of Ra and Bn on laminar Rayleigh-Bénard convection of Bingham fluids in square cross-sectioned cylindrical annular enclosures for different values of r_i / L .
- ii) To provide physical explanations for the observed Ra , Bn and r_i / L dependence of mean Nusselt number in the aforementioned configuration.

The rest of the paper will be organised as follows. The mathematical background behind this work will be discussed in the next section. This will be followed by the presentation of results and the subsequent discussion. The main findings will be summarised and conclusions will be drawn in the final section of this paper.

2. MATHEMATICAL BACKGROUND

The Bingham model is the simplest model which describes the interrelation between the shear stress and strain rate in yield stress fluids [20]. The strain rate dependence of viscous stresses for Bingham fluids (i.e. fluids obeying the Bingham model) can be expressed as:

$$\underline{\dot{\gamma}} = 0 \quad \text{for} \quad \tau \leq \tau_y, \quad (1)$$

$$\underline{\tau} = (\mu + \tau_y / \dot{\gamma}) \underline{\dot{\gamma}} \quad \text{for} \quad \tau > \tau_y, \quad (2)$$

where $\dot{\gamma}_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)$ are the components of the rate of strain tensor $\underline{\dot{\gamma}}$, $\underline{\tau}$ is the stress tensor, τ_y is the yield stress, μ is the plastic viscosity, τ and $\dot{\gamma}$ are the second invariants

of the stress and the rate of strain tensor in a pure shear flow respectively, which are expressed as:

$$\underline{\underline{\tau}} = \left[\frac{1}{2} \underline{\underline{\dot{\tau}}} : \underline{\underline{\dot{\tau}}} \right]^{1/2}, \quad (3)$$

$$\dot{\gamma} = \left[\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \right]^{1/2}. \quad (4)$$

For the current analysis the bi-viscosity regularisation proposed by O'Donovan and Tanner [21] has been used to mimic the shear rate dependence of viscous stress for a Bingham fluid:

$$\underline{\underline{\tau}} = \mu_{yield} \underline{\underline{\dot{\gamma}}} \quad \text{for } \dot{\gamma} \leq \tau_y / \mu_{yield}, \quad (5i)$$

$$\underline{\underline{\tau}} = \tau_y (\underline{\underline{\dot{\gamma}}} / \dot{\gamma}) + \mu \underline{\underline{\dot{\gamma}}} \quad \text{for } \dot{\gamma} > \tau_y / \mu_{yield}, \quad (5ii)$$

where $\underline{\underline{\tau}}_y$ is the yield stress tensor, μ_{yield} is the yield viscosity, and μ is the plastic viscosity.

O'Donovan and Tanner [21] indicated that a value of μ_{yield} equal to 1000μ mimics the true Bingham model in a satisfactory manner but here $\mu_{yield}/\mu = 10^4$ is taken to ensure high fidelity of the simulations. Here, a limited number of simulations have also been carried out based on the regularization proposed by Papanastasiou [22] in order to assess the sensitivity of the simulations on the choice of regularization. Papanastasiou's regularisation [22] takes the following form:

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}_y (1 - \exp(-m\dot{\gamma})) + \mu \underline{\underline{\dot{\gamma}}} \quad (6)$$

where m is the stress growth exponent which has the dimension of time (i.e. L^2/α). The value of m is taken as $m \gg 10^5 L^2/\alpha$ to ensure the true Bingham model in a satisfactory manner. Both Eqs. 5 and 6 transform the “unyielded” region to a zone of high viscosity. The typical difference between the mean Nusselt number obtained from these two regularizations remains of the same order as typical experimental/numerical uncertainties ($\sim 2\%$).

The schematic diagram of the configuration is shown in Fig.1, which demonstrates that both constant wall temperature (CWT) constant wall heat flux (CWHF) boundary conditions have been considered for the differentially heated horizontal walls while the vertical walls are assumed to be adiabatic. According to Buckingham's pi theorem, it is possible to show that the Nusselt number for natural convection of Bingham fluids in square cross-sectioned cylindrical annular enclosures can be expressed as: $Nu = f_1(Ra, Pr, Bn, r_i/L)$, where the nominal Rayleigh, Prandtl and Bingham numbers for Bingham fluids can be defined for CWT and CWHF boundary configurations in the following manner:

For CWT

$$Ra = \rho g \beta (T_H - T_C) L^3 / \mu \alpha \text{ and } Bn = \tau_y L / \mu \sqrt{g \beta (T_H - T_C) L} \quad (7i)$$

For CWHF

$$Ra = \rho g \beta q L^4 / k \mu \alpha \text{ and } Bn = \tau_y / \mu \sqrt{g \beta q / k} \quad (7ii)$$

The Bingham number represents the ratio of yield stress to viscous stress. The current analyses have been carried out for a single representative value of nominal Prandtl number (i.e. $Pr = \mu c_p / k = 500$) since practical yield stress fluids exhibit Prandtl numbers order of 100 [3-5]. For example, a recent experimental analysis [5] based on experimental analysis of Rayleigh-Bénard convection of yield stress fluids in rectangular enclosures reported that 0.05% by mass Carpool solution in water shows a yield stress behaviour with a nominal Prandtl number of $Pr \approx 350$.

The local heat transfer coefficient h is defined as:

$$h = \left| -k \left(\frac{\partial T}{\partial z} \right)_{z=0} \times 1 / (T_{z=0} - T_{z=L}) \right| \quad (8)$$

The mean heat transfer coefficient $\bar{h}$ and the mean Nusselt number $\overline{Nu}_{cy}$ are evaluated as:

$$\bar{h} = \int_{r_i}^{r_i+L} 2\pi h(r) dr / [\pi(r_i + L)^2 - \pi r_i^2], \quad \overline{Nu}_{cy} = \bar{h} L / k \quad (9)$$

The current analysis has been carried out in non-dimensional form for the sake of generalisation. The spatial co-ordinates, velocity components, pressure and temperature can be non-dimensionalised in the following manner:

$$r_i^+ = r_i/L, z^+ = z/L, u_i^+ = u_i/U_{ref}, P^+ = P/\rho U_{ref}^2, \Theta = (T - T_{ref})/\Delta T_{ref} \quad (10)$$

where the reference velocity scale U_{ref} is taken to be equal to $\sqrt{g\beta\Delta T_{ref}L}$ based on the equilibrium of inertial and the buoyancy forces [11-18] where ΔT_{ref} is a reference temperature difference. For CWT configuration ΔT_{ref} can be taken to be $\Delta T = T_H - T_C$ while it can be taken to be $\Delta T_{ref} = qL/k$ for the CWHF configuration. Additionally, the reference temperature is taken to be temperature at the centre of the domain T_{cen} for the CWHF boundary condition, whereas it is taken to be the cold wall temperature T_C for the CWT boundary condition. Accordingly, the steady-state non-dimensional forms of governing equations for mass, momentum and energy for constant temperature-independent thermo-physical properties take the following form under the assumption of axisymmetry:

Non-dimensional mass conservation equation:

$$\frac{1}{r^+} \frac{\partial(r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0 \quad (11)$$

Non-dimensional momentum conservation equations

Radial direction:

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = -\frac{\partial P^+}{\partial r^+} + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rr}^+)}{\partial r^+} - \frac{\tau_{\phi\phi}^+}{r^+} + \frac{\partial(\tau_{rz}^+)}{\partial z^+} \right] \quad (12i)$$

Vertical direction:

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = -\frac{\partial P^+}{\partial z^+} + \Theta + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rz}^+)}{\partial r^+} + \frac{\partial(\tau_{zz}^+)}{\partial z^+} \right] \quad (12ii)$$

Non-dimensional energy conservation equation:

$$u^+ \frac{\partial \Theta}{\partial r^+} + w^+ \frac{\partial \Theta}{\partial z^+} = \frac{1}{Pr^{1/2} Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial}{\partial r^+} \left(r^+ \frac{\partial \Theta}{\partial r^+} \right) + \frac{\partial^2 \Theta}{\partial z^+ \partial z^+} \right] \quad (13)$$

In Eq. 12 τ_{ij}^+ is the non-dimensional stress tensor which is given by:

$$\tau_{ij}^+ = \tau_{ij}L/\mu\sqrt{g\beta\Delta T_{ref}L} \quad (14)$$

where r is the radial coordinate, z axis is taken to align with the vertical direction, and the axisymmetric flow is independent of the azimuthal direction ϕ . The radial and vertical velocity components are given by u and w respectively. However, the strain rate in the circumferential direction $\dot{\gamma}_{\phi\phi}$ takes the form $\dot{\gamma}_{\phi\phi} = u/r$ in axisymmetric configuration [23] and thus $\tau_{\phi\phi}$ appears in Eq. 12i. The components of viscous stress tensor (i.e. τ_{rr} , $\tau_{\phi\phi}$, τ_{rz} and τ_{zz}) are expressed according to Eq. 5. Equations 11-14 are solved in a coupled manner in conjunction with the following boundary conditions. The two vertical walls are kept under adiabatic conditions (i.e. $\partial\Theta/\partial r^+ = 0$ at $r^+ = r_i/L$ and $r^+ = r_i/L + 1$), and both velocity components (i.e. u^+ and w^+) are identically zero on each boundary because of the no-slip condition and impenetrability of rigid walls. For the CWHF configuration, the heat fluxes for horizontal hot and cold walls are specified (i.e. $-\partial\Theta/\partial z^+ = 1$ at $z^+ = 0.0$ and $z^+ = 1.0$ respectively). By contrast, the temperatures of horizontal walls are specified (i.e. $\Theta = 1$ and $\Theta = 0$ at $z^+ = 0.0$ and $z^+ = 1.0$ respectively) for CWT configuration.

3. NUMERICAL IMPLEMENTATION

The governing equations of mass, momentum and energy have been numerically solved in the context of finite-volume methodology using a commercial package ANSYS-FLUENT. This commercial package was previously used successfully for simulating Bingham fluid flows [11-18]. A second-order central difference scheme is used for the discretization of the diffusive terms and a second-order up-wind scheme is used for the convective terms. The well-known SIMPLE [24] (Semi-Implicit Method for Pressure-Linked Equations) algorithm is used for coupling of the pressure and velocity components. The convergence criteria were set to 10^{-6} for all the relative (scaled) residuals. Furthermore, grid-independence of $\overline{Nu}_{cy}$ and streamline

pattern for both Newtonian and Bingham fluids has been established based on three different non-uniform meshes M1 (100×100), M2 (160×160), and M3 (220×220). The non-dimensional minimum grid spacing ($\Delta_{min,cell}/L$) and grid expansion ratio (r_g) values for the Cartesian meshes are 4.01×10^{-3} , 2.51×10^{-3} , 1.83×10^{-3} and 1.016, 1.01, 1.007 respectively. It has been ensured that a change in the computational grid did not lead to a change in flow pattern within the enclosure during the course of grid independency analysis. The numerical uncertainty levels for the mean Nusselt number $\overline{Nu}_{cy}$ and maximum non-dimensional vertical velocity component W_{max} on the horizontal mid-plane for meshes M2 and M3 have been found to be smaller than 1% for the range of $10^3 \leq Ra \leq 10^5$ for $r_i/L = 0 - 16$ for both Newtonian and Bingham fluids. Therefore, the simulations have been conducted using the mesh M2 in this analysis for the sake of high accuracy and computational efficiency.

The mean Nusselt numbers $\overline{Nu}$ for laminar Rayleigh- Bénard convection of Newtonian fluids in square enclosures for $10^3 \leq Ra \leq 10^6$ and $Pr = 0.71$ have been compared to the benchmark data [25] in Table 1. It is evident from Table 1 that an excellent agreement has been achieved between the present results and the benchmark data [25]. It is worth noting that the mean Nusselt number reported in Table 1 have been calculated using the heat flux obtained for a second order spatial discretisation scheme and the same approach has been adopted for the rest of this analysis. The maximum difference in the mean Nusselt number values due to first and second order spatial discretisation has been found to be less than 1% for both Newtonian (i.e. $Bn = 0$) and Bingham fluids. The Bingham fluid simulations also have been compared to the benchmark data reported by Vola *et al.* [26] for natural convection of Bingham fluids in square enclosures with vertical walls with different uniform temperatures. It is worth noting that Vola *et al.* [26] reported only the values of yield stress τ_y . According to eq. 7, the yield stress values reported in Table 5 of Vola *et al.* [26] give rise to $Bn = 3.0, 0.95$ and 0.3 for

$Ra = 10^4$, 10^5 and 10^6 respectively. The mean Nusselt number values obtained for the current numerical methodology for the aforementioned values of Bn and Ra for $Pr = 1.0$ (as used in Ref. [26]) are found to be in good agreement with the corresponding values reported by Vola *et al.* [26] (maximum difference in $\overline{Nu}$ is found to be less than 3%). Furthermore, the same numerical methodology was successfully used in several previous analyses [11-18].

It has been reported in [7] that Bingham fluid flows are unconditionally linearly stable under quiescent starting conditions. Thus, the steady-state solutions for Newtonian (i.e. $Bn = 0$) fluids for a given set of values of nominal Rayleigh and Prandtl numbers are used as the starting conditions for Bingham fluid simulations. Some magneto- and electro-rheological fluids exhibit yield stress behaviour, and it is possible to modify the yield stress by applying electrical or magnetic fields. Therefore, the steady-state Newtonian solutions can be considered as realistic starting conditions with practical relevance for the Bingham fluid simulations.

4. RESULTS AND DISCUSSION

Effects of Bingham number

The contours of non-dimensional temperature $\theta = (T - T_{cen})/\Delta T_{ref}$ and stream function $\Psi = \psi/\alpha^1$ for different values of Bn for $r_i/L = 1$, $Ra = 10^5$ and $Pr = 500$ in the CWHF configuration are shown in Fig. 2. It is clear from Fig. 2 that the magnitude of Ψ decreases and isotherms become increasingly parallel to the horizontal walls with increasing Bn . This indicates that convection weakens with increasing Bn and thermal transport takes place purely due to conduction for large values of Bn . The grey regions on the stream function distribution

¹It is also possible to define the non-dimensional stream function as $\Psi' = \psi/(\mu/\rho)$ but this definition remains directly proportional to Ψ (i.e. $\Psi' = \psi/\nu = \Psi/Pr$) and does not change the physics of problem. The current definition for Ψ (i.e. $\Psi = \psi/\alpha$) was used in several previous analyses [11-18].

in Fig. 2 indicate the Apparently Unyielded Regions (AURs) (regions where $|\tau| \leq \tau_y$ [27]). The AURs can be considered as the regions of slow moving fluids because of the high viscosity ($\mu_{yield} \geq 10^3 \mu$) in the context of bi-viscosity regularisation. The shapes and sizes of AURs are dependent on the choice of μ_{yield} [11-18] but the qualitative and quantitative distributions of stream function and isotherms remain independent of the value of μ_{yield} for $\mu_{yield} \geq 10^3 \mu$, and thus a fixed value $\mu_{yield} = 10^4 \mu$ is used here for all the simulations for the sake of high-fidelity. The precise shapes and sizes of AURs do not affect the mean Nusselt number in the present study. The size of the AURs increases, and thermal transport becomes increasingly conduction-dominated as the value of Bn increases. Fluid flow practically stops for large values of Bn and thermal transport takes place purely due to thermal conduction (see Fig. 2). The relative contribution of convection to conduction on the overall thermal transport can be quantified in terms of the mean Nusselt number. For example, $\overline{Nu}_{cy} = 1.0$ in this configuration indicates that the thermal transport is principally conduction-driven, whereas $\overline{Nu}_{cy} > 1.0$ suggests that convection influences the thermal transport, and the strengthening of convection in comparison to thermal conduction is reflected in the increase of the mean Nusselt number $\overline{Nu}_{cy}$. It will be shown later that $\overline{Nu}_{cy}$ indeed decreases with increasing Bn due to weakening of convection and settles to $\overline{Nu}_{cy} = 1.0$ for large values of Bingham number (see Fig. 6 later in the paper).

Although Fig. 2 shows the results only for the CWHF boundary condition, the distributions of θ and stream function Ψ for the CWT boundary condition remain qualitatively similar to those obtained for the CWHF boundary condition, and thus are not explicitly shown here for the sake of conciseness. Furthermore, the effects of Bn on the distributions of θ and stream function Ψ for the CWT boundary condition are qualitatively similar to those in the case of CWHF boundary condition. This is consistent with previous findings by Turan *et al.* [14] who reported

qualitatively similar behaviour for laminar Rayleigh-Bénard convection of Bingham fluids in square enclosures (i.e. $r_i \rightarrow \infty$) for both CWT and CWHF boundary conditions.

Effects of r_i/L and Ra

The variations of θ and W with $(r - r_i)/L$ at $z/L = 0.5$ for different values of r_i/L at $Ra = 10^4$, 10^5 and $Pr = 500$ are shown in Fig. 3 for $Bn = 0$ and $Bn = 0.03$ case. The corresponding variation of U with z/L at $0.5(r_i + r_o)/L$ for $Bn = 0$ and $Bn = 0.03$ is shown in Fig. 3 for both CWT and CWHF boundary conditions at $Ra = 10^4$, 10^5 and $Pr = 500$. It can be seen from Fig. 3 that the temperature difference between hot and cold walls is smaller in the CWHF configuration than that for the CWT boundary condition for a given set of values of Ra , Pr , Bn . For the CWHF boundary condition, the wall heat flux can be estimated as $q \sim k\Delta T/\delta_{th}$ where ΔT and δ_{th} are the characteristic temperature difference and the thickness of thermal boundary layer on horizontal walls, respectively. This suggests that the non-dimensional temperature θ in the case of CWHF boundary condition scales as $\theta \sim \Delta Tk/qL \sim q\delta_{th}k/qLk \sim O(\delta_{th}/L)$, whereas $\theta \sim O(1)$ for the CWT boundary condition. The above discussion suggests that $\theta \sim O(\delta_{th}/L)$ for the CWHF boundary condition is expected to increase with increasing δ_{th} , as the thermal boundary layer thickness increases with an increase in Bingham number due to the weakening of convection strength (see Fig. 3). The convection strength also weakens with decreasing Ra , which is reflected in the thickening of thermal boundary layer thickness. Thus, the temperature difference between horizontal walls also increases with decreasing Ra for the CWHF boundary condition. Higher temperature difference between the horizontal walls leads to stronger convection in the CWT configuration than in the corresponding CWHF configuration for the same set of values of r_i/L , Ra , Pr and Bn . This can be verified from Fig. 3 which shows that the magnitudes of U and W are greater in the CWT configuration than in the CWHF configuration for a given set of values of r_i/L , Ra , Pr and Bn .

Figure 3 further shows that r_i/L does not significantly affect the temperature difference between horizontal walls for both $Bn = 0$ and $Bn = 0.03$ regardless of the boundary condition at low nominal Ra values (i.e. $Ra = 10^4$). However, the temperature difference between horizontal walls is significantly influenced by r_i/L for high values of Ra (i.e. $Ra = 10^5$) for $r_i/L < 1$ for both CWT and CWHF boundary conditions. This behaviour arises due to a change in the streamline pattern for $r_i/L < 1$ and $Ra \geq 5 \times 10^4$ (see in Fig. 4), which leads to a modification of temperature distribution between hot and cold walls regardless of the boundary condition. It is worth noting that the magnitudes of U and W for $Ra = 10^5$ are greater than the corresponding values for $Ra = 10^4$ due to strengthening of buoyancy forces with increasing Ra for both CWT and CWHF configuration.

It can further be seen from Fig. 3 that non-dimensional axial velocity component W decreases while the non-dimensional radial velocity component U increases with increasing r_i/L for both $Bn=0$ and $Bn=0.03$ cases. Additionally, the magnitudes of W and U are smaller in the Bingham fluid cases than in the corresponding Newtonian fluid cases for a given set of values of nominal Rayleigh and Prandtl numbers.

One obtains $w \sim \sqrt{g\beta(T_H - T_C)L}$ ($w \sim \sqrt{g\beta q \delta_{th} L/k}$) by equating the order of magnitudes of inertial and buoyancy terms for the CWT (CWHF) boundary condition. Alternatively, these scaling can be expressed as: $w \sim (\alpha/L)\sqrt{RaPr}$ ($w \sim (\alpha/L)\sqrt{RaPr(\delta_{th}/L)}$) for the CWT (CWHF) boundary condition, which suggests that the magnitude of W is expected to be greater in the CWT configuration than in the case of CWHF boundary condition for a given set of values of Ra and Pr (because $\delta_{th} \ll L$ in boundary layer transport). Based on the continuity equation for axisymmetric geometry one obtains:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left(\frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{L} \quad (15)$$

which leads to the following relation based on scaling estimate of u :

$$u \sim \frac{wr}{(r+L)} \sim \frac{w}{(1+L/r_i)} \sim \frac{\alpha}{L} \frac{\sqrt{RaPr}}{(1+L/r_i)} \quad \text{for CWT} \quad (16i)$$

$$u \sim \frac{wr}{(r+L)} \sim \frac{w}{(1+L/r_i)} \sim \frac{\alpha}{L} \frac{\sqrt{RaPr}}{(1+L/r_i)} \sqrt{\frac{\delta_{th}}{L}} \quad \text{for CWHF} \quad (16ii)$$

As $\delta_{th}/L < 1$, the magnitude of U is expected to be greater in the CWT configuration than in the case of CWHF boundary condition for a given set of values of Ra and Pr (because $\delta_{th} \ll L$). According to Eq. 16 (i.e. $u \sim w/(1 + L/r_i)$) the magnitude of the radial velocity component u increases with increasing r_i/L which is consistent with numerical findings for both Newtonian and Bingham fluid cases as shown in Fig. 3. The contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for $Bn = 0$ (i.e. Newtonian fluid) and $Bn = 0.15$ fluid cases for different values of r_i/L for $Ra = 10^5$ and $Pr = 500$ are shown in Fig. 4 for the CWT boundary condition. It is evident from the Fig. 4 that the flow pattern (i.e. number of cells/rolls) changes for $r_i/L \leq 1$ for Newtonian fluids. However, this flow pattern reverts to the one cell structure with increasing Bn .

Following previous analyses [11-18] a scaling analysis can be performed to elucidate the expected behaviours in response to the changes in key parameters (e.g. r_i/L , Ra and Bn). One obtains the following expression by equating the order of magnitudes of inertial and viscous terms in the radial direction:

$$\rho \frac{u^2}{L} \sim \frac{1}{\delta_1} \left(\tau_y + \mu \frac{u}{\delta_1} \right) \quad (17i)$$

Using Eqs. 16i and 16ii in eq. 17i leads to:

$$\delta_1 \sim L \left[\sqrt{\left(\frac{Bn}{2} \right)^2 \frac{Pr}{Ra} (1 + L/r_i)^4 + \sqrt{\frac{Pr(1+L/r_i)^2}{Ra}}} + \left(\frac{Bn}{2} \right) \sqrt{\frac{Pr}{Ra}} (1 + L/r_i)^2 \right] \quad \text{for CWT} \quad (17ii)$$

$$\delta_1 \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \text{ for CWHF} \quad (17iii)$$

where δ_1 is the hydrodynamic boundary layer thickness on the horizontal wall. Similarly, the hydro-dynamic boundary layer thickness on the vertical wall δ can be scaled in the following manner by equating the magnitudes of inertial and viscous terms in the vertical direction (i.e. $\rho w^2/L \sim (\tau_y + \mu w/\delta)/\delta$) [13,14]:

$$\delta \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \sqrt{\frac{Pr}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \text{ for CWT} \quad (18i)$$

$$\delta \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \text{ for CWHF} \quad (18ii)$$

Equations 17 and 18 indicate that the boundary layer thickness increases with decreasing (increasing) nominal Rayleigh number Ra (nominal Bingham number Bn). It is worth noting that Eq. 17 approaches Eq. 18 for square enclosures (i.e. $r_i/L \rightarrow \infty$), and Eq. 18 was previously obtained for square enclosures by Turan *et al.* [13,14].

Based on the scaling estimates in Eqs. 17 and 18, it is possible to estimate the effective viscosity in both vertical and horizontal boundary layers (i.e. μ_{eff}^V and μ_{eff}^H) in the following manner ($\mu_{eff}^V \sim \mu + \tau_y \delta/w$ and $\mu_{eff}^H \sim \mu + \tau_y \delta_1/u$):

For CWT configuration [13]

$$\frac{\mu_{eff}^V}{\mu} \sim 1 + Bn \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \sqrt{\frac{Pr}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \quad (19i)$$

$$\frac{\mu_{eff}^H}{\mu} \sim 1 + \frac{Bn}{1+L/r_i} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} (1+L/r_i)^4 + \sqrt{\frac{Pr(1+L/r_i)^2}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} (1+L/r_i)^2 \right] \quad (19ii)$$

For CWHF configuration [13,14]

$$\frac{\mu_{eff}^V}{\mu} \sim 1 + Bn \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \quad (20i)$$

$$\frac{\mu_{eff}^H}{\mu} \sim 1 + \frac{Bn}{1+L/r_i} \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \quad (20ii)$$

Using Eqs. 19 and 20, the effective Rayleigh numbers (i.e. Ra_{eff}^V and Ra_{eff}^H) in both vertical and horizontal boundary layers can be estimated in the following manner:

For CWT configuration (i.e. $Ra_{eff} = \rho g \beta (T_H - T_C) L^3 / \mu_{eff} \alpha$):

$$Ra_{eff}^V \sim Ra / \left\{ 1 + Bn \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \frac{Pr}{Ra}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \right\} \quad (21i)$$

$$Ra_{eff}^H \sim Ra / \left\{ 1 + \frac{Bn}{1+L/r_i} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} (1+L/r_i)^4 + \frac{Pr(1+L/r_i)^2}{Ra}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} (1+L/r_i)^2} \right] \right\} \quad (21ii)$$

For CWHF configuration (i.e. $Ra_{eff} = \rho g \beta q L^4 / k \mu_{eff} \alpha$):

$$Ra_{eff}^V \sim Ra / \left\{ 1 + Bn \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \right\} \quad (22i)$$

$$Ra_{eff}^H \sim Ra / \left\{ 1 + \frac{Bn}{1+L/r_i} \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \right\} \quad (22ii)$$

Equations 21 and 22 indicate that the effective Rayleigh number remains smaller than the nominal Rayleigh number and the effective Rayleigh number decreases with increasing Bn for both CWT and CWHF boundary conditions. Furthermore, the effective Rayleigh number for the CWHF boundary condition is smaller than the CWT boundary condition for same set of values of Ra , Pr and Bn because δ_{th} is expected to be smaller than L . The scaling estimates for Ra_{eff}^V (i.e. eqs. 21i and 22i) are consistent with previous findings in the context of square enclosures.

Equations 19 and 20 indicate that $\delta_1 > \delta$ and $u < w$, thus μ_{eff}^H is expected to be greater than μ_{eff}^V in cylindrical enclosures. This further implies that Ra_{eff}^H is expected to be smaller than Ra_{eff}^V in cylindrical enclosures as it is shown in Eqs. 21 and 22. However, they approach each other for $r_i/L \rightarrow \infty$ (i.e. in the limit of square enclosures). It is worth noting that the velocity magnitudes in both vertical and horizontal directions remain of the same order for a two-dimensional square enclosure, which leads to $\delta \sim \delta_1$ and $Ra_{eff}^V \sim Ra_{eff}^H$.

Behavior on mean Nusselt number of cylindrical enclosures $\overline{Nu}_{cy}$

The variation of $\overline{Nu}_{cy}$ with r_i/L for different Ra is shown in Fig. 5 for Newtonian fluids ($Bn = 0$) at $Pr = 500$.² It can be seen from Fig. 5 that $\overline{Nu}_{cy}$ increases with increasing Ra but remains insensitive to the changes in r_i/L for $r_i/L > 1$ in the nominal Rayleigh number range for both CWT and CWHF boundary conditions. However, $\overline{Nu}_{cy}$ increases sharply with increasing r_i/L for $r_i/L < 1$ in the case of $Ra = 10^5$. This behaviour is particularly prevalent in the CWT case. This can be explained by the change in the flow pattern, as shown in Figs. 4-5. The changes in the flow patterns for $Ra \geq 5 \times 10^4$ and $r_i/L \leq 1$ modify the distributions of isotherms between hot and cold walls, which is reflected in the sharp increase in $\overline{Nu}_{cy}$ (see Figs. 4-5).

The wall heat flux $q \sim h\Delta T$ can be scaled as: $q \sim h\Delta T \sim k\Delta T/\delta_{th} \sim k(\Delta T/\delta_1)f_2$ [11-18], so the mean Nusselt number can be scaled as: $\overline{Nu} \sim hL/k \sim qL/\Delta Tk \sim L/\delta_{th} \sim (L/\delta_1)f_2$, where $f_2(Ra, Pr, r_i/L, Bn)$ is a function which accounts for the ratio of hydro-dynamic and thermal boundary layer thicknesses (i.e. $f_2 \sim \delta_1/\delta_{th}$). However, the exact form of $f_2(Ra, Pr, r_i/L, Bn)$ is not important for the following discussion. Using Eqs. 17ii and 17iii it is possible to estimate

² The mean Nusselt number $\overline{Nu}_{cy}$ from computational simulation results is evaluated using eq. 9 for Figs. 5, 6 and 8.

the mean Nusselt number for natural convection of Newtonian fluids by putting $Bn = 0$ as:

$$\overline{Nu}_{cy} \sim (Ra/Pr)^{0.25} f_2(Ra, Pr) / (1 + L/r_i)^{0.5} \quad (\overline{Nu}_{cy} \sim (Ra/Pr)^{0.2} \{f_2(Ra, Pr)\}^{0.8} / (1 + L/r_i)^{0.4})$$

for the CWT (CWHF) boundary condition. These scalings suggest that $\overline{Nu}_{cy}$ is expected to increase with increasing Ra and r_i/L , which are consistent with the observations made from Fig. 5. Furthermore, one obtains the mean Nusselt number estimates for square enclosures

$$\overline{Nu}_{sq} \quad \text{by using} \quad Bn = 0 \quad \text{and} \quad L/r_i = 0 \quad \text{as:} \quad \overline{Nu}_{sq} \sim (Ra/Pr)^{0.25} f_2(Ra, Pr)$$

$$(\overline{Nu}_{sq} \sim (Ra/Pr)^{0.2} \{f_2(Ra, Pr)\}^{0.8}) \quad \text{for the CWT (CWHF) boundary condition. The following}$$

correlations for the mean Nusselt number were proposed in [13,14] for natural convection of

Newtonian fluids in square enclosures $\overline{Nu}_{sq}$:

$$\overline{Nu}_{sq} = 0.178Ra^{0.269}[Pr/(1 + Pr)]^{0.02} \quad \text{for CWT} \quad (23i)$$

$$\overline{Nu}_{sq} = 0.289Ra^{0.214}[Pr/(1 + Pr)]^{0.017} \quad \text{for CWHF} \quad (23ii)$$

Equations 23i and 23ii are consistent with the scaling estimates $\overline{Nu}_{sq} \sim (Ra/Pr)^{0.25} f_2(Ra, Pr)$

and $\overline{Nu}_{sq} \sim (Ra/Pr)^{0.2} \{f_2(Ra, Pr)\}^{0.8}$, respectively. A comparison between $\overline{Nu}_{sq} =$

$$0.178Ra^{0.269}[Pr/(1 + Pr)]^{0.02} \quad (\text{or} \quad \overline{Nu}_{sq} = 0.289Ra^{0.214}[Pr/(1 + Pr)]^{0.017}) \quad \text{and}$$

$$\overline{Nu}_{sq} \sim (Ra/Pr)^{0.25} f_2(Ra, Pr) \quad (\text{or} \quad \overline{Nu}_{sq} \sim (Ra/Pr)^{0.2} \{f_2(Ra, Pr)\}^{0.8}) \quad \text{reveals that} \quad f_2 =$$

$$0.178Ra^{0.019}Pr^{0.27}/(1 + Pr)^{0.02} \quad (\text{or} \quad f_2 = 0.212Ra^{0.0175}Pr^{0.27}/(1 + Pr)^{0.02}) \quad \text{for the CWT}$$

(or CWHF) boundary condition. The predictions of Eq. 23 are shown in Fig. 5, which indicates

that $\overline{Nu}_{cy}$ can be satisfactorily predicted with the help of Eq. 23 for all cases considered here

except for $Ra = 10^5$ and $r_i/L \leq 1$. Equation 23 is valid for the flow pattern with one cell within

the enclosure but two cells have been found for $Ra = 10^5$ and $r_i/L \leq 1$, which lead to a

reduction in $\overline{Nu}_{cy}$ in comparison to the values obtained for $r_i/L > 1$. The variation of the mean

Nusselt number $\overline{Nu}_{cy}$ with Bn for different values of r_i/L at $Ra = 10^5$ and $Pr = 500$ for both

CWT (top) and CWHF (bottom) boundary conditions is shown in Fig. 6 for Bingham fluids. It

is evident from Fig 6 that the $\overline{Nu}_{cy}$ decreases with increasing Bn which is indicative of the

weakening of convective transport due to strengthening of additional flow resistance arising from yield stress. Furthermore, Fig. 6 further shows that $\overline{Nu}_{cy}$ settles to unity once a threshold value of Bingham number $(Bn_{max})_{cy}$ is surpassed. Here $(Bn_{max})_{cy}$ is taken to be the nominal Bingham number where $\overline{Nu}_{cy} = 1.01$, so that $\overline{Nu}_{cy} < 1.01$ for $Bn > (Bn_{max})_{cy}$. The fluid becomes totally un-yielded for $Bn > (Bn_{max})_{cy}$, and the thermal transport takes place purely due to conduction under this situation. Thus, there is no point in simulating cases with $Bn > (Bn_{max})_{cy}$ because the heat transfer remains purely conduction-driven and $\overline{Nu}_{cy}$ remains unity. The range of Bingham number (i.e. $0 < Bn < (Bn_{max})_{cy}$) depends on Ra and r_i/L for a given value of Pr . During the course of this analysis, the simulations have been conducted by increasing (decreasing) Bn until $(Bn_{max})_{cy}$ ($Bn = 0$) is reached from $Bn = 0$ ($Bn = (Bn_{max})_{cy}$) in order to assess if the variation of $\overline{Nu}_{cy}$ with Bn shows any hysteresis. It has been found that hysteresis occurs (i.e. $\overline{Nu}_{cy}$ is indeed multi-valued for a given value of Bn and different flow patterns are obtained for a given value of Bn) for $Ra \geq 5 \times 10^4$ and $r_i/L \leq 1$ depending on the starting condition (see Fig.6) for both CWT and CWHF conditions. By contrast, starting condition has no influence on the results for $r_i/L > 1$ for a given value of Ra for both CWT and CWHF configurations.

The variation of $(Bn_{max})_{cy}$ with r_i/L for different values of Ra at $Pr = 500$ for both CWT and CWHF boundary conditions is shown in Fig. 7. It can be seen from Fig. 7 that $(Bn_{max})_{cy}$ increases with increasing Ra . The effects of buoyancy force strengthen with increasing Ra and thus convection could be sustained up to high values of Bingham number for large values of nominal Rayleigh number. This is reflected in the increasing trend of $(Bn_{max})_{cy}$ with increasing Ra . Figure 7 further shows that $(Bn_{max})_{cy}$ for a given value of Ra for the CWT boundary condition is greater than in the case of CWHF boundary condition due to stronger

convection in the CWT configuration as a result of higher temperature difference between active walls (see Fig. 3). Figure 7 shows that the $(Bn_{max})_{cy}$ increases with increasing r_i/L before approaching the corresponding value for a square enclosure $(Bn_{max})_{sq}$ in the limit of $r_i/L \rightarrow \infty$. This indicates that although $\overline{Nu}_{cy}$ remains independent of r_i/L for Newtonian fluids ($Bn = 0$) for $r_i/L > 1$, convection strengthens with increasing r_i/L for Bingham fluids for both CWT and CWHF boundary conditions.

The quantity $(Bn_{max})_{cy}$ can be estimated for the CWT boundary condition by considering $\overline{Nu}_{cy} \sim L/\delta_{th} \sim Lf_2/\delta_1 \sim O(1)$. This along with Eq. 17ii yields for the CWT boundary condition:

$$(Bn_{max})_{cy} \sim \frac{f_2 \sqrt{Ra/Pr}}{(1+L/r_i)^2} - \frac{1}{(1+L/r_i)f_2} \quad (24i)$$

Using $r_i/L \rightarrow \infty$ in Eq. 24i yields the estimate of the corresponding threshold value of Bingham number for square enclosures for the CWT boundary condition:

$$(Bn_{max})_{sq} \sim f_2 \sqrt{Ra/Pr} - \frac{1}{f_2} \quad (24ii)$$

It can be seen from Eq. 24i that $(Bn_{max})_{cy}$ is expected to increase with increasing Ra and r_i/L , which are consistent with the observations made from Fig. 7. Although δ_1 and $(Bn_{max})_{cy}$ cannot be analytically obtained from Eq. 17iii, the same qualitative behaviour as that of the CWT boundary condition can be expected for the CWHF boundary condition.

The quantity $(Bn_{max})_{sq}$ is parameterised in [17] in the following manner for $10 \leq Pr \leq 500$:

$$(Bn_{max})_{sq} = \left[\left(\frac{0.00048Pr+0.132}{Pr+18.15} \right) Ra^{0.525} \right] - \frac{0.0389Pr+7.687}{Pr+13.56} \text{ for CWT} \quad (24iii)$$

$$(Bn_{max})_{sq} = \left[\left(\frac{Pr+236}{2707Pr+40280} \right) Ra^{0.525} \right] - \frac{Pr+128}{44.6Pr+271} \text{ for CWHF} \quad (24iv)$$

Equation 24iii is consistent with the scaling estimate presented in Eq. 24ii. Using Eq. 24, $(Bn_{max})_{cy}$ is correlated here in the following manner:

$$(Bn_{max})_{cy} = (Bn_{max})_{sq}/[1 + a_o(L/r_i)^{b_o}] \quad (25i)$$

$$\text{where } a_o = 0.0127Ra^{0.118} \text{ and } b_o = 0.142Ra^{0.125} \text{ for CWT} \quad (25ii)$$

$$a_o = 0.05 \text{ and } b_o = 0.292Ra^{0.077} \text{ for CWHF} \quad (25iii)$$

The predictions of Eq. 25 are shown in Fig. 7 which shows this correlation satisfactorily predicts ($R^2=0.99$) $(Bn_{max})_{cy}$ for Bingham fluids in square cross-sectioned cylindrical enclosures for $10^3 \leq Ra \leq 10^5$, $0.125 \leq r_i/L \leq 16$ at $Pr = 500$. It is worth noting that $(Bn_{max})_{cy}$ becomes equal to $(Bn_{max})_{sq}$ in the limit of $r_i/L \rightarrow \infty$ which ensures that asymptotic condition $(Bn_{max})_{cy} = (Bn_{max})_{sq}$ for $r_i/L \rightarrow \infty$ is satisfied.

Using Eq. 17ii one obtains the following estimate of mean Nusselt number $\overline{Nu}_{cy} \sim (L/\delta_1)f_2$ for the CWT boundary condition:

$$\overline{Nu}_{cy} \sim f_2 \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} (1 + L/r_i)^4 + \sqrt{\frac{Pr(1+L/r_i)^2}{Ra}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \left(1 + \frac{L}{r_i}\right)^2} \right]^{-1} \quad (26i)$$

Equation 26i indicates that $\overline{Nu}_{cy}$ is expected to increase with increasing Ra and r_i/L , and decrease with increasing Bn . It is not possible to obtain an analytical scaling estimate of $\overline{Nu}_{cy}$ from Eq. 17iii but the mean Nusselt number behaviour in response to Ra, Bn and r_i/L for CWHF boundary condition is expected to be qualitatively similar to that in the case of CWT boundary condition. A correlation for $\overline{Nu}_{sq}$ was proposed in [13] for natural convection of Bingham fluids in square enclosures in the range $0 \leq Bn \leq (Bn_{max})_{sq}$. The methodology of [13] has been utilised here along with Eq. 26i and $(\overline{Nu}_{cy})_{Bn=0} \sim (Ra/Pr)^{0.25} f_2(Ra, Pr)/(1 + L/r_i)^{0.5}$ to propose a correlation of $\overline{Nu}_{cy}$ for $0 \leq Bn \leq (Bn_{max})_{cy}$ in the following manner for the CWT boundary condition:

$$\frac{\overline{Nu}_{cy}-1}{(\overline{Nu}_{cy})_{Bn=0}-1} = \frac{2[1-(Bn^*/Bn_{max}^*)^b]^c}{Bn^*+\sqrt{Bn^{*2}+4}} \text{ when } (\overline{Nu}_{cy})_{Bn=0} > 1 \quad (26ii)$$

$$\overline{Nu}_{cy} = 1 \text{ when } (\overline{Nu}_{cy})_{Bn=0} = 1 \quad (26iii)$$

where $Bn_{max}^* = Bn_{max}(1 + L/r_i)^{1.5}(Ra/Pr)^{-1/4}$, b and c are the correlation parameters. Following Ref. [13], the functional forms given by Eqs. 26ii and 26iii are taken to be valid also for the CWHF boundary condition. Turan *et al.* [13,14] proposed $b_{sq} = 0.6$ and $c_{sq} = 0.025Ra^{0.171}Pr^{0.095}$ ($b_{sq} = 0.75$ and $c_{sq} = 0.0818Ra^{0.0191}Pr^{0.054}$) for the CWT (CWHF) boundary condition [13,14]. Accordingly in the current study b and c have been parameterized in the range of $2 \leq r_i/L \leq 16$ ($0.25 \leq r_i/L \leq 16$) as:

$$b_{cy} = b_{sq} \text{ and } c_{cy} = c_{sq}/[1 + m_o(L/r_i)^{n_o}] \quad (27i)$$

$$m_o = 4.616 - 0.385\ln Ra \text{ and } n_o = 0.2 \text{ for } 2 \leq r_i/L \leq 16 \text{ for CWT} \quad (27ii)$$

$$m_o = 0.1 \text{ and } n_o = 0.021Ra^{0.274} \text{ for } 0.25 \leq r_i/L \leq 16 \text{ for CWHF} \quad (27iii)$$

The predictions of Eqs. 26ii and iii are shown in Fig. 8 which shows that this correlation satisfactorily predicts ($R^2=0.99$ and 3% maximum percentage error) $\overline{Nu}_{cy}$ for natural convection of Bingham fluids in square-cross sectioned cylindrical enclosures. It is also worth noting that $(c)_{cy}$ becomes equal to $(c)_{sq}$ in the limit of $r_i/L \rightarrow \infty$ which ensures that asymptotic condition $\overline{Nu}_{cy} = \overline{Nu}_{sq}$ for $r_i/L \rightarrow \infty$ is satisfied in Eqs. 26ii-iii. Note that the correlation given by Eqs. 26ii-iii is only valid for the range of r_i/L where one cell convection pattern is observed. Therefore, the values corresponding to $r_i/L = 0.25$ and 1.0 for the CWT boundary condition are not shown in Fig. 8 since multiple cell patterns are obtained under these conditions, and Eqs. 26ii-iii are valid for the range of $2 \leq r_i/L \leq 16$ ($0.25 \leq r_i/L \leq 16$) for the CWT (CWHF) boundary condition.

Turan *et al.* [13,14] analysed the influences of Pr for the range of $0.1 \leq Pr \leq 100$ on $\overline{Nu}_{sq}$ for laminar Rayleigh-Bénard convection of Bingham fluids in square enclosures (i.e. $r_i \rightarrow \infty$)

for both CWT and CWHF boundary conditions. It has been found that the $\overline{Nu}_{sq}$ decreases with increasing Pr for large values of Bn values whereas, $\overline{Nu}_{sq}$ increases with increasing Pr for small values of Bn values for both CWT and CWHF boundary conditions. Furthermore, Bn_{max} decreases with increasing Pr [13,14]. Interested readers are requested to refer to Refs. [13,14] for detailed information in this regard. As the functional relations given by Eqs. 26ii and 26iii are valid for natural convection of Bingham fluids in annular enclosures irrespective of the value of Pr , and the parameterisation of Bn_{max} is proposed here based on the expressions originally proposed by Yigit *et al.* [17] for $10 \leq Pr \leq 500$, it can be expected the correlation of $\overline{Nu}_{cy}$ given by Eqs. 26ii-iii likely to valid for a range of different Prandtl numbers and a limited number of simulations have been carried out for $Pr = 100$ and Eqs. 26ii-iii have indeed been found to predict $\overline{Nu}_{cy}$ satisfactorily also for these cases.

5. CONCLUSIONS

Steady-state laminar Rayleigh-Bénard convection of Bingham fluids in cylindrical annular enclosures has been analysed for the range of nominal Rayleigh number $10^3 \leq Ra \leq 10^5$ at a representative value of nominal Prandtl number (i.e. $Pr = 500$) for different values of normalised inner radius $0 \leq r_i/L \leq 16$ for both constant wall temperature and constant wall heat flux boundary conditions. It has been found that the buoyancy-driven transport strengthens with increasing nominal Rayleigh number Ra for both Newtonian and Bingham fluids but the mean Nusselt number $\overline{Nu}_{cy}$ for Bingham fluids has been found to be smaller than the corresponding values obtained for Newtonian fluids for both CWT and CWHF boundary conditions due to additional flow resistance arising from yield stress in the case of Bingham fluids. Moreover, $\overline{Nu}_{cy}$ decreases with increasing Bn and it assumes a value equal to unity (i.e. $\overline{Nu}_{cy} = 1.0$) for large values of Bn which indicates that thermal transport takes

place purely due to thermal conduction as a result of high flow resistance arising from yield stress. Although the buoyancy force strengthens with increasing Ra , the mean Nusselt number $\overline{Nu}_{cy}$ does not show a monotonic increase with increasing Ra for small values of r_i/L . This behaviour arises due to the change in flow pattern (i.e. number of convection rolls/cells) for small values of r_i/L for large values of Ra . In contrast, only one cell flow pattern is obtained, and $\overline{Nu}_{cy}$ monotonically increases with increasing Ra for large values of r_i/L . Furthermore, $\overline{Nu}_{cy}$ has been found to increase with increasing r_i/L but approaches the corresponding value obtained for square enclosures ($r_i \rightarrow \infty$) for both CWT and CWHF boundary conditions for large values of r_i/L . It has also been found that flow pattern and the mean Nusselt number $\overline{Nu}_{cy}$ are dependent on the starting conditions for Bingham fluid cases since hysteresis is evident for small values of r_i/L for both CWT and CWHF boundary conditions. It has been found that convection could be sustained up to a higher value of Bingham number due to stronger convection arising from higher temperature difference between horizontal walls in the case of CWT boundary condition than in the case of CWHF configuration. Numerical findings have been used to propose a correlation for $\overline{Nu}_{cy}$ in the range of $2 \leq r_i/L \leq 16$ for CWT and $0.25 \leq r_i/L \leq 16$ for CWHF boundary conditions.

REFERENCES

- [1] I. Catton, P.S. Ayyaswamy, and R.M. Clever, Natural convection flow in a finite, rectangular slot arbitrarily oriented with respect to the gravity vector, *Int. J. Heat Mass Trans.* 17 (1974) 173-184. Doi: 10.1016/0017-9310(74)90079-9
- [2] S. Ostrach, Natural Convection in Enclosures, *Journal of Heat Transfer* 110 (1988) 1175–1190. Doi: 10.1115/1.3250619
- [3] M. A. Hassan, M. Pathak, M.K. Khan, Rayleigh-Bénard convection in Herschley-Bulkley fluid, *J. Non-Newt. Fluid Mech.* 226 (2015) 32-45. Doi:10.1016/j.jnnfm.2015.10.003
- [4] M. Darbouli, C. Métivier, J.M. Piau, A. Magnin and A. Abdelali, Rayleigh-Bénard convection for viscoplastic fluids, *Physics of Fluids* 25 (2013) 023101. Doi: 10.1063/1.4790521
- [5] Z. Kebiche, C. Castelain, T. Burghlea, Experimental investigation of the Rayleigh-Bénard convection in a yield stress fluid, *J. Non-Newt. Fluid Mech.* 203 (2014) 9-23. Doi:10.1016/j.jnnfm.2013.10.005
- [6] N.M. Wereley *Magnetorheology: Advances and Applications*, Royal Society of Chemistry, RSC Smart materials, Cambridge, UK, 2014.
- [7] J. Zhang, D. Vola, I. A. Frigaard, Yield stress effects on Rayleigh-Bénard convection, *J. Fluid Mech.* 566 (2006) 389-419. Doi: 10.1017/S002211200600200X
- [8] N. J. Balmforth and A.C. Rust, Weakly nonlinear viscoplastic convection, *J. Non-Newt. Fluid Mech.* 158 (2009) 36-45. Doi: 10.1016/j.jnnfm.2008.07.012
- [9] A. Vikhansky, Thermal convection of a viscoplastic liquid with high Rayleigh and Bingham numbers, *Phys. Fluids* 21 (2009) 103103. Doi: 10.1063/1.3256166
- [10] A. Vikhansky, On the onset of Bingham liquid in rectangular enclosures, *J. Non-Newt. Fluid Mech.* 165 (2010) 1713-1716. Doi: 10.1016/j.jnnfm.2010.09.003

- [11] O. Turan, N. Chakraborty, R.J. Poole, Laminar natural convection of Bingham fluids in a square enclosure with differentially heated side walls, *J. Non-Newt. Fluid Mech.* 165 (2010) 901-913. Doi: 10.1016/j.jnnfm.2010.04.013
- [12] O. Turan, A. Sachdeva, R.J. Poole, N. Chakraborty, Laminar natural convection of Bingham fluids in a square enclosure with vertical walls subjected to constant heat flux, *Num. Heat Trans. A* 60 (2011) 381-409. Doi:10.1080/10407782.2011.594417
- [13] O. Turan, A. Sachdeva, R.J. Poole, N. Chakraborty, Laminar Rayleigh-Bénard convection of yield stress fluids in a square enclosure, *J. Non-Newt. Fluid Mech.* 172 (2012) 83-96. doi:10.1016/j.jnnfm.2012.01.006
- [14] O. Turan, R.J. Poole, N. Chakraborty, Boundary condition effects on natural convection of Bingham fluids in a square enclosure with differentially heated horizontal walls, *J. Comput. Therm. Sci.* 4 (2012) 77-97. Doi: 10.1615/ComputThermalScien.2012004759
- [15] O. Turan, N. Chakraborty, R.J. Poole, “Aspect ratio effects in laminar natural convection of Bingham fluids in rectangular enclosures with differentially heated side walls, *J. Non-Newt. Fluid Mech.* 166 (2011) 208-230. Doi:10.1016/j.jnnfm.2010.12.002
- [16] S. Yigit, R. J. Poole, N. Chakraborty, Laminar natural convection of Bingham fluids in inclined differentially heated square enclosures subjected to uniform wall temperatures, *J. Heat Transfer Transactions of ASME* 137 (2015) 052504. Doi:10.1115/1.4029763
- [17] S. Yigit, R. J. Poole, N. Chakraborty, Effects of aspect ratio on natural convection of Bingham Fluids in rectangular enclosures with differentially heated horizontal walls heated from below, *Int. J. Heat Mass Transfer* 80 (2015) 727-736. Doi:10.1016/j.ijheatmasstransfer.2014.09.046
- [18] O. Turan, R.J. Poole, N. Chakraborty, Influences of boundary conditions on laminar natural convection of Bingham fluids in rectangular enclosures with differentially heated side walls, *Heat Trans. Eng.* 35 (2014) 822-849. Doi:10.1080/01457632.2014.852870

- [19] M.A. Hassan, M. Pathak, M.K. Khan, Natural convection of viscoplastic fluids in a square enclosure, J. Heat Transfer Transactions of ASME 135 (2013) 122501. Doi: 10.1115/1.4024896
- [20] H.A. Barnes, The yield stress—a review or ‘*παντα ρει*’—everything flows?, J Non-Newt. Fluid Mech. 81 (1999) 133–178. Doi: 10.1016/S0377-0257(98)00094-9
- [21] E. J. O'Donovan, R. I. Tanner, Numerical study of the Bingham squeeze film problem, J. Non-Newt. Fluid Mech. 15 (1984) 75-83. Doi: 10.1016/0377-0257(84)80029-4
- [22] T. C. Papanastasiou, Flow of materials with yield, J. Rheol. (31) 1987 385. Doi: 10.1122/1.549926.
- [23] N. Kim, Remarks for the axisymmetric Navier-Stokes equations, J. Differential Equations, 187 (2003) 226-239. Doi: 10.1016/S0022-0396(02)00077-3
- [24] S. V. Patankar, Numerical Heat Transfer and Fluid Flow, Hemisphere, Washington, D.C., 1980.
- [25] N. Quertatani, N. B. Cheikh, B. B. Beya, T. Lili, Numerical simulation of two dimensional Rayleigh- Bénard convection in an enclosure, C. R. Mecanique 336 (2008) 464-470. Doi: 10.1016/j.crme.2008.02.004
- [26] D. Vola, L. Boscardin, J.C. Latché, Laminar unsteady flows of Bingham fluids: a numerical strategy and some benchmark results, J. Comput. Phys. 187 (2003) 441-456. Doi: 10.1016/S0021-9991(03)00118-9
- [27] E. Mitsoulis and T. Zisis, Flow of Bingham plastics in a lid-driven square cavity, J. Non-Newt. Fluid Mech. 101 (2001) 173-180. Doi: 10.1016/S0377-0257(01)00147-1

Table 1: Comparison of the mean Nusselt number $\overline{Nu}$ for Newtonian fluid with the benchmark data [25] for square enclosure ($AR = 1$) at $Pr = 0.71$.

Ra	Present study	[25]
1×10^3	1.000	1.000
1×10^4	2.154	2.158
1×10^5	3.907	3.910
1×10^6	6.363	6.309

FIGURE CAPTIONS

Figure 1: Schematic diagram of simulation domain: a) CWT, b) CWHF configuration.

Figure 2: Contours of θ and Ψ with AURs (shown in grey) for $r_i/L = 1.0$, $Ra = 10^5$ and $Pr = 500$ in the CWHF configuration.

Figure 3: Variations of (a) θ and (b) W with $(r - r_i)/L$ at $z/L = 0.5$ at $Ra = 10^4$ and 10^5 . (c) Variations of U with z/L at $0.5(r_i + r_o)/L$ at $Ra = 10^4$ and 10^5 for $Bn = 0$ and $Bn = 0.03$.

Figure 4: Contours of θ and Ψ with AURs (shown in grey) for $Bn = 0$ and $Bn = 0.15$ in case of $r_i/L =$ a) 0.25 , b) 1 , c) 4 , d) 16 for $Ra = 10^5$ at $Pr = 500$ for CWT configuration.

Figure 5: Variation of $\overline{Nu}_{cy}$ with r_i/L for Newtonian fluids ($Bn = 0$). The mean Nusselt number values evaluated from simulation data using Eq. 9 are shown by symbols. The predictions of Eq. 23 are shown by broken lines.

Figure 6: Variation of $\overline{Nu}_{cy}$ (evaluated from simulation data using Eq. 9) with Bn for Bingham fluids at $Ra = 10^5$ for a) CWT (top row), b) CWHF (bottom row) boundary conditions.

Figure 7: Variation of $(Bn_{max})_{cy}$ with r_i/L along with the prediction of Eq. 25.

Figure 8: Variation of $\overline{Nu}_{cy}$ with Bn for different values of Ra a) 10^4 , b) 5×10^4 , c) 10^5 along with the prediction of Eqs. 26ii and 26iii. The mean Nusselt number values evaluated from simulation data using Eq. 9 are shown by symbols. The predictions of Eqs. 26ii and 26iii are shown by continuous lines.

Fig. 1: Schematic diagram of simulation domain: a) CWT, b) CWHF configuration.

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