Yigit S, Foxon T, Chakraborty N.

Influences of Boundary Condition on Laminar Natural Convection of Bingham
Fluids in Square Cross-Sectioned Cylindrical Annular Enclosures with
Differentially Heated Vertical Walls.
Heat Transfer Engineering 2017

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This is an Accepted Manuscript of an article published by Taylor & Francis in Heat Transfer Engineering on 13/3/2017, available online: <http://www.tandfonline.com/10.1080/01457632.2017.1305821>

DOI link to article:

<https://doi.org/10.1080/01457632.2017.1305821>

Date deposited:

19/07/2017

Embargo release date:

13 March 2018

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Influences of boundary condition on laminar natural convection of Bingham fluids in square cross-sectioned cylindrical annular enclosures with differentially heated vertical walls

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ABSTRACT

Numerical simulations have been carried out to analyse steady-state laminar natural convection of yield stress fluids obeying Bingham model in square cross-sectioned cylindrical annular enclosures with differentially heated vertical walls for both constant wall temperature and constant wall heat flux boundary conditions for active walls. The simulations have been performed under the assumption of axisymmetry for a nominal Rayleigh number range of 10^3 to 10^6 and nominal Prandtl number range of 10 to 10^3 for different ratio of internal cylinder radius to cylinder height range of 0.125 to 16. The mean Nusselt number on the inner periphery for the constant wall heat flux configuration has been found to be smaller than that in the case of constant wall temperature configuration for a given set of values of nominal Rayleigh and Prandtl numbers for both Newtonian and Bingham fluid cases. The mean Nusselt number normalized by the corresponding value obtained for pure conductive transport increases with increasing internal radius before approaching the corresponding mean Nusselt number for square enclosures regardless of the boundary conditions. Detailed physical explanations have been provided for the effects of the aforementioned parameters on the mean Nusselt number on the inner periphery. Finally, the new Nusselt number correlations have been proposed for laminar natural convection of both Newtonian and Bingham fluids in square cross-sectioned cylindrical annular enclosures for both constant wall temperature and constant wall heat flux boundary conditions.

INTRODUCTION

Natural convection of yield stress fluids in rectangular enclosed [1-18] and open [19] spaces have been investigated recently because of their relevance to several engineering applications such as chemical preparation, food processing, oil extraction, cryogenic storage etc. However, in most engineering applications natural convection of yield stress fluids takes place in cylindrical annular spaces instead of rectangular enclosures. Natural convection in cylindrical annular space with differentially heated vertical walls has previously been analysed for Newtonian fluids by computational [20,21] and experimental [22,23] means. Recently, Yigit and Chakraborty [24] numerically analysed natural convection of yield stress fluids obeying Bingham model in square cross-sectioned vertical annular spaces for constant heat flux (CWHF) boundary condition for differentially heated sidewalls. It is also worth noting that the experimental analysis of Keyhani *et al.* [23] involving Newtonian fluids was also carried out for CWHF boundary condition but the aspect ratio of the cylinder was different from the one used by Yigit and Chakraborty [24]. It has been found that convective thermal transport strengthens with increasing internal cylinder radius in the cylindrical annular enclosures for both Newtonian and Bingham fluids [20-24]. Several previous analyses demonstrated that the boundary condition for the active vertical walls significantly affect the heat transfer rate and the mean Nusselt number for same numerical values of nominal Rayleigh and Prandtl numbers for natural convection of Bingham fluids in rectangular enclosures [5-14]. For example, the difference between the mean Nusselt number for constant wall temperature and constant wall heat flux boundary conditions in square enclosures with differentially heated sidewalls can be up to 80% for the same set of numerical values of nominal Rayleigh and Prandtl numbers [6]. However, the effects of boundary condition on natural convection of Bingham fluids in cylindrical annular enclosures with differentially heated vertical walls have not been addressed in existing literature. Thus, the present study aims to address this deficit for analysing boundary

condition effects on natural convection of Bingham fluids in cylindrical annular enclosures. Here natural convection of yield stress fluids obeying the Bingham model (i.e. a linear strain rate dependence of viscous stress) in square cross-sectional cylindrical annular spaces have been numerically investigated for different values of internal radius normalised by enclosure height r_i/L , nominal Rayleigh, Bingham and Prandtl numbers (i.e. Ra , Bn and Pr) for differentially heated vertical walls under the assumption of axisymmetry for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions for vertical walls. In this respect, the main objectives of the current analysis are:

1. To analyse the effects of r_i/L , Ra , Pr and Bn on natural convection of Bingham fluids in cylindrical annular space with square cross-section and differentially heated vertical walls for both CWT and CWHF boundary conditions.
2. To explain the differences in heat transfer rate due to wall boundary condition.
3. To propose correlation of the mean Nusselt number based on inner periphery $\overline{Nu}_i$ accounting for the effects of r_i/L , Ra , Pr and Bn for both CWT and CWHF boundary conditions.

The rest of the paper will be organised as follows. The mathematical background and numerical implementation pertaining to the current analysis are presented in next two sections. Following this, results will be presented and subsequently discussed. The main findings will be summarised and conclusions will be drawn in the final section of this paper.

MATHEMATICAL BACKGROUND

The strain rate dependence of viscous stresses for Bingham fluids in the Bingham model [25] can be expressed as:

$$\underline{\underline{\dot{\gamma}}} = 0 \quad \text{for} \quad \tau \leq \tau_y \quad (1)$$

$$\underline{\underline{\tau}} = (\mu + \tau_y/\dot{\gamma})\underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \tau > \tau_y \quad (2)$$

where $\dot{\gamma}_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ are the components of the rate of strain tensor $\underline{\underline{\dot{\gamma}}}$, $\underline{\underline{\tau}}$ is the stress tensor, τ_y is the yield stress, μ is the plastic viscosity, τ and $\dot{\gamma}$ are the second invariants of the stress and the rate of strain tensors respectively, which are expressed as:

$$\tau = \left[\frac{1}{2} \underline{\underline{\tau}} : \underline{\underline{\tau}} \right]^{1/2} \quad (3)$$

$$\dot{\gamma} = \left[\frac{1}{2} \underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} \right]^{1/2} \quad (4)$$

For the current analysis, the bi-viscosity regularisation proposed by O'Donovan and Tanner [26] has been used to mimic the stress-shear rate characteristics for Bingham fluids:

$$\underline{\underline{\tau}} = \mu_{yield} \underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \dot{\gamma} \leq \tau_y/\mu_{yield} \quad (5a)$$

$$\underline{\underline{\tau}} = \tau_y (\underline{\underline{\dot{\gamma}}}/\dot{\gamma}) + \mu \underline{\underline{\dot{\gamma}}} \quad \text{for} \quad \dot{\gamma} > \tau_y/\mu_{yield} \quad (5b)$$

where μ_{yield} is the yield viscosity and μ is the plastic viscosity. O'Donovan and Tanner [26] indicated that a value of μ_{yield} equal to 1000μ mimics the true Bingham model in a satisfactory manner but here $\mu_{yield}/\mu = 10^4$ is kept fixed to ensure higher fidelity of the simulations. Moreover, a limited number of simulations have also been carried out for the current configuration for $Ra = 10^6$ and $Pr = 10$ based on the regularization proposed by Papanastasiou [27] in order to assess the sensitivity of the simulations on the choice of regularization. Papanastasiou's regularisation [27] takes the following form:

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}_y (1 - \exp(-m\dot{\gamma})) + \mu \underline{\underline{\dot{\gamma}}} \quad (6)$$

where m is the stress growth exponent which has the dimension of time (i.e. $m \gg 10L^2/\alpha$). The stress growth exponent has been chosen equal to $m = 10^5 L^2/\alpha$ in here to mimic true Bingham model in a satisfactory manner. Both Eqs. (5) and (6) transform the “unyielded” region to a zone of high viscosity so that the numerical solutions predict vanishingly small magnitudes of velocity in the unyielded regions, and under this condition heat transfer takes place principally due to conduction, and fluid flow does not influence the thermal transport. The maximum difference between the mean Nusselt numbers obtained from these two regularizations remains less than 2% where details provided in Table 1 in Ref. [24]. Additionally, several recent studies numerically investigated natural convection of Bingham fluids in square cavity with differentially heated sidewalls obeying Bingham model without using any regularisation [28, 29]. Comparisons have been made between the results obtained using the bi-viscosity regularisation [5] with the ones without regularisation [28, 29]. It has been reported by Refs. [28, 29] that the mean Nusselt number values obtained for low Ra (i.e. $Ra = 10^4$) remain in good agreement, but small differences have been reported for high values of Ra (i.e. $Ra = 10^5$) between the Bingham model without regularisation and bi-viscosity regularisation (i.e. 5% difference in the mean Nusselt number has been reported for $Bn = 3$, $Ra = 10^5$ at $Pr = 1$ between Refs. [28] and [5] in the Table 2 of Ref. [28]). However, this difference remains within typical computational uncertainty and may also arise because of the differences in the mesh and numerical schemes.

Using Buckingham’s π theorem, it is possible to express the Nusselt number for natural convection of Bingham fluids in square cross-sectional cylindrical annular enclosures as: $Nu = f_1(Ra, Pr, Bn, r_i/L)$, where r_i/L represents slenderness ratio of the square cross-sectioned cylindrical annular enclosures. The nominal Rayleigh, Prandtl and Bingham (i.e. ratio of yield

stress to viscous stress) numbers, which can be defined for CWT and CWHF boundary configurations in the following manner:

$$Ra_{CWT} = \frac{\rho g \beta (T_H - T_C) L^3}{\mu \alpha} = Gr_{CWT} Pr \text{ and } Bn_{CWT} = \frac{\tau_y L}{\mu \sqrt{g \beta (T_H - T_C) L}} \quad (7a)$$

$$Ra_{CWHF} = \frac{\rho g \beta q_i L^4}{k \mu \alpha} = Gr_{CWHF} Pr \text{ and } Bn_{CWHF} = \frac{\tau_y}{\mu \sqrt{g \beta q_i / k}} \quad (7b)$$

where $Gr_{CWT} = \rho^2 g \beta (T_H - T_C) L^3 / \mu^2$ ($Gr_{CWHF} = \rho^2 g \beta q_i L^4 / k \mu^2$) is the nominal Grashof number in the CWT (CWHF) configuration. The local heat transfer coefficient h_i on the internal radius can be defined as:

$$h_i = \left| -k \left(\frac{\partial T}{\partial r} \right)_{r=r_i} \times 1 / (T_{r=r_i} - T_{r=r_i+L}) \right| \quad (8)$$

The mean heat transfer coefficient $\bar{h}_i$ and the mean Nusselt number $\overline{Nu}_i$ based on internal radius are evaluated as:

$$\bar{h}_i = \frac{1}{L} \int_0^L h_i dz \text{ and } \overline{Nu}_i = \frac{\bar{h}_i L}{k} \quad (9)$$

According to steady state it is possible to obtain:

$$\frac{q_o}{q_i} = \frac{r_i}{r_o} \quad (10)$$

where q_i and q_o are the mean heat flux magnitudes at the internal and external radius respectively, which can be defined as:

$$q_i = \frac{1}{L} \int_0^L \left| k \frac{\partial T}{\partial r} \right|_{r=r_i} dz \text{ and } q_o = \frac{1}{L} \int_0^L \left| k \frac{\partial T}{\partial r} \right|_{r=r_o} dz \quad (11)$$

This give rise to:

$$\bar{h}_o = \bar{h}_i \frac{r_i}{r_o} \text{ and } \overline{Nu}_o = \overline{Nu}_i \frac{r_i}{r_o} \quad (12)$$

It is worth noting that the heat flux due to pure conduction on the internal surface can be expressed as:

$$(q_i)_{cond} = \frac{k |T_{r=r_i} - T_{r=r_o}|}{r_i \ln(1 + L/r_i)} \quad (13)$$

Using Eqs. (12) and (13), the Nusselt number on the internal surface due to pure conduction can be expressed as:

$$(\overline{Nu}_i)_{cond} = \frac{(q_i)_{cond}L}{k(T_H - T_C)} = \frac{(L/r_i)}{\ln(1+L/r_i)} \quad (14)$$

The current analysis has been carried out in non-dimensional form for the sake of generalisation. The spatial co-ordinates, velocity components, pressure and temperature can be non-dimensionalised in the following manner:

$$r^+ = r/L; z^+ = z/L; u^+ = u/U_{ref}; w^+ = w/U_{ref}; P^+ = P/\rho U_{ref}^2; \Theta = (T - T_{ref})/\Delta T_{ref} \quad (15)$$

where the reference velocity scale U_{ref} is taken to be equal to $\sqrt{g\beta\Delta T_{ref}L}$ based on the equilibrium of inertial and the buoyancy forces [5-13] where ΔT_{ref} is a reference temperature difference. For CWT configuration ΔT_{ref} can be taken to be $\Delta T = (T_H - T_C)$ while it can be taken to be $\Delta T_{ref} = q_i L/k$ for the CWHF configuration. Additionally, the reference temperature is taken to be temperature at the centre of the domain T_{cen} for CWHF boundary condition whereas the cold wall temperature T_C is taken to be the reference temperature for CWT boundary condition. Accordingly, the steady-state non-dimensional forms of governing equations for mass, momentum and energy for constant temperature-independent thermo-physical properties take the following form under the assumption of axisymmetry:

Non-dimensional mass conservation equation:

$$\frac{1}{r^+} \frac{\partial(r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0 \quad (16)$$

Non-dimensional momentum conservation equations

Radial direction:

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = -\frac{\partial P^+}{\partial r^+} + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rr}^+)}{\partial r^+} - \frac{\tau_{\phi\phi}^+}{r^+} + \frac{\partial(\tau_{rz}^+)}{\partial z^+} \right] \quad (17a)$$

Vertical direction:

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = -\frac{\partial P^+}{\partial z^+} + \Theta + \frac{Pr^{1/2}}{Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial(r^+ \tau_{rz}^+)}{\partial r^+} + \frac{\partial(\tau_{zz}^+)}{\partial z^+} \right] \quad (17b)$$

Non-dimensional energy conservation equation:

$$u^+ \frac{\partial \Theta}{\partial r^+} + w^+ \frac{\partial \Theta}{\partial z^+} = \frac{1}{Pr^{1/2} Ra^{1/2}} \left[\frac{1}{r^+} \frac{\partial}{\partial r^+} \left(r^+ \frac{\partial \Theta}{\partial r^+} \right) + \frac{\partial^2 \Theta}{\partial z^+ \partial z^+} \right] \quad (18)$$

In Eq. (17) τ_{ij}^+ is the non-dimensional stress tensor which is given by:

$$\tau_{ij}^+ = \tau_{ij} L / \mu \sqrt{g \beta \Delta T_{ref} L} \quad (19)$$

where r is the radial coordinate, z axis is taken to align with the vertical direction, and the axisymmetric flow is independent of the azimuthal direction ϕ . The components of viscous stress tensor (i.e. τ_{rr} , $\tau_{\phi\phi}$, τ_{rz} and τ_{zz}) are expressed according to Eq. (5).

A schematic diagram of the configuration is shown in Fig.1, which demonstrates both constant wall temperature (CWT) and constant wall heat flux (CWHF) configurations. Accordingly, the Eqs. (16-19) are solved in a coupled manner where the two horizontal walls are kept under adiabatic conditions (i.e. $\partial \Theta / \partial z^+ = 0$ at $z^+ = 0.0$ and $z^+ = 1.0$), and both velocity components (i.e. u^+ and w^+) are identically zero on each boundary because of the no-slip condition and impenetrability of rigid walls. For the CWHF configuration, the heat fluxes for vertical hot and cold walls are specified (i.e. $-\partial \Theta / \partial r^+ = 1$ and $-\partial \Theta / \partial r^+ = 1/(1 + L/r_i)$ at $r^+ = r_i/L$ and $r^+ = r_i/L + 1$ respectively). By contrast, the temperatures of vertical walls are specified (i.e. $\Theta = 1$ and $\Theta = 0$ at $r^+ = r_i/L$ and $r^+ = r_i/L + 1$ respectively) for the CWT configuration.

NUMERICAL IMPLEMENTATION

The non-dimensionalised equations of mass, momentum and energy have been numerically solved using the finite-volume methodology using a commercial package ANSYS-FLUENT. This commercial package was previously used successfully for simulating non-Newtonian fluid flows [5-15,30,31]. A second-order central difference scheme is used for the discretisation of the diffusive terms and a second-order up-wind scheme is used for the convective terms. The well-known SIMPLE [32] (Semi-Implicit Method for Pressure-Linked Equations)

algorithm is used for coupling of the pressure and velocity components. The convergence criteria were set to 10^{-6} for all the relative (scaled) residuals. The present numerical scheme has been validated for natural convection of both Newtonian and Bingham fluids in square enclosures with respect to the results reported by de Vahl Davis [33] and Vola *et al.* [34] respectively and excellent agreement has been obtained. Interested readers are referred to Refs. [5,24] for further information.

SCALING ANALYSIS

The velocity component in the vertical direction (i.e. w) can be scaled by equating the order of magnitudes of inertial and buoyancy terms as: $w \sim \sqrt{g\beta(T_H - T_C)L}$ ($w \sim \sqrt{g\beta q_i \delta_{th} L/k}$) for CWT (CWHF) boundary conditions where δ_{th} is the thermal boundary layer thickness on vertical walls. Based on the continuity equation for axisymmetric geometry one obtains:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left(\frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{L} \quad (20)$$

which leads to:

$$u \sim \frac{wr}{(r+L)} \sim \frac{w}{(1+L/r_i)} \sim \frac{\alpha}{L} \frac{\sqrt{RaPr}}{(1+L/r_i)} \quad \text{for CWT} \quad (21a)$$

$$u \sim \frac{wr}{(r+L)} \sim \frac{w}{(1+L/r_i)} \sim \frac{\alpha}{L} \frac{\sqrt{RaPr}}{(1+L/r_i)} \sqrt{\frac{\delta_{th}}{L}} \quad \text{for CWHF} \quad (21b)$$

Similarly, equating the order of magnitudes of inertial and viscous terms in the radial direction yields:

$$\rho \frac{u^2}{L} \sim \frac{1}{\delta_1} \left(\tau_y + \mu \frac{u}{\delta_1} \right) \quad (22a)$$

Using Eqs. (21a) and (21b) in Eq. (22a) leads to:

$$\delta_1 \sim L \left[\sqrt{\left(\frac{Bn}{2} \right)^2 \frac{Pr}{Ra} (1 + L/r_i)^4 + \sqrt{\frac{Pr(1+L/r_i)^2}{Ra}}} + \left(\frac{Bn}{2} \right) \sqrt{\frac{Pr}{Ra}} (1 + L/r_i)^2 \right] \quad \text{for CWT} \quad (22b)$$

$$\delta_1 \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \text{ for CWHF} \quad (22c)$$

where δ_1 is the hydrodynamic boundary layer thickness on the horizontal wall. Similarly, the hydro-dynamic boundary layer thickness on the vertical wall δ can be scaled in the following manner by equating the magnitudes of inertial and viscous terms in the vertical direction (i.e. $\rho w^2/L \sim (\tau_y + \mu w/\delta)/\delta$):

$$\delta \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \sqrt{\frac{Pr}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \text{ for CWT} \quad (23a)$$

$$\delta \sim L \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \text{ for CWHF} \quad (23b)$$

Equations (22) and (23) indicate that the boundary layer thickness increases with decreasing (increasing) nominal Rayleigh number Ra (nominal Bingham number Bn). Also, based on the scaling estimates in Eqs. (22) and (23), it is possible to estimate the effective viscosity in both vertical and horizontal boundary layers (i.e. μ_{eff}^V and μ_{eff}^H) in the following manner ($\mu_{eff}^V \sim \mu + \tau_y \delta/w$ and $\mu_{eff}^H \sim \mu + \tau_y \delta_1/u$):

For CWT configuration:

$$\frac{\mu_{eff}^V}{\mu} \sim 1 + Bn \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \sqrt{\frac{Pr}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \quad (24a)$$

$$\frac{\mu_{eff}^H}{\mu} \sim 1 + \frac{Bn}{1+L/r_i} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} (1+L/r_i)^4 + \sqrt{\frac{Pr(1+L/r_i)^2}{Ra}}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} (1+L/r_i)^2 \right] \quad (24b)$$

For CWHF configuration:

$$\frac{\mu_{eff}^V}{\mu} \sim 1 + Bn \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \quad (25a)$$

$$\frac{\mu_{eff}^H}{\mu} \sim 1 + \frac{Bn}{1+L/r_i} \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \quad (25b)$$

Using Eqs. (24) and (25), the effective Rayleigh numbers (i.e. Ra_{eff}^V and Ra_{eff}^H) in both vertical and horizontal boundary layers can be estimated in the following manner:

For CWT configuration (i.e. $Ra_{eff} = \rho g \beta (T_H - T_C) L^3 / \mu_{eff} \alpha$):

$$Ra_{eff}^V \sim Ra / \left\{ 1 + Bn \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} + \frac{Pr}{Ra}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra}} \right] \right\} \quad (26a)$$

$$Ra_{eff}^H \sim Ra / \left\{ 1 + \frac{Bn}{1+L/r_i} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} (1+L/r_i)^4 + \frac{Pr(1+L/r_i)^2}{Ra}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} (1+L/r_i)^2} \right] \right\} \quad (26b)$$

For CWHF configuration (i.e. $Ra_{eff} = \rho g \beta q L^4 / k \mu_{eff} \alpha$):

$$Ra_{eff}^V \sim Ra / \left\{ 1 + Bn \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{1}{(\delta_{th}/L)}} \right] \right\} \quad (27a)$$

$$Ra_{eff}^H \sim Ra / \left\{ 1 + \frac{Bn}{1+L/r_i} \sqrt{\frac{L}{\delta_{th}}} \left[\sqrt{\left(\frac{Bn}{2}\right)^2 \frac{Pr}{Ra} \frac{(1+L/r_i)^4}{(\delta_{th}/L)^2} + \frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} + \left(\frac{Bn}{2}\right) \sqrt{\frac{Pr}{Ra} \frac{(1+L/r_i)^2}{(\delta_{th}/L)}} \right] \right\} \quad (27b)$$

Equations (26) and (27) indicate that the effective Rayleigh number remains smaller than the nominal Rayleigh number and the effective Rayleigh number decreases with increasing Bn for both CWT and CWHF boundary conditions. Furthermore, the effective Rayleigh number for CWHF boundary condition is smaller than in the case of CWT boundary condition for same set of values of Ra , Pr and Bn because δ_{th} is expected to be smaller than L .

Equations (22) and (23) indicate that $\delta_1 > \delta$ and $u < w$, thus μ_{eff}^H is expected to be greater than μ_{eff}^V in cylindrical enclosures. This further implies that Ra_{eff}^H is expected to be smaller than Ra_{eff}^V in cylindrical enclosures as it is shown in Eqs. (26) and (27). However, they approach each other for $r_i/L \rightarrow \infty$ (i.e. in the limit of square enclosures). It is worth noting that the velocity magnitudes in both vertical and radial directions remain of the same order for a two-dimensional square enclosure, which leads to $\delta \sim \delta_1$ and $Ra_{eff}^V \sim Ra_{eff}^H$.

RESULTS & DISCUSSION

Effects of nominal Rayleigh number Ra

The variations of non-dimensional temperature $\theta = (T - T_{cen})/\Delta T_{ref}$, vertical ($W = wL/\alpha$) and radial ($U = uL/\alpha$) velocity components in the radial (axial) at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) directions are shown in Fig. 2 for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids for both CWT and CWHF boundary conditions in the range of $Ra = 10^3 - 10^6$ at $r_i/L = 1.0$ and $Pr = 10^2$. The distributions of θ for pure conduction solutions (where fluid flow does not affect thermal transport) have been shown in Fig. 2 by the triangle (circle) for CWT (CWHF) configurations. It is worth noting that the pure conduction solution is same for both CWT and CWHF boundary conditions. However, the distributions of θ for pure condition solutions for CWT and CWHF boundary conditions are different due to differences in the expressions for ΔT_{ref} (see Eq. (15)). It can be easily seen from Fig. 2 that the distribution of θ deviates significantly from the pure conduction solution with increasing Ra , which bears the signature of the strengthening of convection for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluids irrespective of the boundary condition. This can also be confirmed from the rising magnitudes of W and U with increasing Ra , as shown in Fig. 2 for both CWT and CWHF configurations. Figure 2 further shows that the magnitudes of θ , W and U in the CWHF configuration are smaller than in the CWT configuration. This is an indicative of stronger

convective thermal transport in CWT configuration than in the CWHF configuration. Such behaviour can be explained by scaling arguments in the following manner. The wall heat flux on the inner periphery wall can be scaled as $q_i \sim k\Delta T/\delta_{th}$ where ΔT and δ_{th} are the characteristic temperature difference and the thickness of thermal boundary layer on vertical walls, respectively. This suggests that the non-dimensional temperature θ in the case of CWHF boundary condition scales as $\theta \sim \Delta Tk/q_i L \sim q_i \delta_{th} k/q_i L k \sim O(\delta_{th}/L)$, whereas $\theta \sim O(1)$ for CWT boundary condition. This indicates that the magnitude of θ for CWHF boundary condition is expected to decrease with increasing Ra because of thinning of boundary layer thickness, which is consistent with the scaling estimation given by Eq. (23). This smaller temperature difference between vertical walls in the CWHF configuration than in the CWT configuration leads to weaker convection strength in the case of CWHF boundary condition. Furthermore, Eqs. (26) and (27) indicate that the effective Rayleigh number in the CWHF configuration remains smaller than the corresponding value in the CWT configuration for a given set of values of r_i/L , Ra , Pr and Bn . This leads to smaller magnitudes of W and U for CWHF boundary condition than in the case of CWT boundary condition. It is also worth noting that the magnitudes of W and U for Bingham fluid cases remain smaller than the values obtained for Newtonian ($Bn = 0$) fluid cases for same set of values of Ra , Pr and r_i/L . The viscous resistance strengthens with increasing Bn , which is reflected in the smaller magnitudes of W and U in Bingham fluid cases than in the corresponding Newtonian fluid cases.

The contours of non-dimensional temperature θ and non-dimensional stream function $\Psi = \psi/\alpha$ are shown in Fig. 3 for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for different values of Ra at $r_i/L = 1.0$ and $Pr = 10^2$ for CWT boundary condition. A flow structure consisting of a large convection cell can be seen from Fig. 3 which is qualitatively consistent with previous findings for natural convection of both rectangular [5,6,9,10] and

cylindrical [22-24] enclosures. Figure 3 clearly shows that the magnitude of Ψ increases and isotherms become increasingly curved with increasing Ra owing to enhanced convective transport for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases. The grey regions on the streamline plot in Fig. 3 shows the Apparently Unyielded Regions (AURs) (regions where $|\tau| \leq \tau_y$ [35]). The bi-viscosity regularization (eq. 5) transforms the “unyielded” region to a zone of high viscosity here in order to avoid singularities arising from purely conduction solution (i.e. where $\dot{\gamma} = 0$). As indicated by eq. (5), the value of μ_{yield} determines the critical shear rate (i.e. or $\dot{\gamma}_c = \tau_y / \mu_{yield}$) beyond which the linear stress-strain rate relation (i.e. $\underline{\underline{\tau}} = \tau_y (\underline{\underline{\dot{\gamma}}} / \dot{\gamma}) + \mu \underline{\underline{\dot{\gamma}}}$) according to the Bingham model is invoked. The value of μ_{yield} affects only the size and distributions of AURs (regions where $|\tau| \leq \tau_y$ [35])). However, qualitative and quantitative distributions of stream function and isotherms remain independent of the value of μ_{yield} / μ for $|\tau| > \tau_y$. Therefore, the shapes and sizes of AURs are dependent on the choice of μ_{yield} [5-15, 24] but qualitative and quantitative distributions of stream functions remain independent of the numerical magnitude of μ_{yield} (at least for $\mu_{yield} \geq 10^3$). Therefore, the precise shape and size of AURs do not influence $\overline{Nu}_i$.

Effects of Bingham number Bn

The variations of θ , W (U) in the radial (axial) direction at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) directions are shown in Fig. 4 for different values Bn for $r_i/L = 0.125$ and 16 at $Ra = 10^6$ and $Pr = 10^2$ for both CWT and CWHF boundary conditions. The distributions of θ for pure conduction solutions have been shown by the triangle (circle) for CWT (CWHF) configurations in Fig. 4. It can be seen from Fig. 4 that θ approaches pure conduction solution with increasing Bn regardless of the boundary condition due to weakening of buoyancy force relative to the viscous resistance. This can further be substantiated by decreasing magnitudes of W and U

with increasing Bn as shown in Fig. 4 for both CWT and CWHF configurations. The contours of non-dimensional temperature θ and non-dimensional stream function Ψ for different values of Bn are shown in Fig. 5 for $r_i/L = 0.125$ and 16 at $Ra = 10^6$ and $Pr = 10^2$ for CWT boundary condition. Figure 5 shows that the magnitude of Ψ decreases and isotherms tend to be parallel to the active walls (i.e. hot and cold) due to weakening of convective thermal transport with increasing Bn . Accordingly, the size of AURs increases with increasing Bn and heat transfer takes place purely due to conduction for large values of Bn since flow practically stops under such a condition. Similar behaviour has been reported by Yigit and Chakraborty [24] for CWHF configuration for the same set of parameters considered here.

Effects of nominal Prandtl number Pr

The variations of the mean Nusselt number based on the inner periphery $\overline{Nu}_i$ normalised by the corresponding Nusselt number for pure conductive transport (i.e. $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$) for different values of Ra , Pr and r_i/L are shown in Fig. 6 for Newtonian fluids ($Bn = 0$) in the case of both CWT and CWHF boundary conditions. It is worth noting that Nusselt number for pure conductive transport $(\overline{Nu}_i)_{cond}$ also depends on r_i/L as shown in Eq. (14) (i.e. $(\overline{Nu}_i)_{cond} = (L/r_i)/\ln(1 + L/r_i)$). It can be seen from Fig. 6 that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ does not change significantly with a change in Pr for Newtonian fluids ($Bn = 0$) irrespective of the choice of boundary condition. For Newtonian fluids, a change in Pr for $Pr \gg 1$ modifies the relative balance between the buoyancy and viscous forces because the thermal boundary layer remains much thinner than the hydro-dynamic boundary layer. Thus, thermal transport within thermal boundary layer does not get affected by a change in Pr for $Pr \gg 1$ in the case of Newtonian fluids (i.e. $Bn = 0$) [36].

The variations of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different values Pr and r_i/L at $Ra = 10^6$ are shown in Fig. 7 for CWT configuration. Figure 7 demonstrates that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Pr for large values of Bn , however an opposite trend has been observed for small values of Bn cases for CWT boundary condition. This behaviour is qualitatively consistent with the previous findings for rectangular [5,6] and cylindrical [24] enclosures with differentially heated walls. It is evident from Eqs. (26) and (27) that both effective Rayleigh and Grashof numbers (i.e. $Ra_{eff} = \rho g \beta \Delta T_{ref} L^3 / \mu_{eff} \alpha$; $Gr_{eff} = Ra_{eff} / Pr = \rho^2 g \beta \Delta T_{ref} L^3 / \mu_{eff}^2$) decrease with increasing Pr for large values of Bn since μ_{eff} increases sharply for large values of Bn . However, an increase in Pr leads to a decrease in the thermal boundary thickness for small values of Bn for a given value of Ra . This gives rise to a small increase in $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for small values of Bn because the thinning of thermal boundary layer overcomes the strengthening of viscous resistance under this condition. Figure 7 further shows that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Bn and attains a value of unity for either equal to or greater than a threshold value of Bingham number Bn which is termed as Bn_{max} in the current analysis. For the current paper, Bn_{max} is estimated based on the value of Bingham number where $\overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1.01$. It can also be noted from Fig. 7 that Bn_{max} decreases with increasing Pr for the range of Ra considered here for CWT boundary condition. The nominal Prandtl number dependence of the variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn in the case of CWHF boundary condition (shown in Ref. [24]) has been found to be qualitatively similar to the corresponding variation in the case of CWT boundary condition.

Effects of normalised internal radius r_i/L

The variations of θ and W (U) in the radial (axial) direction at $z/L = 0.5$ ($(r - r_i)/L = 0.5$) are shown in Fig. 8 for different values r_i/L for $Bn = 0$ and $Bn = 0.5$ at $Ra = 10^6$ and $Pr = 10^2$ for both CWT and CWHF boundary conditions. Figure 8 indicates that the temperature

difference between vertical walls increases with increasing r_i/L for both Newtonian ($Bn = 0$) and Bingham ($Bn = 0.5$) cases for the CWHF configuration. It was previously shown in Ref. [24] that the temperature difference between hot and cold vertical walls for CWHF boundary condition depends on r_i/L , and pure conductive transport the temperature difference between vertical walls ΔT_{cond} can be expressed according to Eq. (13) (i.e. $(q_i)_{cond} = k(\Delta T_{cond}/r_i)\ln(1 + L/r_i)$) in the following manner:

$$\Delta T_{cond} = \frac{q_i(r_i/L)}{k} \ln \left(1 + \frac{L}{r_i} \right) \quad (28)$$

Equation (28) indicates that ΔT_{cond} decreases with increasing r_i/L for a given enclosure height L . As $h_i = q_i/\Delta T > (h_i)_{cond} = q_i/\Delta T_{cond}$, the temperature difference between vertical walls for the CWHF configuration remains smaller for the convection-dominated regime remains smaller than that in the case of purely conductive thermal transport regime. Figure 8 further shows that U increases, whereas W decreases with increasing r_i/L for Newtonian (i.e. $Bn = 0$) case. The isotherms and streamline patterns for different values r_i/L for $Bn = 0$ and $Bn = 0.5$ at $Ra = 10^6$ and $Pr = 10^2$ in Fig. 9 for the CWT configuration. It can be seen from Fig. 9 that the magnitude of Ψ increases and the size of the AURs decreases with increasing r_i/L for both $Bn = 0$ and $Bn = 0.5$, which indicates that convective thermal transport strengthens with increasing r_i/L . This can also be verified from Eqs. (24) and (25), which indicate that the effective viscosity in the horizontal direction μ_{eff}^H is expected to decrease with increasing r_i/L , which leads to strengthening of convection with increasing r_i/L .

Behaviour of mean Nusselt Number $\overline{Nu}_i$

The wall heat flux for the internal radius q_i can be scaled as: $q_i = h_i \Delta T \sim k \Delta T / \delta_{th}$ where δ_{th} is the thermal boundary layer on the vertical wall. Accordingly, the scaling estimates of the mean Nusselt number can be expressed as $\overline{Nu}_i \sim h_i L / k \sim L / \delta_{th}$ or $\overline{Nu}_i \sim (L/\delta) f_2(Ra, Pr, Bn, r_i/L)$

where the function $f_2(Ra, Pr, Bn, r_i/L)$ represents the ratio of hydro-dynamic to thermal boundary layer thicknesses (i.e. δ/δ_{th}) on the vertical walls. It is not possible to obtain analytical relation for δ_{th} from Eq. (23) for CWHF boundary condition but it is possible to obtain a scaling estimate of $\overline{Nu}_i$ using Eq. (23) for CWT boundary condition in the following manner:

$$\overline{Nu}_i \sim \text{Max} \left[(\overline{Nu}_i)_{cond}, \frac{(Ra/Pr)^{1/2}}{\left[\frac{Bn}{2} + \frac{1}{2} \sqrt{Bn^2 + 4 \left(\frac{Ra}{Pr} \right)^{1/2}} \right]} f_2(Ra, Pr, Bn, r_i/L) \right] \quad (29)$$

The variations of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with r_i/L for different Ra at $Pr = 10^2$ are shown in Fig. 10 for Newtonian fluids ($Bn = 0$) for both CWT and CWHF configurations. It can be seen from Fig. 10 that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ increases with increasing r_i/L and approaches the value of mean Nusselt number for square enclosure (i.e. $\overline{Nu}_{sq}$) in the limit of $r_i/L \rightarrow \infty$. The quantity $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ provides the relative contributions of thermal transport due to convection to conduction because $(\overline{Nu}_i)_{cond}$ remains a function of r_i/L and $(\overline{Nu}_i)_{cond}$ asymptotically approaches unity in the limit of $r_i/L \rightarrow \infty$ (i.e. for square enclosures). The results shown in Fig. 10 indicate that convective transport strengthens in comparison to the conductive transport with increasing r_i/L . Figure 10 further shows that comparable values of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ are obtained for both CWT and CWHF configurations for small values of Ra but $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ assumes greater values in the CWT configuration than in the CWHF configuration for high values of Ra for a given set of values of r_i/L and Pr in the case of Newtonian fluids (i.e. $Bn = 0$). Putting $Bn = 0$ in Eq. (28) yields $\overline{Nu}_i \sim (Ra/Pr)^{1/4} f_2$ ($\overline{Nu}_i \sim (Ra/Pr)^{1/5} f_2^{4/5}$) for CWT (CWHF) boundary condition. The difference between the numerical values of $(Ra/Pr)^{1/4}$ and $(Ra/Pr)^{1/5}$ remains small for small values of Ra but the difference between $(Ra/Pr)^{1/4}$ and $(Ra/Pr)^{1/5}$ is significant for large values of Ra . These scaling relations suggest that $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ is expected to be comparable for CWT and CWHF boundary conditions for

small values of Ra , whereas $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ is likely to be greater for CWT boundary condition than in CWHF boundary condition for large values of Ra , which are consistent with numerical findings shown in Fig. 10.

Turan *et al.* [5,6] utilised Eq. (29) to propose a correlation for mean Nusselt number for $\overline{Nu}_{sq}$ for natural convection of Bingham fluids in square enclosures for both CWT [5] and CWHF [6] boundary conditions in the range of $0 \leq Bn \leq Bn_{max}$:

$$\frac{\overline{Nu}_{sq}-1}{(\overline{Nu}_{sq})_{Bn=0}-1} = \frac{2[1-(Bn^*/Bn_{max}^*)^b]^c}{Bn^*+\sqrt{Bn^{*2}+4}} \text{ when } (\overline{Nu}_{sq})_{Bn=0} > 1 \quad (30a)$$

$$\overline{Nu}_{sq} = 1 \text{ when } (\overline{Nu}_{sq})_{Bn=0} = 1 \quad (30b)$$

where $Bn_{max}^* = Bn_{max}(Ra/Pr)^{-1/4}$, b and c are the correlation parameters. Furthermore, Turan *et al.* [5,6] proposed a correlation for the mean Nusselt number for natural convection of Newtonian fluids in square enclosures for both CWT [5] and CWHF [6] boundary conditions for $10^3 \leq Ra \leq 10^6$ and $0.1 \leq Pr \leq 10^2$:

$$(\overline{Nu}_{sq})_{Bn=0} = \left[0.162Ra^{0.293} \left(\frac{Pr}{1+Pr} \right)^{0.091} \right] \text{ for CWT} \quad (31a)$$

$$(\overline{Nu}_{sq})_{Bn=0} = \left[0.209Ra^{0.249} \left(\frac{Pr}{1+Pr} \right)^{0.037} \right] \text{ for CWHF} \quad (31b)$$

The correlation proposed in Eq. (31) can be used for $r_i/L \gg 1$ as $\overline{Nu}_i$ is expected to approach $\overline{Nu}_{sq}$ and $(\overline{Nu}_i)_{cond}$ approaches unity (i.e. $\lim_{r_i/L \rightarrow \infty} [(L/r_i)/\ln(1 + L/r_i)] = 1.0$) in the limit of $r_i/L \rightarrow \infty$. In this study, the mean Nusselt number $(\overline{Nu}_i)_{Bn=0}$ for natural convection of Newtonian fluids in the cylindrical enclosures is correlated with using Eq. (31) in following manner:

$$(\overline{Nu}_i)_{Bn=0} = (\overline{Nu}_{sq})_{Bn=0} \frac{[m_0(L/r_i)]^d}{\ln[1+(m_0(L/r_i))^d]} \quad (32a)$$

For CWT configuration:

$$m_0 = [(1 - 0.164 \ln Ra) / (0.555 - 0.139 \ln Ra)]^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \text{ and } d = 0.95^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \quad (32b)$$

For CWHF configuration:

$$m_0 = \{0.914 / [1 + \exp((9.25 - \ln Ra) / 2.15)]\}^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \text{ and } d = 0.927^{(\overline{Nu}_{sq})_{Bn=0}^{-1}} \quad (32c)$$

Equation (32) is not only valid for $0.125 \leq r_i/L \leq 16$ but also valid for square enclosures because $\overline{Nu}_i$ will be equal to the $\overline{Nu}_{sq}$ and $(\overline{Nu}_i)_{cond}$ will be equal to unity (i.e. $\lim_{r_i/L \rightarrow \infty} [(L/r_i) / \ln(1 + L/r_i)] = 1.0$) in the limit of $r_i/L \rightarrow \infty$. The predictions of Eq. (32) are shown in Fig. 10, which indicates that this correlation satisfactorily predicts $\overline{Nu}_i$ (i.e. $R^2 > 0.99$) for Newtonian fluids in square cross-sectional cylindrical annular enclosures for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

The variations of $(Bn_{max})_i$ with r_i/L for different values of Ra at $Pr = 10^2$ are shown in Fig. 11 for both CWT and CWHF configurations. Figure 11 indicates that $(Bn_{max})_i$ increases with increasing r_i/L and approaches the corresponding value for square enclosure (i.e. $(Bn_{max})_{sq}$) in the limit of $r_i/L \rightarrow \infty$ regardless of the boundary condition. The findings based on Fig. 11 indicate that convective thermal transport strengthens with increasing r_i/L and $(Bn_{max})_i$ approaches that of the rectangular enclosures for large values of internal radius irrespective of the boundary condition.

The quantity $(Bn_{max})_i$ can be scaled for CWT boundary condition by considering $\overline{Nu}_i \sim L/\delta_{th} \sim Lf_2/\delta \sim (\overline{Nu}_i)_{cond}$. This along with Eq. (29) leads to: $(Bn_{max})_i \sim f_2(\overline{Nu}_i)_{cond}^{-1} \sqrt{Ra/Pr} - (\overline{Nu}_i)_{cond}/f_2$ for CWT boundary condition. This scaling relation indicates that $(Bn_{max})_i$ is expected to increase (decrease) with increasing Ra (Pr) for a given value of r_i/L , which is consistent with the observations made from Fig. 11. Although

δ and $(Bn_{max})_i$ cannot be analytically obtained from Eq. (23b), their behaviour for CWHF boundary condition is found to be qualitatively to that in the case of CWT boundary condition [24]. It is worth noting that Bn_{max} is the Bingham number for which $\overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1.01$ because it is not suitable to estimate the critical Bingham number at which the flow stops using a bi-viscosity regularisation. This is due to the fact that the flow always remains present for bi-viscosity regularisation but it no longer affects thermal advection for large values of Bn [10]. The above definition of Bn_{max} leads to: $1.01 = f(Ra, Pr, Bn_{max})$ for square enclosures, which in turn can be recast as: $(Bn_{max})_{sq} = g_1(Ra, Pr)$ where g_1 is a function. Turan *et al.* [5,6] also proposed an empirical correlation for $(Bn_{max})_{sq}$, which is relevant to natural convection of Newtonian fluids in square enclosures for both CWT [5] and CWHF [6] boundary conditions for $10^3 \leq Ra \leq 10^6$ and $0.1 \leq Pr \leq 10^2$:

$$(Bn_{max})_{sq} = 0.019Ra^{0.56}Pr^{-0.46} \quad (33)$$

It can be expected from $(Bn_{max})_i \sim f_2(\overline{Nu}_i)_{cond}^{-1} \sqrt{Ra/Pr} - (\overline{Nu}_i)_{cond}/f_2$ that $(Bn_{max})_i$ is likely to be proportional to $\sqrt{Ra/Pr}$. The empirical correlation in Eq. (33) yields $(Bn_{max})_{sq} \propto Ra^{0.56}Pr^{-0.46}$ which is not very far from $\sqrt{Ra/Pr}$. It is also worth noting that Bn_{max} is different from the critical Bingham number Bn_c at which the buoyancy force is just sufficient to overcome the yield stress. Under this situation the fluid is unyielded throughout the domain and $\overline{Nu}_i$ remains equal to $(\overline{Nu}_i)_{cond}$. For $Bn = Bn_c$, the equilibrium of buoyancy and yield stress effects (i.e. $\rho g \beta \Delta T \sim \tau_y/\delta \sim \tau_y/L f_2(Bn_c, Pr)$) gives rise to $Bn_c \sim \sqrt{Ra/Pr} f_2(Bn_c, Pr)$ [6]. This is consistent with the recent analytical results of Karimfazli *et al.* [29] and Vikhansky [4] for the onset of natural convection of Bingham fluids in rectangular enclosures with differentially heated vertical sidewalls. It is worth noting that $Bn > Bn_c$ does not ensure that $\overline{Nu}_i > (\overline{Nu}_i)_{cond}$ because the fluid flow may not be strong enough to significantly influence convective heat transfer. However, $\overline{Nu}_i$ will deterministically be $(\overline{Nu}_i)_{cond}$ for $Bn < Bn_c$. In

this regard it is worth noting that $\overline{Nu}_l = (\overline{Nu}_l)_{cond}$ does not indicate that whole flow field is unyielded (some weak flow, albeit insufficient to affect the thermal transport, may still be occurring). For $\overline{Nu}_l = (\overline{Nu}_l)_{cond}$, the thermal boundary layer thickness δ_{th} becomes of the same order as the enclosure width L (i.e. $\delta_{th} \sim L$). It has been shown in Fig.9 (d) by Turan *et al.* [10] that Bn_c remains greater than Bn_{max} for all values of Ra .

Here, $(Bn_{max})_i$ in the square cross-sectional cylindrical annular enclosures is parameterised using Eq. (33) as:

For CWT configuration:

$$(Bn_{max})_i = (Bn_{max})_{sq} - [a_o(L/r_i)^{b_o}] \quad (34a)$$

where $r_i/L \leq 1$;

$$a_o = \left[\frac{1+0.0028Pr}{656.8+20.4Pr} \right] Ra^{0.615} \text{ and } b_o = \left[\frac{8.82+0.35Pr}{21.8+Pr} \right] - 0.017 \ln Ra \quad (34b)$$

and where $r_i/L > 1$;

$$a_o = \left[\frac{1+0.003Pr}{565.2+18.15Pr} \right] Ra^{0.6} \text{ and } b_o = \left[\frac{1+0.006Pr}{1.397+0.038Pr} \right] - \left[\frac{1+0.006Pr}{19.85+0.547Pr} \right] \ln Ra \quad (34c)$$

For CWHF configuration:

$$(Bn_{max})_i = (Bn_{max})_{sq} / [1 + a_o(L/r_i)^{b_o}] \text{ where } r_i/L \leq 1 \quad (35a)$$

$$a_o = \left[\frac{50.8+0.97Pr}{77.6+Pr} \right] Ra^{0.048} ; b_o = \left[\frac{1.145+0.117Pr}{1+0.117Pr} \right] - 0.02 \ln Ra \quad (35b)$$

$$(Bn_{max})_i = (Bn_{max})_{sq} - [a_o(L/r_i)^{b_o}] \text{ where } r_i/L > 1 \quad (35c)$$

where;

$$a_o = (0.077 - 0.01 \ln Pr) + \left[\frac{1+0.0028Pr}{311.4+11Pr} \right] Ra^{0.585} \text{ and } b_o = \left[\frac{1+0.153Pr}{0.064+0.372Pr} \right] - \left[\frac{1-0.226Pr}{63.76-9.63Pr} \right] \ln Ra \quad (35d)$$

It can be seen Fig. 11 that Eq. (34) and Eq. (35) satisfactorily capture (i.e. $R^2 > 0.99$) the variation of $(Bn_{max})_i$ for different values of r_i/L for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

Finally, Eq. (30) is extended to correlate $\overline{Nu}_i$ for natural convection of Bingham fluids in square cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions for $0 \leq Bn \leq Bn_{max}$:

$$\frac{[\overline{Nu}_i/(\overline{Nu}_i)_{cond}]-1}{[(\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond}]-1} = \frac{2[1-(Bn^*/Bn_{max}^*)^{b_i}]^{c_i}}{Bn^* + \sqrt{Bn^{*2} + 4}} \text{ when } (\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond} > 1 \quad (36a)$$

$$\overline{Nu}_i/(\overline{Nu}_i)_{cond} = 1 \text{ when } (\overline{Nu}_i)_{Bn=0}/(\overline{Nu}_i)_{cond} = 1 \quad (36b)$$

where $Bn_{max}^* = Bn_{max}(Ra/Pr)^{-1/4}$, b and c are the correlation parameters. Turan *et al.* [5,6] proposed $b_{sq} = 1$ and $c_{sq} = 0.42Ra^{0.13}Pr^{0.12}$ for CWT boundary condition [5] and $b_{sq} = 0.643$ and $c_{sq} = 0.143Ra^{0.226}Pr^{0.062}$ for CWHF boundary condition [6] in the case of square enclosures. Here, b_i and c_i have been parameterised here as follows:

For CWT configuration:

$$b_i = b_{sq} \text{ and } c_i = c_{sq} - [k_o(L/r_i)^n] \quad (37a)$$

$$k_o = \frac{1.047Pr^{0.2}}{1+(Ra/16880)^{-0.53}} \text{ and } n = \frac{4.807+0.031Pr}{51+Pr} \quad (37b)$$

For CWHF configuration:

$$b_i = b_{sq} \left[\left(\frac{\ln Ra - 15.48}{0.512 \ln Ra - 9.76} \right) (L/r_i)^{0.01} \right] \text{ and } c_i = c_{sq} - [k_o(L/r_i)^n] \quad (38a)$$

$$k_o = \left[\frac{0.015}{1-0.05 \ln Pr} \right] Ra^{0.37} - \left[\frac{0.733}{1+0.267 \ln Pr} \right] \text{ and } n = \left[\frac{1-0.072 \ln Ra}{\ln Ra - 7.643} \right] + \left[\frac{1-0.06 \ln Ra}{0.22 \ln Ra - 1.98} \right] \exp(-n_o \ln Pr) \quad (38b)$$

$$n_o = \frac{1-0.064 \ln Ra}{0.148 \ln Ra - 0.994} \quad (38c)$$

The predictions of Eq. (38) are shown in Fig. 12, which shows that the correlation satisfactorily predicts ($R^2 > 0.99$ according to non-linear regression reported by Brown [37]) $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ for Bingham fluids in square cross-sectional cylindrical annular enclosures for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

CONCLUSIONS

Laminar natural convection of yield stress fluids obeying Bingham model in square cross-sectional cylindrical annular enclosures with differentially heated vertical walls has been investigated for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions. The laminar steady-state simulations have been carried out under the assumptions of axisymmetry for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$. It is found that $\overline{Nu}_i$ increases with increasing Ra due to the strengthening of buoyancy forces for both Newtonian and Bingham fluids regardless of the boundary condition. The mean Nusselt number $\overline{Nu}_i$ remains comparable for CWT and CWHF configurations for small values of Ra . By contrast, greater values of $\overline{Nu}_i$ have been found in the CWT configuration than in the CWHF configuration for high values of Ra . Moreover, $\overline{Nu}_i$ has been found to decrease with increasing values of Bn for both CWT and CWHF configurations. The mean Nusselt number $\overline{Nu}_i$ remains smaller for Bingham fluids than in the case of Newtonian fluids due to stronger viscous resistance in Bingham fluids. The mean Nusselt number normalised by its value for pure conductive transport $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ has been found to increase with increasing r_i/L and it approaches the value of the corresponding mean Nusselt number for a square enclosure $\overline{Nu}_{sq}$ in both CWT and CWHF boundary conditions. The value of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ decreases with increasing Bn and attains a value of unity for large values of Bn as the fluid flow practically stops in such a condition irrespective of the boundary condition. Moreover, $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ does not change significantly with increasing Pr for Newtonian fluids, but it decreases with increasing Pr for large values of Bingham number for both CWT and CWHF boundary conditions. A detailed scaling analysis has been utilised to explain the observed Ra, Pr, Bn and r_i/L dependences of $\overline{Nu}_i$. The scaling relations have been utilised to propose a new correlation for $\overline{Nu}_i$ for natural convection of both Newtonian and Bingham fluids for both CWT and

CWHF boundary conditions in square cross-sectional cylindrical annular enclosures for $10^3 \leq Ra \leq 10^6$, $10 \leq Pr \leq 10^3$ and $0.125 \leq r_i/L \leq 16$.

It is worth noting that the temperature dependence of thermo-physical properties (e.g. yield stress τ_y and plastic viscosity μ) have not been taken into consideration in the current analysis for the sake of simplicity and also for the purpose of generalisation (e.g. different fluids show different level of temperature dependences of thermo-physical properties). Experimental evidence [38] suggests that the yield stress remains approximately independent of temperature and the plastic viscosity is only a weakly decreasing function of temperature (similar to Newtonian fluids) for Carbopol (i.e. a yield stress fluid) in the temperature range 0 to 90°C. Thus, the results presented in this paper are expected to remain qualitatively valid but the numerical values of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ and $(Bn_{max})_i$ can be affected by the temperature dependence of thermo-physical properties. Thus, further investigation with temperature-dependent thermo-physical properties of viscoplastic fluids will be necessary for deeper understanding and more accurate quantitative predictions.

ACKNOWLEDGEMENTS

The authors are grateful to Mr. P. Wang from Newcastle University for his help in developing the computational database.

NOMENCLATURE

$AURs$	Apparently Unyielded Regions
a_o	Correlation parameters, dimensionless
b, b_o	Correlation parameters, dimensionless
Bn	Bingham number, dimensionless
Bn_c	Critical Bingham number, dimensionless
Bn_{max}	Threshold value of Bingham number above which $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ settles to unity, dimensionless
Bn_{max}^*	Scaled threshold value of Bingham number, dimensionless
c	Correlation parameter, dimensionless
CWT	Constant wall temperature
$CWHF$	Constant wall heat flux
d	Correlation parameter, dimensionless
f_1, f_2	Functions, dimensionless
g	Gravitational acceleration, m/s^2
g_1	Function, dimensionless
Gr	Grashof number, dimensionless
h	Heat transfer coefficient, W/m^2K
k	Thermal conductivity, W/mK
k_o	Correlation parameter, dimensionless
L	height of the enclosure and difference between inner and outer radius, m
m	Stress growth exponent, s
m_o	Correlation parameter, dimensionless
n, n_o	Correlation parameter, dimensionless
Nu	Nusselt number, dimensionless
P	Pressure, Pa
Pr	Prandtl number, dimensionless
q	Heat flux, W/m^2
r	Radial coordinate, m
r_i	Inner radius, m
r_o	Outer radius, m
R^2	Coefficient of determination, dimensionless
Ra	Rayleigh number, dimensionless
T	Temperature, K

T_C	Cold wall temperature, K
T_H	Hot wall temperature, K
u, w	Radial, vertical velocity component, m/s
U, W	Dimensionless radial ($U = u L / \alpha$) and vertical velocity ($W = w L / \alpha$), dimensionless
U_{ref}	Reference velocity scale, m/s
z	Coordinate in vertical direction, m

Greek Symbols

α	Thermal diffusivity, m ² /s
β	Coefficient of thermal expansion, 1/K
δ_{th}	Thermal boundary-layer thickness, m
δ	Hydro-dynamic boundary layer thickness for vertical walls, m
δ_l	Hydro-dynamic boundary layer thickness for horizontal walls, m
θ	Dimensionless temperature, dimensionless
μ	Dynamic viscosity, Ns/m ²
ρ	Density, kg/m ³
$\dot{\gamma}$	Second invariant of strain rate tensor, s ⁻¹
$\dot{\gamma}_c$	Critical strain rate, s ⁻¹
$\underline{\underline{\dot{\gamma}}}$	Strain rate tensor, s ⁻¹
τ	Second invariant of viscous stress tensor, Pa
τ_{ij}	Components of viscous stress tensor, Pa
τ_y	Yield stress, Pa
$\underline{\underline{\tau_y}}$	Yield stress tensor, Pa
$\underline{\underline{\tau}}$	Viscous stress tensor, Pa
ϕ	Azimuthal coordinate, dimensionless
ψ	Stream function, m ² /s
Ψ	Dimensionless stream function, dimensionless
ΔT	Temperature difference, K

Subscripts

cen	Geometrical centre of the domain
$cond$	Conduction
$CWHF$	Constant wall heat flux
CWT	Constant wall temperature
eff	Effective value
max	Maximum value

<i>ref</i>	Reference value
<i>sq</i>	Square
<i>i</i>	Inner periphery wall
<i>o</i>	Outer periphery wall

Superscripts

—	Mean value
+	Non-dimensionalised value
<i>V</i>	Representative value in the vertical boundary layer
<i>H</i>	Representative value in the horizontal boundary layer

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FIGURE CAPTIONS

Figure 1: Schematic diagram of the simulation domain: a) CWT, b) CWHF configurations.

Figure 2: Variations of non-dimensional temperature θ and non-dimensional axial (radial) $W = wL/\alpha$ ($U = uL/\alpha$) velocity components along the horizontal (vertical) mid-plane for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for different values of Ra for $r_i/L = 1$ at $Pr = 10^2$. The pure conduction solution is shown by the triangle (circle) for CWT (CWHF) configurations.

Figure 3: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.1$) fluid cases for $Ra =$ a) 10^3 , b) 10^4 , c) 10^5 , d) 10^6 at $r_i/L = 1$ and $Pr = 10^2$ for CWT boundary condition.

Figure 4: Variations of non-dimensional temperature θ and non-dimensional axial (radial) W (U) velocity components along the horizontal (vertical) mid-plane for different Bn values for $r_i/L = 0.125$ (left column) and $r_i/L = 16$ (right column) at $Ra = 10^6$ and $Pr = 10^2$. The pure conduction solution is shown by the triangle (circle) for CWT (CWHF) configuration.

Figure 5: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for different Bn values for $r_i/L = 0.125$ (left column) and $r_i/L = 16$ (right column) at $Ra = 10^6$ and $Pr = 10^2$ for CWT boundary condition.

Figure 6: Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Pr for different values of Ra and $r_i/L =$ a) 0.125, b) 1, c) 16 for Newtonian fluids ($Bn = 0$). The values for CWT (CWHF) boundary condition are shown by the solid (broken) line.

Figure 7: Variation of the $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different values of Pr in the case of $r_i/L =$ a) 0.125, b) 1, c) 16 at $Ra = 10^6$ for Bingham fluids for CWT configuration.

Figure 8: Variations of (a) θ and (b) W with $(r - r_i)/L$ at $z/L = 0.5$ for different values of r_i/L at $Ra = 10^6$ and $Pr = 10^2$. (c) Variations of non-dimensional radial velocity component U with z/L at $0.5(r_i + r_o)/L$ different values of r_i/L at $Ra = 10^6$ and $Pr = 10^2$ for

Newtonian ($Bn = 0$) and Bingham ($Bn = 0.5$) fluids for both CWT and CWHF boundary conditions. The pure conduction solution for θ is shown by the dashed lines for both CWT CWHF boundary conditions.

Figure 9: Contours of non-dimensional temperature θ and non-dimensional stream function Ψ with AURs (shown in grey) for Newtonian ($Bn = 0$) and Bingham ($Bn = 0.5$) fluid cases for $r_i/L =$ a) 0.125 , b) 1 , c) 4 , d) 16 at $Ra = 10^6$ and $Pr = 10^2$ for CWT boundary condition.

Figure 10 : Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with r_i/L for different values of Ra at $Pr = 10^2$ for Newtonian Fluids ($Bn = 0$) along with the prediction of Eq. (32). The mean Nusselt number $\overline{Nu}_{sq}$ is also shown with the dashed line (Eq. (31)).

Figure 11 : Variation of the $(Bn_{max})_i$ with r_i/L for different values of Pr and $Ra =$ (a) 10^4 , (b) 5×10^5 , (c) 10^6 along with the predictions of the correlation (Eqs. (34) and (35)) for both CWT and CWHF configurations. The corresponding $(Bn_{max})_{sq}$ values are shown by dashed lines.

Figure 12: Variation of $\overline{Nu}_i/(\overline{Nu}_i)_{cond}$ with Bn for different values of r_i/L and Ra at $Pr = 10^2$ for both CWT and CWHF configurations along with the prediction of Eq. (36).

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