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***In: Recent Advances in Financial Engineering - Proceedings of the TMU
Finance Workshop 2014. World Scientific, Singapore, 2016, pp.1-27.***

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Moment Properties of Probability Distributions Used in Stochastic Financial Models, Stoyanov J,
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DOI link to article:

https://doi.org/10.1142/9789814730778_0001

Date deposited:

04/08/2017

In: **Advances in Financial Engineering**. (Proc. Workshop, TMU, Tokyo, November 2014). Eds: Y. Muromachi, T. Shibata and M. Kijima.
World Scientific Publishers, Singapore, October 2015.

Moment Properties of Probability Distributions Used in Stochastic Financial Models

Jordan Stoyanov¹

School of Mathematics & Statistics, Newcastle University, U.K.

Email: stoyanovj@gmail.com

Our goal in this paper is to look at stochastic financial models and especially on those properties of the involved distributions which are expressed in terms of the moments. The questions discussed and the results presented reveal the role which the moments play in the analysis of distributions. Interesting conclusions can be derived in both cases when available are finitely many moments and when we know all moments.

Among the results included in the paper are sharp lower and/or upper bounds for option prices in terms of a finite number of moments. However the main attention is paid to the determinacy of distributions by their moments. While any light tailed distribution is uniquely determined by its moments, the uniqueness may fail for heavy tailed distributions. And, here is the point. Heavy tailed distributions are essentially involved in stock market modelling, and most of them are non-unique in terms of the moments. This is why the phenomenon *non-uniqueness of distributions* deserves a special attention.

We treat distributions on the positive half-line used to describe, e.g., stock prices and option prices, and distributions on the whole real line describing log-returns. For reader's convenience we give a brief and unified general picture of existing results about uniqueness and non-uniqueness of distribution in terms of their moments. Some of the results are classical and well-known. In several cases we provide here new arguments along with presenting new recent results on the moment determinacy of random variables and their non-linear transformations and also of stochastic processes which are solutions to SDEs.

Key words: probability distributions, moments, uniqueness, non-uniqueness, Cramér's condition, Carleman's condition, Krein's condition, Hardy's condition, Lin's condition, Stieltjes class, rate of growth of moments, option price bounds, Black-Scholes model, stochastic differential equations, stationarity

¹The final version of this paper was completed after I moved recently to Ljubljana, Slovenia. I would like to thank the Mathematics Department, University of Ljubljana for the hospitality and the provided status of Guest-Professor.

1. Introduction

The evolution of stock prices, derivative prices, log-returns, etc., is diverse, dynamic and random. This is why for modelling we use a variety of stochastic processes with discrete or continuous time, with trajectories which are continuous or have jumps, and with specific distributional properties. The distributions are essential for describing financial instruments. In many cases the questions of interest and hence the answers are given in terms of the moments.

Notice, the moments are regarded as, and they indeed are, the simplest characteristics of a distribution. This is why we pay much attention to those properties of a distribution which are expressed in terms of the moments.

It is known and easy to imagine that, e.g., we cannot identify a distribution by knowing only a finite number of its moments. Hence the question is: Can we determine a distribution by knowing all moments? For many distributions the answer is “yes” and they are called *M-determinate*. However for quite popular distributions used in stochastic financial models the answer is “no”. These are *M-indeterminate* distributions. The basics are summarized in Section 2.

It is well-known that the log-normal distribution is non-unique in terms of the moments. Why is this so, how to show the non-uniqueness and how this property affects conclusions involving the log-normal distribution? For reader’s convenience we provide in Section 3 the necessary details.

In Section 4 we present a brief, systematic and unified picture of the most useful and workable conditions in the area of moment determinacy of probability distributions. Some of the results are classical and well-known, others are quite recent. We apply these tools in Section 5 to some distributions frequently used in stochastic modelling. We answer completely the question about the moment determinacy of all power transformations (usually called Box-Cox transformations) of the exponential and the normal distributions. New arguments are given to prove previously known statements.

The goal of Section 6 is to present interesting and non-trivial results which are based on knowing only a finite number of moments. The upper and/or lower bounds for option prices are important not only in the context of stochastic finance but in general. These are universal sharp bounds for probability distributions.

Moment determinacy of the solutions of stochastic differential equations are discussed in Sections 7 and 8. We describe examples of SDEs showing different kind of moment determinacy of their solutions. We also present a few new results providing explicit constructions of stochastic processes with prescribed moments.

The list of References, papers and books, is extensive. Most items are published over the last years, some are new and available only online. Useful comments are given at the end of the paper.

Readers who wish to become better familiar with the topic ‘determinacy of probability distributions’ may use the present paper as a gentle introduction. We have included the most important ideas, old and very new results, techniques and illustrations. Useful comments are provided in many places in the text. Outlined are challenging conjectures and open questions.

2. Basic notions and notations

We assume that all random variables and stochastic processes considered in this paper are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For some purposes we may need this to be a ‘usual filtered probability space’.

We use $\mathbf{E}[X]$ to denote the expectation of the random variable X with respect to the probability measure $\mathbf{P}$. Also we use $\tilde{\mathbf{E}}_t$ for the time- t expectation with respect to a martingale probability measure.

We write $X \sim F$, if the distribution function of X is F . We use f for the density of X , and of F , $f = F'$, if it exists. If X is such that $\mathbf{E}[|X|^k] < \infty$ for all $k = 1, 2, \dots$, then $m_k = \mathbf{E}[X^k]$ is the *moment of order k* and $\{m_k, k = 1, 2, \dots\}$ the *moment sequence* of X , and of F .

Clearly, the moments are simple characteristics of a distribution and it is remarkable that important properties of F and X can be derived or expressed only in terms of the moments. Notice, the moment sequence $\{m_k, k = 1, 2, \dots\}$ is a discrete and countable object. This is just a sequence of numbers. If X is a positive random variable, i.e. its values are in $\mathbb{R}_+ = [0, \infty)$, the moments of X increase and tend to infinity. (Exceptions are random variables with values in $[0, 1]$, which is an easy case.) If X takes values in $\mathbb{R} = (-\infty, \infty)$, then the even order moments increase and tend to infinity.

It is non-trivial that based on such a discrete information, the moment sequence $\{m_k, k = 1, 2, \dots\}$, we can derive essential properties or even exactly identify the unknown distribution $F(x)$, $x \in \mathbb{R}$. Notice, F is obviously an infinite and uncountable object. (In Functional Analysis, this is equivalent to the non-trivial problem of finding exactly an unknown function f (density or arbitrary function) by knowing the values at zero of all derivatives of $\hat{f}$, the Fourier transform of f .)

If the distribution function F is the only one with the moment sequence $\{m_k\}$, we say that F is ***M-determinate***, or that F is uniquely determined by its moments. Otherwise, F is ***M-indeterminate***, or non-unique in terms of the moments. In the latter case there must be at least one distribution function, say G , such that G has the same moments as F , however $G \neq F$. There is a deep analytical and non-trivial result which can be extracted from works by C. Berg and collaborators; for references, see, e.g., [89].

Fundamental Theorem. *Suppose that F is M-indeterminate. Then there are infinitely many absolutely continuous distributions and infinitely many purely discrete distributions all having the same moments as F .*

Examples 3.2 and 7.3 are a partial but good illustration of this result.

It is well-known that there are conditions which are necessary and sufficient for uniqueness of a distribution by its moments. These are important theoretical results. However, from practical point of view they are uncheckable and hence not so useful. Thus a great attention was paid to find conditions which are only sufficient, or only necessary, but which are easy to check. In Section 3 we give a compact summary of what is available nowadays.

The moment problem has a specific name depending on the support of the

distribution, $\text{supp}(F)$, or the range of values of the random variable:

$$\begin{aligned}\text{supp}(F): \quad & [0, 1] \quad (\text{Hausdorff moment problem}); \\ & \mathbb{R}_+ = [0, \infty) \quad (\text{Stieltjes moment problem}); \\ & \mathbb{R} = (-\infty, \infty) \quad (\text{Hamburger moment problem}).\end{aligned}$$

Most important for the M-(in)determinacy is the asymptotic behavior of its tail(s). We will be also looking at another property of a distribution, the infinite divisibility. Most of the distributions used in stochastic financial models are heavy tailed and infinitely divisible. The interplay between these two properties is quite delicate. Some illustrations are given in the next sections.

What to say about distributions used in financial models? It looks we have a lot of ‘freedom’. And indeed a variety of distributions are involved. Since stock prices and derivatives prices are positive, we use for them random variables or stochastic processes with values in $\mathbb{R}_+$. Respectively, the log-returns are random variables with values in $\mathbb{R}$. The specific choice of a model depends on characteristics observed for the financial phenomenon we want to study. The models themselves are always interesting because their study brings to us new and challenging theoretical questions. However important is that some models, as intended, are ‘good’ for describing adequately aspects of the reality in the financial world.

3. First illustrations

We start with the normal distribution $\mathcal{N}$ and the log-normal distribution $\text{Log}\mathcal{N}$ both widely used in stochastic financial modelling. Consider two random variables,

$$Z \sim \mathcal{N} \quad \text{and} \quad S \sim \text{Log}\mathcal{N}.$$

We are interested in the M-determinacy of Z , S and their non-linear transformations. Look first at something classic.

Example 3.1. There are two coefficients, β_1 and β_2 , *skewness* and *kurtosis*, introduced for any random variable $X \sim F$ with finite the first four moments and defined as follows:

$$\beta_1 = \frac{\mu_3}{\sigma^3}, \quad \beta_2 = \frac{\mu_4}{\sigma^4}.$$

Here $\sigma^2 = \mathbf{E}[(\xi - m_1)^2]$ and $\mu_j = \mathbf{E}[(\xi - m_1)^j]$ for $j = 3, 4$; $m_1 = \mathbf{E}[\xi]$. The role of these two coefficients when ‘describing’ the shape of a distribution is well-known in statistical theory and practice.

For a random variable $Z \sim \mathcal{N}(0, 1)$, the first four moments are $0, 1, 0, 3$, hence Z has skewness $\beta_1 = 0$ and kurtosis $\beta_2 = 3$.

Suppose that from a large data set you have calculated the empirical moments of order up to four and then found that $\hat{\beta}_1 \approx 0$ and $\hat{\beta}_2 \approx 3$. Are you willing to accept that your data come from a normal distribution?

It is known that the answer is ‘no’. It is useful and instructive, however, to have a couple of examples. Indeed, consider a purely discrete random variable Y :

$$\mathbf{P}[Y = \pm\sqrt{3}] = \frac{1}{6}, \quad \mathbf{P}[Y = 0] = \frac{2}{3}.$$

It is easy to see that the first four moments of Y are $0, 1, 0, 3$, the same as the first four moments of Z . (Even more, Z and Y have coinciding all odd order moments, they are just all equal to zero.) Hence Y has skewness 0 and kurtosis 3.

There are purely discrete random variables with infinitely many values in $\mathbb{R}$, not just three as above, such that the skewness is 0 and the kurtosis is 3.

The next illustration involves absolutely continuous distributions. Consider the family $DG(a)$ of double-gamma distributions with parameter a , $a > 0$. A random variable $Y \sim DG(a)$ if its density is

$$g(x) = \frac{1}{2\Gamma(a)} |x|^{a-1} e^{-|x|}, \quad x \in \mathbb{R}.$$

For a particular value of a , $a = \frac{1}{2}(\sqrt{13} + 1)$, we find that $\beta_1 = 0$ and $\beta_2 = 3$.

Hence there are infinitely many distributions, continuous and discrete, all different from the normal, and all having the same skewness 0 and kurtosis 3.

After the above we may think that if knowing all moments we are in a better position allowing us to identify uniquely a distribution. In general, this is not the case as shown in the next example involving the log-normal distribution.

Example 3.2. The density f of the random variable $X \sim \text{Log } \mathcal{N}(0, 1)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp \left[-\frac{1}{2}(\ln x)^2 \right], \quad x > 0; \quad f(x) = 0, \quad x \leq 0. \quad (1)$$

It can be shown that all positive integer order moments of X are finite and

$$m_k = \mathbf{E}[X^k] = e^{k^2/2}, \quad k = 1, 2, \dots$$

Interestingly, the order k can also be any negative integer, and we have the relation $\mathbf{E}[X^k] = \mathbf{E}[X^{-k}]$. This follows from the fact that X^{-1} is a well-defined random variable with the same $\text{Log } \mathcal{N}$ distribution as X .

Let us define two infinite sets of random variables, call them *Stieltjes classes*:

$$\mathbf{S}_c = \{X_\varepsilon, \varepsilon \in [-1, 1]\}, \quad X_\varepsilon \sim f_\varepsilon, \quad f_\varepsilon(x) = f(x) [1 + \varepsilon \sin(2\pi \ln x)], \quad x \in \mathbb{R};$$

$$\mathbf{S}_d = \{Y_a, a > 0\}, \quad \mathbf{P}[Y_a = ae^n] = a^{-n} e^{-n^2/2} / A, \quad n = 0, \pm 1, \pm 2, \dots$$

Here A is the normalizing constant. In the notations $\mathbf{S}_c$ and $\mathbf{S}_d$, $\mathbf{S}$ stands for Stieltjes, c for continuous and d for discrete.

Amazing property: With $X_0 = X \sim \text{Log } \mathcal{N}(0, 1)$, for any $\varepsilon \in [-1, 1]$ and any $a > 0$, the following relations hold:

$$\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[Y_a^k] = \mathbf{E}[X^k] = e^{k^2/2}, \quad \text{for } k = 1, 2, \dots$$

Conclusion: The log-normal distribution is M-indeterminate! We see explicitly that there are so ‘many’ other distributions all with the same moments. Notice,

the family $\mathbf{S}_c$ consists of *absolutely continuous* variables, while $\mathbf{S}_d$ consists of *purely discrete* variables. Then, obviously, we can take mixtures.

The above explicit constructions are two different ways to show that $\text{Log}\mathcal{N}$ is M-indeterminate. There are at least four other methods.

Another immediate conclusion is that in the Black-Scholes model, the stock price S_t being log-normally distributed, is M-indeterminate for any $t \in (0, T]$.

Two other important properties are that $\text{Log}\mathcal{N}$ is unimodal and infinitely divisible. It is easy to see that in the Stieltjes class $\mathbf{S}_c$ above, for any $\varepsilon \neq 0$, f_ε has infinitely many modes.

Conjecture 3.1. It is known that for $\varepsilon = 1$, X_1 is not infinitely divisible. We conjecture that for any $\varepsilon \in (-1, 1)$ and $\varepsilon \neq 0$, f_ε is not infinitely divisible.

Open Question 3.1. Suppose that F is an absolutely continuous distribution function on $\mathbb{R}_+$ with all moments finite. Let the following two ‘properties’ of F hold: (i) $m_k(F) = \int_0^\infty x^k dF(x) = e^{k^2/2}$, $k = 1, 2, \dots$; (ii) F is unimodal. Prove that $F = \text{Log}\mathcal{N}$. Otherwise give a counterexample.

It is worth mentioning that of interest is also a class, denoted by $LSN(\lambda)$, and called *logarithmic skew-normal distributions*. Here λ can be any real number. We say that $X \sim LSN(\lambda)$ if its density is

$$f_\lambda(x) = \varphi(x) \Phi(\lambda x), \quad x \in \mathbb{R},$$

where φ and Φ are the standard normal density and distribution function. If $\lambda = 0$, we get the usual $\text{Log}\mathcal{N}$ distribution. The class $LSN(\lambda)$ is used in stochastic modelling, it is wider than $\text{Log}\mathcal{N}$. However, any random variable $X_\lambda \sim LSN(\lambda)$ is M-indeterminate. Available are also explicit Stieltjes classes; see [66].

4. Classical and recent conditions

For a reader’s convenience we give below in a compact form essential and workable conditions for either uniqueness or non-uniqueness of a distribution in terms of the moments. Historically these conditions appeared at a different time and in a different form. Sometime they were ‘ready’ to be used, sometime needed to be extracted from the original purely analytical forms and reformulated and given in the language of modern probability and statistics.

4.1. Cramér’s condition

We start with a classical result which appeared in the first decades of 20th century and described by H. Cramér in the middle of 30-ties.

Let X be a random variable with values in $\mathbb{R}$ such that its moment generating function exists, i.e., for some $t_0 > 0$,

$$M(t) = \mathbf{E}[e^{tX}] < \infty \quad \text{for all } t \in (-t_0, t_0) \quad (\text{Cramér’s condition}).$$

In such a case we say that the distribution of X has *light tails*, or that X has an exponential moment.

Statement 4.1. *Let Cramér's condition hold for the random variable X . Then all moments of positive integer order of X are finite and X is M-determinate.*

An easy consequence is that any random variable with values in a bounded interval is M-determinate. This is the Hausdorff case.

If there is no moment generating function, $M(t) = \mathbf{E}[e^{tX}] = \infty$ for all $t \neq 0$, we say that the distribution of X has *heavy tails*. Distributions such as Pareto and Student, frequently used in financial modelling, not only do not have moment generating functions, but they are 'very heavy', only a finite number of their moments are finite. If, however, a heavy tailed distribution has all moments finite, it is then reasonable to ask about its moment determinacy. There are two possibilities, either X is M-determinate, or it is M-indeterminate.

Note that Cramér's condition is the strongest sufficient condition for uniqueness. There are heavy tailed distributions, i.e., no Cramér's condition, which are M-determinate. Later we illustrate this.

4.2. Hardy's condition

The next condition, applicable to positive random variables, is based on two old papers by G.H. Hardy published in 1917/1918. In its precise modern form, as given below, it appeared explicitly in the literature only recently; see [94].

Statement 4.2. *Consider a random variable $X > 0$, $X \sim F$, where F is an arbitrary distribution function. Assume that $\sqrt{X}$ has a moment generating function (equivalently, Cramér's condition holds for $\sqrt{X}$), namely: for some $t_0 > 0$,*

$$\mathbf{E}[e^{t\sqrt{X}}] < \infty \quad \text{for all } t \in [0, t_0) \quad (\textbf{Hardy's condition}).$$

Then X has all moments finite, i.e. $m_k = \mathbf{E}[X^k] < \infty$, $k = 1, 2, \dots$, and moreover, F is the only distribution function with the moment sequence $\{m_k, k = 1, 2, \dots\}$.

This condition is sufficient for uniqueness. It is weaker than Cramér's condition. In fact, we have the following general statement: If X is a random variable with values in $\mathbb{R}_+$ or $\mathbb{R}$, and Cramér's condition holds for X , then X^2 is M-determinate. Notice, the condition is on X , and the conclusion is for its square X^2 .

4.3. Carleman's condition

Consider a random variable X with distribution function F and denote its moment sequence by $\{m_k, k \in \mathbb{N}\}$. If the range of values of X , i.e. $\text{supp}(F)$, is bounded, X is M-determinate by Cramér's condition. Thus, assume that $\text{supp}(F)$ is unbounded. In the Hamburger case $\text{supp}(F) = \mathbb{R}$ or is its subset, while in the Stieltjes case $\text{supp}(F) = \mathbb{R}_+$ or is its subset. We use the moments $\{m_k\}$ to define an infinite series, called usually *Carleman's series*:

$$C = \begin{cases} \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}} & (\text{Hamburger case}), \\ \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}} & (\text{Stieltjes case}). \end{cases}$$

Statement 4.3a. *The condition $C = \infty$, called **Carleman's condition**, implies that X , and also F , is M-determinate.*

This result, established by T.J. Carleman in 1926, became a very useful tool, ‘easy’ to apply and as soon as $C = \infty$, to conclude that a distribution is M-determinate. Carleman’s condition uses explicitly all moments. Since important is the divergence of the Carleman’s series, this can be derived by knowing only the asymptotic behavior of the moments m_k for large k . Sometime this small ‘trick’ helps to establish the determinacy property.

Note, however, that Carleman’s condition is only sufficient for the uniqueness. If a distribution is M-indeterminate, then necessarily $C < \infty$. There are M-determinate distributions with Carleman’s series $C < \infty$; see [93].

We need also a result which is converse to Statement 4.3. For details see [81].

Statement 4.3b. *Suppose that Carleman’s condition is not satisfied: $C < \infty$. Assume that F is absolutely continuous and there is $x_0 > 0$ such that its density $f = F'$ is positive for $x > x_0$ and the function $-\ln f(e^x)$ is convex for $y > y_0$, where $y_0 = \ln x_0$. This condition together with $C < \infty$ imply that X is M-indeterminate.*

4.4. Krein’s condition

Suppose now that X is an absolutely continuous random variable with distribution function G , its density is $g = G'$, and that the moments of all orders are finite. Another assumption is that X takes values in an unbounded domain, and that in this domain $g(x) > 0$. In the Hamburger case the domain is $\mathbb{R}$. In the Stieltjes case the domain is $\mathbb{R}_+$ if g is bounded on $\mathbb{R}_+$; however, if g is unbounded near zero, we take (x_0, ∞) , for some $x_0 > 0$, to avoid the ‘singularity’ of g . We define the following normalized logarithmic integral (called also *Krein’s integral*):

$$K[g] = \begin{cases} \int_{-\infty}^{\infty} \frac{-\ln g(y)}{1+y^2} dy & \text{(Hamburger case),} \\ \int_{x_0}^{\infty} \frac{-\ln g(y^2)}{1+y^2} dy & \text{(Stieltjes case).} \end{cases}$$

Statement 4.4a. *The condition $K[g] < \infty$, called **Krein’s condition**, implies that X is M-indeterminate.*

Krein’s condition is only sufficient for non-uniqueness of a distribution. If a distribution is M-determinate, and its density is positive, then necessarily the Krein’s integral is divergent. Statement 4.4a in the Hamburger case is one of the famous results obtained by M.G. Krein in 1944. Both cases, Hamburger and Stieltjes, were further studied and extended in works by N.I. Akhiezer, H. Pedersen, E. Slud, G.D. Lin and A.G. Pakes; details can be found in [65, 81, 89, 93].

We also need a result which is converse to Statement 4.4a.; see [65].

Statement 4.4b. *Suppose that the density g of the random variable X is as above and that Krein’s integral is divergent: $K[g] = \infty$. In the Hamburger case we require g to be symmetric. Let in both Hamburger and Stieltjes cases the following **Lin’s condition** be satisfied: for some $x_0 > 0$, g is differentiable for $x > x_0$ and the ratio $L(x) := -x g'(x)/g(x)$ is ultimately monotone increasing and tending to infinity as $x \rightarrow \infty$. The two conditions, $K[g] = \infty$ and Lin’s condition, imply that X is M-determinate.*

4.5. Explicit Stieltjes classes

The way we illustrated in Section 3 the non-uniqueness of the log-normal distribution gives a clear idea of the term *Stieltjes class*. Look at the sets $\mathbf{S}_c$ and $\mathbf{S}_d$. Each one is a parametrized infinite family of different random variables, hence different distributions, all having the same moments as one initial distribution. The idea of constructing such a family goes back to T.J. Stieltjes (1894), who was the first to describe explicit distributions supported by the positive half-line which are M-indeterminate.

We deal, as in [91], with an absolutely continuous random variable X with distribution function G and density g assuming that all moments are finite.

Suppose that we have found a function $h = (h(x), x \in \mathbb{R})$, a sign function, such that $|h(x)| \leq 1$ for all x and the product $g(x)h(x)$, $x \in \mathbb{R}$, has ‘vanishing moments’ in the sense that $\int x^k g(x)h(x) dx = 0$, $k = 0, 1, 2, \dots$. The function h is called a *perturbation* and the **Stieltjes class** based on the density g and the perturbation h is defined as follows:

$$\mathbf{S} = \mathbf{S}(g, h) = \{g_\varepsilon(x) : g_\varepsilon(x) = g(x)[1 + \varepsilon h(x)]; x \in \mathbb{R}, \varepsilon \in [-1, 1]\}.$$

Statement 4.5. *For any $\varepsilon \in [-1, 1]$, the function g_ε is a density, hence there is a random variable, say X_ε such that $X_\varepsilon \sim G_\varepsilon$, where $G'_\varepsilon = g_\varepsilon$. While the distributions in $\mathbf{S}$ are obviously different, all moments of X_ε are the same as those of $X = X_0$:*

$$\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[X^k] \text{ for all } k = 1, 2, \dots \text{ and any } \varepsilon \in [-1, 1].$$

To have written an explicit Stieltjes class is equivalent to the statement that all members in this class are M-indeterminate. However, to construct a Stieltjes class is a delicate analytical problem. If we are successful, this would confirm the claim that we deal with distributions which are M-indeterminate.

The distribution function G , also its density g we started with, play the role of a center in the class $\mathbf{S}$. Important is that all these distributions are moment equivalent. Clearly, we can use different perturbations h and the same density g . The most important is the ‘vanishing moments’ property for the product hg . This property is equivalent to the fact that given g , we are looking for a function h which in the Hilbert space $L_2[g]$ (g is the ‘weight function’) is orthogonal to the set of all polynomials. If there is no such h , the Stieltjes class is trivial, it consists of only one element, G itself, then G is M-determinate. Further details about Stieltjes classes can be found in [91, 82, 80, 79].

4.6. Rate of growth of the moments

Intuitively, the tails of a distribution are related to the growth of its moments. If the tails are heavier, the moments grow ‘faster’. We will see that there is a critical boundary of growing of the moments which separates the distributions into two groups, M-determinate and M-indeterminate. We treat separately positive random variables and variables with values in $\mathbb{R}$. We provide a little more details since the results are recent and available only online; see [67, 95, 68].

Stieltjes case. For a random variable X with values in $\mathbb{R}_+$ and moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \dots$, we consider the ratio of two consecutive moments:

$$\Delta_k = \frac{m_{k+1}}{m_k}.$$

We can easily check that Δ_k increases in k , $k \geq k_0$ for some fixed integer $k_0 \geq 1$. Suppose that there is a number $\gamma > 0$ such that

$$\Delta_k = \mathcal{O}((k+1)^\gamma) \quad \text{as } k \rightarrow \infty.$$

The number γ is called a **rate of growth of the moments** of X .

Statement 4.6a. *If the rate $\gamma \leq 2$, then X is M-determinate.*

The natural question to ask is about the role, or sharpness of the value $\gamma = 2$. The answer is given by the next result.

Statement 4.6b. *The value $\gamma = 2$ is the best possible constant for which X is M-determinate. There is a positive random variable, Y , with finite moments such that $\Delta_k = \mathcal{O}((k+1)^{2+\delta})$ for some $\delta > 0$ and Y is M-indeterminate.*

The question now is about the moment determinacy of X if the rate is $\gamma > 2$. We may suggest that such an X is M-indeterminate. We need however an additional condition and here is one possible answer.

Statement 4.6c. *Suppose Y is a positive random variable with finite moments and its rate of growth of the moments is $\gamma > 2$. Suppose in addition that Y is absolutely continuous with smooth density g such that Lin's condition is satisfied: for some $y_0 > 0$, the ratio $-yg'(y)/g(y)$, $y > y_0$ is monotone increasing and tending to infinity as $y \rightarrow \infty$. Then Y is M-indeterminate.*

Hamburger case. Similarly to the above, we can study random variables with values in $\mathbb{R}$ and finite moments. We work now with the even order moments. First we define the rate of growth γ as follows:

$$\Delta_{2(k+1)} = \frac{m_{2(k+1)}}{m_{2k}} = \mathcal{O}((k+1)^\gamma) \quad \text{as } k \rightarrow \infty.$$

We can formulate three statements which are analogous to those in the Stieltjes case. We do not give details here and leave this task to the reader with a reference to another recent paper; see [95].

Remark 4.1. The conditions and the results described above provide us with diverse tools allowing, in principle, to analyze the moment determinacy of any distribution. However there are cases when to find the answer is not so straightforward. It is always useful to work with specific distributions frequently used in statistical theory and practice.

Remark 4.2. Relevant questions and results can be found in many papers, see, e.g., [39, 40, 41, 52, 72, 77, 84, 92, 96].

5. Details about the exponential and the normal distributions

Let us use the knowledge from Section 4 and give all details about the moment determinacy of two random variables, $X \sim \text{Exp}(1)$ and $Z \sim \mathcal{N}(0, 1)$, and also of their power transformations.

5.1. Exponential distribution

The random variable $X \sim \text{Exp}(1)$ has density e^{-x} , $x > 0$ and moments $m_k(X) = k!$, $k = 1, 2, \dots$. Since X satisfies Cramér's condition, X is M-determinate. Another way is to check that Carleman's condition holds, hence again X is M-determinate. The third proof uses the Krein-Lin result, Statement 4.4b. The forth proof is to apply Statement 4.6a and see that X has a rate of growth of the moments $\gamma = 1$.

What about X^2 , X^3 , or any power, X^r , for real r ? In each of these cases we can write explicitly the density and use it in the analysis. The answer is well-known: X^2 is heavy tailed, but is M-determinate. This is true because, e.g., its moments $m_k(X^2) = (2k)!$, $k = 1, 2, \dots$ satisfy Carleman's condition. We can also refer to Hardy's condition: X satisfies Cramér's condition, hence X^2 is M-determinate. At least two more ways can be suggested to prove the claim.

In general, X^r is M-determinate for any real $r \in [0, 2]$ and M-indeterminate for all $r > 2$. In particular, X^3 is M-indeterminate, i.e. $r = 3$ is the smallest integer power such that X^r obeys this property. The moments are $m_k(X^3) = (3k)!$, $k = 1, 2, \dots$, they grow 'very fast'. Since the density f of X^3 is

$$f(x) = \frac{1}{3} x^{-2/3} e^{-x^{1/3}}, \quad x > 0,$$

Krein's condition (in the Stieltjes case) is satisfied, so indeed X^3 is M-indeterminate.

We can make one step more. Namely, to write an explicit *Stieltjes class* for the cube $Y = X^3$ of $X \sim \text{Exp}(1)$:

$$\mathbf{S}(f, h) = \{f_\varepsilon = f[1 + \varepsilon h], \varepsilon \in [-1, 1]\}.$$

Here f is the density of X^3 and the perturbation is $h(x) = \sin\left(\frac{\pi}{6} - \sqrt{3}x^{1/3}\right)$, $x > 0$.

Notice, in $\mathbf{S}(f, h)$, the function f_ε , for each $\varepsilon \in [-1, 1]$, is a density of a random variable, say Y_ε , and we have $\mathbf{E}[Y_\varepsilon^k] = m_k = (3k)!$, $k = 1, 2, \dots$. These are the same as the moments of X^3 .

Open Question 5.1. Find a discrete random variable with values in $\mathbb{R}_+$ such that its moments are $\{(3k)!, k = 1, 2, \dots\}$.

The Fundamental Theorem, see Section 2, states only the existence of such distributions. We do not know any, yet.

Conjecture 5.1. While it is known that $Y = X^3$ for $X \sim \text{Exp}$ is infinitely divisible, we conjecture here that for any $\varepsilon \neq 0$ in the above Stieltjes class, the random variable Y_ε is not infinitely divisible.

Note finally that the density of X^3 is unimodal, while the density f_ε of Y_ε has infinitely many modes for any $\varepsilon \neq 0$.

5.2. Normal distribution

What can we say about the M-determinacy of a random variable $Z \sim \mathcal{N}(0, 1)$ and then of its powers Z^2 , Z^3 , Z^4 and $|Z|^r$ for arbitrary real r ?

The results given in Section 4 allow to find the answer. Let us just list what is true: Z and Z^2 are M-determinate (by Cramér's condition, and/or Hardy's condition, or to use the existence of the moment generating function of $Z^2 = \chi_1^2$, or the rate of growth of the moments).

More interesting is the next observation. Since Z^2 has light tail, this means that Z^4 satisfies Hardy's condition, so Z^4 is M-determinate. Notice, to use twice Cramér's condition and twice Hardy's condition is the shortest and most beautiful way to show that Z^4 , where $Z \sim \mathcal{N}$, is M-determinate.

However, Z^3 turns to be M-indeterminate. Once again: the M-indeterminate Z^3 is 'between' the M-determinate Z^2 and Z^4 . Even more surprising is that the absolute value $|Z^3|$ is M-determinate. Each of these can be proved in several ways. We see the 'small' difference: the cube Z^3 is a Hamburger case, there are two heavy tails, while $|Z^3|$ is a Stieltjes case with only one heavy tail.

The next fact to mention here is that for any real $r > 4$, $|Z|^r$ is M-indeterminate. Just write the density of $|Z|^r$, this is a Stieltjes case, and apply Krein's condition. Moreover, available are explicit Stieltjes classes for Z^3 and $|Z|^r$, $r > 4$.

Remark 5.2. We can write explicit Stieltjes classes for Z^3 and state an open question and a conjecture as done for the exponential distribution, see the previous sub-section. Similarly for $|Z|^r$, $r > 4$.

6. Relations between stock prices and option prices

The M-indeterminacy of the log-normal distribution is a fact implying that we have to be careful when studying models of Black-Scholes type and using the moments. Besides the geometric Brownian motion, several other heavy tailed distributions, including stable distributions, are essentially used to model different market ingredients. And, not surprisingly, most of them are M-indeterminate.

Any financial model is based on objects with specific distributions. Some of the questions we ask are in terms of the moments. A standard assumption is that there is no arbitrage opportunity in the market. So all expectations of random quantities will be taken with respect to a martingale probability measure.

Let us write a short list of possible and reasonable questions arising when studying any stochastic financial model. Typically, there is one or more underlying assets in such model which is described by a stochastic process S_t , $t \in [0, T]$.

- Direct question: In the pricing of an European contingent claim, there is a payoff f that is payable at the claim's maturity T . In this context, what are the best possible bounds for the price of the claim based on knowing the first n moments of the stock price at maturity?

- Converse question: We do not know the stock price and available to us are observable option prices. Can we derive conclusions about the stock price dynamics?

- Given observable option prices, can we find bounds for the prices of other derivatives on the same stock?
- How to answer the previous questions if there are transaction costs?
- Can we price options by knowing all moments of the stock price at maturity?

The specific results given below are variations of results due to several authors. The results are spread in the literature; see, e.g., [5, 6, 15, 21, 22, 44, 46, 50, 61, 62, 63, 71] and more references therein.

6.1. Upper bound for call option price based on two moments

Suppose that the stock price process S_t , $0 \leq t \leq T$ is quite ‘general’ and such that we do not know its distributions including the distribution at maturity T . We deal with an European call option with strike K . We assume that in the market the bank interest is $r = 0$ and there is no arbitrage, so if Q is a martingale probability measure, we write just $\mathbf{E}$ instead of $\mathbf{E}_Q$. The only information available to the traders is that at maturity T the random variable S_T has finite second moment and explicitly known are the first two moments

$$m_1 = \mathbf{E}[S_T] \quad \text{and} \quad m_2 = \mathbf{E}[S_T^2].$$

Since the fair price of the European call is $C(K) := \mathbf{E}[(S_T - K)^+]$, the question of interest is: *What is the maximal value of $C(K)$?*

Equivalently, what is the maximum value of the expectation $\mathbf{E}[(S_T - K)^+]$ taken over all random variables S_T with fixed the first two moments m_1 and m_2 . Clearly, such an upper bound will depend on K, m_1, m_2 .

Statement 6.1. *Under the above conditions we have the following:*

$$\mathbf{E}[(S_T - K)^+] \leq \begin{cases} \frac{1}{2} (m_1 - K + \sqrt{m_2 - 2m_1K + K^2}), & \text{if } K \geq m_2/(2m_1), \\ m_1 - Km_1^2/m_2, & \text{otherwise.} \end{cases}$$

The remarkable feature is that this bound is valid for all square integrable stock price processes as soon as fixed are the first two moments of S_T .

6.2. Other sharp bounds based on higher order moments

Assume now that the interest rate is $r > 0$ and let the time- t expected payoff of the European call be denoted by $\tilde{\mathbf{E}}_t[(S_T - K)^+]$; the time- t neutral option price will be equal to $e^{-r(T-t)}\mathbf{E}[(S_T - K)^+]$.

Notice first that we can write an upper bound for $\tilde{\mathbf{E}}_t[(S_T - K)^+]$, similar to the above one in Statement 6.1. We only need instead of m_1 and m_2 to use

$$m_1(t) = \tilde{\mathbf{E}}_t[S_T] \quad \text{and} \quad m_2(t) = \tilde{\mathbf{E}}_t[S_T^2].$$

Statement 6.2a. *Let for some $p \geq 1$ the p th order moment of S_T be finite. Then*

$$\tilde{\mathbf{E}}_t[(S_T - K)^+] \leq \begin{cases} (\mathbf{E}[S_T^p])^{1/p} - K, & \text{if } K \leq \frac{p-1}{p} (\mathbf{E}[S_T^p])^{1/p}, \\ (\mathbf{E}[S_T^p])^{1/p} \frac{1}{p} \left(\frac{p-1}{pK}\right)^{p-1}, & \text{otherwise.} \end{cases}$$

In the next result we provide a two-sided bound for the time- t option price. We use the quantity $\sigma(t)$, where $\sigma^2(t) = m_2(t) - m_1^2(t)$, the time- t variance of S_T .

Statement 6.2b. *Suppose we know the time- t tail probability $c_t := \mathbf{P}_t[S_T > K]$. Then the following two-sided relation holds:*

$$(m_1(t) - K) c_t \leq \tilde{\mathbf{E}}_t[(S_T - K)^+] \leq (m_1(t) - K) c_t + \sigma_t \sqrt{c_t - c_t^2}.$$

Remark 6.2. It is important to mention that the bounds in Statements 6.2a and 6.2b are sharp. There are specific distributions for which equality is attained in each case. Hence these statements and other once available in the literature provide *universal bounds* for the expectation of truncated random variables. See the references mentioned before.

Clearly, instead of upper bounds we can look from the very beginning for the lower bound $\min \mathbf{E}[(S_T - K)^+]$ for the European call option. The minimum is taken over all random variables S_T with fixed the first two moments (or, which is the same, with fixed the first moment and the variance). Similarly, we can find upper bound or lower bound of European put option, written on the same stock and having the same strike and maturity. Available are bounds for European type of options at maturity T or at a current time $t < T$.

In most cases, the payoff of options of European type is a *piece-wise linear function*. This is the property allowing to use, e.g., linear programming techniques.

Available in the literature are diverse bounds when known about S_T are, e.g., the first three moments, m_1, m_2, m_3 , or the first n moments, $m_1, \dots, m_n$. More delicate are the non-linear problems arising in portfolio selection. The goal is for a non-linear function $u(\cdot)$ to optimize $\mathbf{E}_Q[u(X)]$ over all random variables with fixed two, three or larger number of moments. Useful references are [6, 33, 61, 62, 63].

6.3. Bounds for call and put option prices

Suppose that the market is as before: the distribution of the stock price S_T at maturity T is unknown, traded in the market are an European call option and European put option with the same strike K and maturity T . Either the interest rate is $r = 0$ or $r > 0$. Both cases are of interest and they are treated in the literature. The first and main step is to analyze models with interest rate $r = 0$. A standard assumption is that there is no arbitrage in the market, so all calculations are performed with respect to a martingale probability measure, the expectations are denoted by $\tilde{\mathbf{E}}$. The call and put option prices are

$$C(K) = \tilde{\mathbf{E}}[(S_T - K)^+] \quad \text{and} \quad P(K) = \tilde{\mathbf{E}}[(K - S_T)^+].$$

Statement 6.3. *Suppose that all positive and all negative order moments of the stock price S_T at maturity T are finite, i.e., $\mathbf{E}[S_T^p] < \infty$ and $\mathbf{E}[S_T^{-q}] < \infty$ for any real $p > 0$ and $q > 0$.*

(a) *For any fixed p and all strikes $K > 0$, we have the call price upper bound:*

$$C(K) \leq \frac{B_0 \mathbf{E}[S_T^{p+1}]}{p+1} \left(\frac{p}{p+1} \right)^p \frac{1}{K^p}.$$

(b) For any fixed q and all strikes K , the put price upper bound is

$$P(K) \leq \frac{B_0 \mathbf{E}[S_T^{-q}]}{q+1} \left(\frac{q}{q+1} \right)^q K^{q+1}.$$

Remark 6.3. We can use the put-call parity relation and in case (a) easily derive lower bound for $P(K)$ in terms of positive $(p+1)$ th order moment of S_T , while in case (b) write a lower bound for $C(K)$ via q th order negative moment of S_T .

In most of the models considered in the literature, but not always, the stock price process at any time has all positive and all negative order moments finite. This is true for any S_T satisfying Cramér's condition, i.e., for light tailed distributions. However this is true also for several heavy tailed distributions. Such an example is the geometric Brownian motion, in which case we have explicit expressions for the moments, positive and negative. This allows to find two-sided bounds for the call option price and for the put option price. We leave this to the reader.

The last comment here is about the role of the strike K in the above bounds. It turns out some bounds 'work' well for small K , others are good for large K .

The upper bound for $C(K)$, see (a), is 'useful' for large K . If $\mathbf{E}[S_T^{p+1}] < \infty$ for some $p > 0$, then $C(K) = \mathcal{O}(K^{-p})$ as $K \rightarrow \infty$.

The upper bound for $P(K)$, see (b), is 'useful' for small K . If $\mathbf{E}[S_T^{-q}] < \infty$ for some $q > 0$, then $P(K) = \mathcal{O}(K^{q+1})$ as $K \rightarrow 0$.

Take $p \downarrow 0$ in (a) to arrive at the familiar bound $C(K) \leq B_0 \mathbf{E}[S_T]$.

Take $q \downarrow 0$ in (b), find another familiar bound $P(K) \leq B_0 K$.

7. Moment determinacy of the solutions of SDEs

In this section we discuss the moment determinacy property of a stochastic process $X = (X_t, t \in [0, T])$ which is the solution of the following Itô's type SDE:

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0, \quad t \in [0, T], \quad \text{or } t \geq 0.$$

As usual $W = (W_t, t \geq 0)$ is a standard Brownian motion, the initial condition X_0 is either a constant or a random variable independent of W . Under some 'general conditions' on the drift $a(\cdot)$, the diffusion $\sigma^2(\cdot)$ and the initial position X_0 , this SDE has a unique solution such that X_t , at any time t , has all moments finite. Since we are basically interested in the moments, it is enough for us to consider uniqueness in a weak sense of the solutions of SDEs.

The initial value X_0 does play a role for the finiteness of the moments of X_t for $t > 0$. It is easy to take $X_0 = \text{const.}$ If, however, X_0 is a random variable, we assume that all its moments are finite and that X_0 is M-determinate. If X_0 is M-indeterminate, this property will be inherited by the solution X_t , $t > 0$, which case does not seem interesting. Thus in what follows we work with the drift and the diffusion coefficients and if necessary, we specify X_0 .

Question 7.1. When are the one-dimensional and the finite-dimensional distributions of the process X uniquely determined by their marginal or multi-indexed moments?

It may happen that a SDE has a unique weak solution which, however, is non-unique in terms of the moments. This should not be surprising since these are quite different properties.

According to a recent result in [56], if all marginal one-dimensional distributions of the process X are M-determinate, then any finite-dimensional distribution of X is M-determinate. Hence enough is to give details just for the random variable X_t , where $t > 0$ is fixed. We assume generally that X_0 , $a(\cdot)$, and $\sigma^2(\cdot)$ are ‘nice’ in the sense that there is a unique weak solution of the corresponding SDE.

Let us exhibit four examples covering a range of SDEs with different moment determinacy and different integrability property.

Example 7.1. If $a(\cdot)$ and $\sigma^2(\cdot)$ are ‘nice’ and $\sigma^2(\cdot)$ is uniformly bounded, then X_t is M-determinate, X_t^2 is M-determinate, however X_t^3 is M-indeterminate. We note first that the moment properties of the solution X_t in this case are the ‘same’ as those of the standard Brownian motion W_t . Then we easily derive the above determinacy from the properties of the powers of a normally distributed random variable, see Section 5. The same moment determinacy properties for X_t , X_t^2 and X_t^3 are valid for any Gaussian process. In particular, such statements hold for Ornstein-Uhlenbeck type processes (particular linear SDEs)

$$dX_t = (\alpha_0 + \alpha_1 X_t)dt + \sigma dW_t,$$

where $\alpha_0, \alpha_1, \sigma$ are constants.

Another case, which is more general, is instead to require $\sigma^2(\cdot)$ to be uniformly bounded, to assume that $a(\cdot)$ and $\sigma^2(\cdot)$ are such that X_t satisfies Cramér’s condition, i.e., X_t to have an exponential moment. Then X_t is M-determinate. Moreover, if $a(\cdot)$ and $\sigma^2(\cdot)$ are such that the random variable $X_T^* = \sup_{0 \leq t \leq T} X_t$ is exponentially integrable, then X_t is M-determinate for any $t \in [0, T]$.

The exponential integrability was intensively studied for several classes of SDEs, martingales, Lévy processes; see, e.g., [26, 43, 51, 69]. One of the consequences in any such a case is that the distributions involved are M-determinate. Conclusions can also be derived for the absolute value $|X_t|$ and its powers $|X_t|^r$. We have to see which of the tools in Section 4 is best to use.

Example 7.2. Consider the familiar linear SDE (*Black-Scholes Model, GBM*):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 = s_0 = \text{const} > 0, \quad t \in [0, T], \quad \text{or } t \geq 0.$$

There is a unique weak solution S_t , $t \in [0, T]$, where

$$S_t \sim \text{Log}\mathcal{N}(a_t, b_t^2), \quad a_t = (\mu - \frac{1}{2}\sigma^2)t, \quad b_t^2 = \frac{1}{2}\sigma^2 t.$$

Hence all moments of S_t , $t > 0$, are finite, they can be written explicitly and the conclusion is that S_t for $t > 0$ is M-indeterminate as we know from Section 3.

Here is another interesting fact. While the moment non-uniqueness implies that S_t is not exponentially integrable, we can claim that even a ‘smaller’ random variable, $\sqrt{S_t}$, is also not exponentially integrable. This follows by Hardy’s condition, discussed in Section 4. Alternative proofs can also be given.

Example 7.3. Let us present a simple example of a *non-linear* SDE with unique non-trivial weak solution such that X_t is M-indeterminate for each $t > 0$. Take this one:

$$dX_t = 3 X_t^{1/3} dt + 3 X_t^{2/3} dW_t, \quad X_0 = 0, \quad t \geq 0.$$

(We ignore the trivial solution $X_t = 0$ for all $t \geq 0$.) How to show the moment non-uniqueness of X_t ? First, by using Itô's formula we find that the unique weak solution is $X_t = W_t^3$. Second, since $W_t \sim \mathcal{N}$, we refer to Section 5 and the fact that the cube of a normally distributed random variable is M-indeterminate.

Example 7.4. Suppose that X_0 , $a(\cdot)$ and $\sigma^2(\cdot)$ are such that X_t does not satisfy Cramér's condition, i.e., X_t is heavy tailed (no exponential moment). We still assume that all moments of X_t are finite. There are two possibilities. The first one, X_t to be M-indeterminate, was illustrated in Examples 7.2 and 7.3. The second possibility is to choose the drift and the diffusion such that the unique (non-trivial) weak solution to the corresponding SDE to be M-determinate, even having heavy tail(s). To see that this is possible, consider the following SDE:

$$dX_t = dt + 2 X_t^{1/2} dW_t, \quad X_0 = 0, \quad t \geq 0.$$

This SDE has a unique non-trivial weak solution with the property that for any $t > 0$, X_t has all moments finite, and it is M-determinate. The reader can easily see that this follows from the fact that $X_t = W_t^2$ and referring to Section 5.

Remark 7.1. Different ideas can be exploited to characterize the moment determinacy of the solutions of general SDEs. For example, an answer can be found if we know properties of the density $p(s, x; t, y)$, the solution of the forward Kolmogorov PDE. Clearly, the drift and the diffusion are essentially involved. We fix the first three arguments s, x, t and analyze $p(\cdot)$ as a function of y , and especially the asymptotic behavior of $p(\cdot)$ as $y \rightarrow \infty$. Then we can use Krein's condition or the converse to Krein's condition together with Lin's condition and make a conclusion about the moment determinacy of X_t . It is interesting and relevant to study also the moment determinacy of X_t as $t \rightarrow \infty$.

In the literature available are upper bounds for the moments $\mathbf{E}[X_t^{2k}]$, $\mathbf{E}[X_t^{2k+1}]$ or $\mathbf{E}[|X_t|^k]$. However they are not so useful since they lead to converging Carleman's series. Hence, no conclusion can be drawn about the moment determinacy.

If we look again at the examples above we see that the diffusion coefficient $\sigma^2(\cdot)$ is most important for the moment determinacy of the solution of a SDE. If we write formally $|\sigma(x)| \approx c|x|^\gamma$, we see that for some γ the solution X_t is M-determinate, for others it is M-indeterminate.

It is desirable to have answers to questions as above for stochastic process which are solutions to SDEs of the following type:

$$dX_t = (\alpha_0 + \alpha_1 X_t) dt + (\beta_0 + \beta_1 X_t) dW_t, \quad X_0;$$

$$dX_t = (\alpha_0 + \alpha_1 X_t) dt + \sigma X_t^\gamma dW_t, \quad X_0;$$

$$dr_t = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3/r_t) dt + \sqrt{\beta_0 + \beta_1 r_t + \beta_2 r_t^\gamma} dW_t, \quad r_0.$$

Such results would be of independent interest, however they can be involved when using SDEs to model stock price dynamics, interest rates, etc.

These problems and related topics are currently under study (a joint project with M. Zhitlukhin).

8. More challenging questions and some new results

Let us ask a few questions in a very general form.

Question 8.1. Do you believe that there is only one SDE such that at any time $t \geq 0$ its solution $X_t, t \geq 0$ has the moments $\mathbf{E}[X_t^k] = k!, k = 1, 2, \dots$?

Question 8.2. Do you believe that there is only one SDE such that at any time $t \geq 0$ its solution has the moments $\mathbf{E}[X_t^k] = (2k)!, k = 1, 2, \dots$?

Question 8.3. Do you believe that there are infinitely many SDEs such that all solutions at any time $t \geq 0$ have the same moments $\{(3k)!, k = 1, 2, \dots\}$?

After seeing the numbers $k!, (2k)!, (3k)!$, the reader may identify them as the moments of the random variables ξ, ξ^2, ξ^3 , where $\xi \sim \text{Exp}(1)$. Since we want these to be the moments at any time $t \geq 0$, this suggests that the processes must be stationary. However, the stationarity of a process requires the marginal distributions to be time-shift invariant and the process to have a specified correlation structure. Hence, in each of the above three cases we have to provide the complete description of the process we want to construct. Below c is a constant, $c > 0$.

Proposition 8.1. *There is only one SDE with explicit drift and diffusion coefficients such that it has a unique weak solution $X = (X_t, t \geq 0)$ which is a stationary diffusion Markov process with correlation function e^{-ct} and at any time $t \geq 0$ the moments of X_t are $\mathbf{E}[X_t^k] = k!, k = 1, 2, \dots$. Moreover, X_t is exponentially integrable.*

Proposition 8.2. *There is only one SDE with explicit coefficients such that it has a unique weak solution $X = (X_t, t \geq 0)$ which is a stationary diffusion Markov process with correlation function e^{-ct} and at any time $t \geq 0$ the moments of X_t are $\mathbf{E}[X_t^k] = (2k)!, k = 1, 2, \dots$. Moreover, in this case X_t is not exponentially integrable.*

Proposition 8.3. *There is a parametrized infinite family of SDEs, $\{SDE^{(\varepsilon)}, \varepsilon \in [-1, 1]\}$, with explicit coefficients such that their solutions are stationary diffusion Markov processes $\{X^{(\varepsilon)}, \varepsilon \in [-1, 1]\}$ each with correlation function e^{-ct} and at any time $t \geq 0$ all $X_t^{(\varepsilon)}$ have the same moments $\mathbf{E}[(X_t^{(\varepsilon)})^k] = (3k)!, k = 1, 2, \dots$. None is exponentially integrability.*

Sketch of Proofs. As mentioned above, we will use our knowledge about the moment determinacy of $\xi \sim \text{Exp}(1)$ and its powers ξ^2 and ξ^3 .

However, we need to know how to construct a stochastic process with a specified distributional and correlation structure. We provide first the necessary details.

Let F be an absolutely continuous distribution function with density $f = F'$ and X a random variable, $X \sim F$. Denote by (a, b) the range of values of X , i.e.,

the support of F : $\text{supp}(F) = (a, b)$. Assume that f is continuous, bounded and strictly positive in the domain (a, b) which can be a finite or infinite interval. One requirement more is that F has a finite second moment, hence finite variance. We use the usual notation m_1 for the first moment of F .

Thus, given F and f and taking any number $c > 0$, we define a ‘new’ function, $v = (v(x), x \in (a, b))$, as follows:

$$v(x) = \frac{2c}{f(x)} \int_a^x (m_1 - u) f(u) du = \frac{2c}{f(x)} \left(m_1 F(x) - \int_a^x u f(u) du \right).$$

It can be checked that the function v is strictly positive in (a, b) , hence its square-root, $\sqrt{v(x)}$, is a well-defined function.

Theorem. (See [7]) *Suppose that F, f and v are as above, W is a standard Brownian motion which is independent of the random variable X_0 and $c > 0$ is an arbitrary number. Then the following SDE*

$$dX_t = -c(X_t - m_1) dt + \sqrt{v(X_t)} dW_t, \quad X_t|_{t=0} = X_0, \quad t \geq 0$$

has a unique weak solution $X = (X_t, t \geq 0)$ which is a diffusion Markov process. Moreover, X is ergodic with ergodic density f . If the density of X_0 is f , the process X is stationary with correlation function e^{-ct} , $t \geq 0$.

This theorem gives a clear idea of what we need to do now. We take $F = \text{Exp}(1)$, its support is $(a, b) = (0, \infty) = \mathbb{R}_+$. Hence we are ready to give the main details in each of the above three cases, Propositions 8.1, 8.2 and 8.3.

Details for Case 1. We work with ξ , $m_k(\xi) = k!$, $f(x) = e^{-x}$ and assume that $X_0 \sim \text{Exp}(1)$. We have all ingredients to calculate $v(x)$. Hence we can write explicitly just one SDE and conclude that its unique weak solution $X = (X_t, t \geq 0)$ is exactly the desired stationary and diffusion Markov process X . The correlation function is e^{-ct} , $t \geq 0$; X_t has k th order moment equal to $k!$, and X_t is M-determinate. The exponential integrability of X_t follows from the Cramér’s condition for $\xi \sim \text{Exp}(1)$.

Details for Case 2. Now we need ξ^2 for which $m_k(\xi^2) = (2k)!$. Take $X_0 \sim f$, where $f(x) = \frac{1}{2} x^{-1/2} e^{-x^{1/2}}$, $x > 0$ is the density of ξ^2 . Use this f to calculate v , write explicitly the SDE and produce the unique weak solution with the desired properties. One difference from Case 1 is that here X_t is not exponentially integrable. This is because ξ^2 is heavy tailed, no Cramér’s condition.

Details for Case 3. Here we work with the cube ξ^3 . The moment of order k is $m_k(\xi^3) = (3k)!$ and the density to be used next is $f(x) = \frac{1}{3} x^{-2/3} e^{-x^{1/3}}$, $x > 0$. Take this f as a density of X_0 , calculate v , write the SDE and get a stationary diffusion Markov process X with all required properties.

However, as we know, ξ^3 is M-indeterminate. Hence we can construct a Stieltjes class $\mathbf{S}(f, h) = \{f_\varepsilon = f[1 + \varepsilon h], \varepsilon \in [-1, 1]\}$ based on the density f of ξ^3 and the perturbation $h(x) = \sin\left(\frac{\pi}{6} - \sqrt{3}x^{1/3}\right)$, $x > 0$; see also Section 5. For any fixed $\varepsilon \in [-1, 1]$ we use the density f_ε and calculate v_ε . Thus we obtain a SDE^(ε)

whose unique weak solution $X^{(\varepsilon)}$ is a stationary diffusion Markov process. Take now $\varepsilon \in [-1, 1]$ and get the desired infinite family of processes. The drift and the diffusion coefficients are different for different ε . However, at any time $t \geq 0$, the solution of any of these SDEs has the moments $\{(3k)!\}$. The non-exponential integrability is obvious because ξ^3 is ‘too heavy tailed’.

Remark 8.1. We have chosen the moments $k!$, $(2k)!$, $(3k)!$, $k = 1, 2, \dots$ not just for curiosity. On one side it was convenient to use our findings in the previous sections. On the other side, more important is that these numbers show exactly where are the boundaries between the following three groups of SDEs:

- SDEs with M-determinacy and exponential integrability;
- SDEs with M-determinacy but no exponential integrability;
- SDEs with M-indeterminacy and hence no exponential integrability.

We have a general result about a SDE which involves the moment sequence $\{m_k\}$ of an arbitrary absolutely continuous distribution F . We do not give details.

9. Final comments

9.1. Briefly on relevant topics not discussed here

The role of the underlying stochastic process is essential for any valuation of financial instruments. However, sometime we do not know the stock price process and instead we have in our disposal a ‘huge’ data set of observable option prices. Based on these we want to ‘find’, ‘identify’ the underlying process. This is called an *Inverse Problem*. These are difficult theoretical problems and they are computationally heavy; see [11, 70, 85] and more references therein.

We have discussed in this paper only a few popular distributions each used in one or another way in stochastic financial models. Many other distributions are also exploited in the literature. To mention just a few: skew-normal and skew-Student distributions, polynomial logistic, negative inverse Gaussian, generalized gamma, generalized hyperbolic; see, e.g., [3, 4, 8, 9, 10, 14, 18, 25, 27, 30, 31, 33, 34, 36, 57, 58, 64, 74, 75, 86].

Special attention deserve the *explosion phenomenon* of solutions of SDEs, see [2, 42, 43], and also the *stochastic volatility processes*, see [21, 59, 88].

There are several methods based on moments and used to analyse stochastic models. This includes variations of the classical Pearson’s method of moments. It turns out that the moments of distributions can be involved in option pricing when following semi-definite optimization methods. Pricing of options based on moments is treated in [1, 24, 32, 38, 60].

An interesting class of problems is to construct a stochastic process with prescribed specific properties. For example, we may require the process to have a special martingale structure, or to follow given one-dimensional and/or multi-dimensional distributions, or to have a given sequence of moments. Sometime we want the process to be a solution to a SDE. This kind of questions and results for general Lévy processes can be found in [7, 12, 19, 28, 29, 35, 37, 47, 49, 53, 73]. The results in Section 8 fall into this category.

9.2. Comments on the References

Well-known are the classical works by N.I. Akhiezer, M.G. Krein, J.A. Shohat and J.D. Tamarkin, B. Simon and C. Berg. They are not included here.

The items included into the list below are chosen to reflect, in one or another way, the fact that the moments of distributions are involved and/or used essentially in the analysis of stochastic financial models. Some papers contain the term ‘moments’ in the title and/or deal explicitly with the moment determinacy of distributions. More than a half of the papers contain ‘moments’ in the abstract and essentially in the main text. And some papers deal with specific distributions without mentioning the term ‘moments’ at all. However, when reading such papers one may get the feeling that implicitly the moments are there. Or, as one may say: ‘The rabbit is behind the bush!’.

Most important is that all papers and books in the References list are indeed relevant and essential in the area of stochastic financial modelling.

Acknowledgments

I am grateful to the Organizers of the ‘Finance Workshop’ (Tokyo Metropolitan University, 6-7 November 2014) for inviting me to attend the event and deliver a talk. I enjoyed a series of fruitful discussions with Professors Takashi Shibata, Yukio Muromachi and Masaaki Kijima and with many young and enthusiastic PhD students, post-docs and visitors at TMU. The present paper is an extended version of the material reported at the Workshop.

It was also a pleasure to deliver seminar and/or colloquium talks at Kyoto University (Professor Kouji Yano), University of Osaka (Professor Masaaki Fukasawa) and Tokyo City University (Professor Shuya Kanagawa).

I would like to acknowledge with thanks the support from the above institutions and the support for my visit to Japan from Emeritus Fellowship provided by Leverhulme Trust (UK).

My thanks are addressed also to G.D. Lin, M. Zhitlukhin, M. Zervos, D. Hobson, the Referee and the Editors for their relevant suggestions and comments well used when preparing the final text of the paper for which I am solely responsible.

References

1. Albanese C., Osseiran A. (2007). Moment methods for exotic volatility derivatives. Available online: arXiv:0710.2991v1 [math.PR] 16 Oct 2007
2. Andersen L.B.G., Piterbarg V.V. (2007). Moment explosions in stochastic volatility models. *Finance & Stochastics* **11**, 29–50.
3. Barndorff-Nielsen O.E. (1997). Normal IG distributions in stochastic volatility modelling. *Scandinavian Journal of Statistics* **24**, 1–13.
4. Basrak B., Davis R.A., Mikosch T. (2002). Regular variation of GARCH processes. *Stochastic Processes & Applications* **99**, 95–115.
5. Beiglböck M., Henri-Labordère P., Penkner F. (2013). Model independent bounds for option prices: a mass transport approach. *Finance & Stochastics* **17**, 477–501.

6. Bertsimas D., Popescu I. (2002) On the relation between option and stock prices: a convex optimization approach. *Operations Research* **50**, 358–374.
7. Bibby B.M., Skovgaard I.M., Sorensen, M. (2005). Diffusion-type models with given marginal distribution and auto-correlation function. *Bernoulli* **11**, 191–220.
8. Bingham N.H., Kiesel, R. (2001). Semi-parametric modelling in finance: theoretical foundations. *Quantitative Finance* **1**, 1–10.
9. Biró T.S., Rosenfeld, R. (2007). Microscopic origin of non-Gaussian distributions of financial returns. Available: arXiv:0705.4112v2 [q-fin.ST] 6 Jul 2007
10. Blasi F., Scarlatti S. (2012). From normal vs skew-normal portfolios: FSD and SSD rules. *Journal of Mathematical Finance* **2**, 90–95.
11. Bouchouev I., Isakov V. (1997). The inverse problem of option pricing. *Inverse Problems* **13**, 11–17.
12. Campi L. (2004). Arbitrage and completeness in financial markets with given N -dimensional distributions. *Decisions & Economics Finance* **27**, 57–80.
13. Carr P., Madan D. (2009). Saddlepoint methods for option prices. *Journal of Computational Finance* **13**, 49–61.
14. Chen J.T., Gupta A.K., Troskie C.G. (2003). The distribution of stock returns when the market is up. *Communications in Statistics: Theory & Methods* **32**, 1541–1558.
15. Chen L., He S., Zhang S. (2011). Tight bounds for some risk measures, with applications to robust portfolio selection. *Operations Research* **59**, 847–865.
16. Chin S., Dufresne D. (2012). A general formula for option prices in a stochastic volatility model. *Applied Mathematical Finance* **19**, 313–340.
17. Concuera J.M., Nualart D., Schoutens W. (2005). Completion of a Lévy market by power-jump assets. *Finance & Stochastics* **9**, 109–127.
18. Cont R., Tankov P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, London.
19. Cox A.M.G., Hobson D., Obloj J. (2011). Time homogeneous diffusions with a given marginals at a random time. *ESAIM Probability Statistics* **15**, 511–524.
20. Cuchiero C., Keller-Ressel M., Teichmann J. (2012). Polynomial processes and their applications to mathematical finance. *Finance & Stochastics* **16**, 711–740.
21. Davis R.A., Mikosch T. (2009). Probabilistic properties of stochastic volatility models. In: Andersen T.G., Davis R.A., Kreiss J.-P., Mikosch T. (eds.) *Handbook of Financial Time Series*, pp.255–267. Springer, New York.
22. De La Peña V.H., Ibragimov R., Jordan S. (2004). Option bounds. *Journal of Applied Probability* **41A**, 145–156.
23. De Schepper, A., Heijnen, B. (2007). Distribution-free option pricing. *Insurance: Mathematics and Economics* **40**, 179–199.
24. Demos A. (2002). Moments and dynamic structure of a time-varying parameter stochastic volatility in mean model. *Econometrics Journal* **5**, 345–357.
25. Dragulescu A.A., Yakovenko V.M. (2002). Probability distribution of returns in the Heston model with stochastic volatility. *Quantitative Finance* **2**, 443–453.
26. Duffie D., Filipović D., Schachermeiher W. (2003). Affine processes and applications in finance. *Annals of Applied Probability* **13**, 98–1053.

27. Eberlein, E., Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli* **1**, 281–299.
28. Ekström E., Hobson D. (2011). Recovering a time-homogeneous stock price process from perpetual option prices. *Annals of Applied Probability* **21**, 1102–1135.
29. Ekström E., Hobson D., Janson S., Tysk J. (2013). Can time-homogeneous diffusions produce any distribution? *Probability Theory and Related Fields* **155**, 493–520.
30. Eling M. (2014). Fitting asset returns to skewed distributions: Are the skew-normal and skew-Student good models? *Insurance: Mathematics and Economics* **59**, 45–56.
31. Embrechts P., Klüppelberg C., Mikosch T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.
32. Eriksson B., Pistorius M. (2011). Method of moments approach to pricing double barrier contracts in polynomial jump-diffusion model. *International Journal of Theoretical & Applied Finance* **14**, 1139–1158.
33. Fabozzi F.J., Rachev S.T., Menn C. (2005). *Fat-Tailed and Skewed Asset Return Distributions: Implications for Risk Management Portfolio Selections and Option Pricing*. Wiley, New York.
34. Fajardo J., Farias A. (2004). Generalized hyperbolic distributions and Brazilian data. *Brazilian Review of Econometrics* **24** 249–271.
35. Filipović D., Larsson M. (2014). Polynomial preserving diffusions and applications to finance. Available online: arXiv:1404.0989v3 [math.PR] 11 Aug 2014
36. Fischer M.J. (2014). *Generalized Hyperbolic Secant Distributions: With Applications to Finance*. Springer, Berlin.
37. Föllmer H., Wu C.-T., Yor M. (2000). On weak Brownian motion of arbitrary order. *Annales Institut H. Poincaré: Probabilité et Statistique* **36**, 447–487.
38. Fusai G., Tagliani A. (2002). An accurate valuation of Asian options using moments. *International Journal of Theoretical & Applied Finance* **5**, 147–169.
39. Gallant A.R., Tauchen G. (1996). Which moments to match? *Econometric Theory* **12**, 657–681.
40. Gavrilidis, P.N. (2008). Moment information for probability distributions, without solving the moment problem. I. Where is the mode? *Communications in Statistics: Theory & Methods* **37**, 671–681.
41. Gavrilidis P.N., Athanassoulis G.A. (2009). Moment information for probability distributions, without solving the moment problem, II: Main-mass, tails and shape approximation. *Journal of Computational & Applied Mathematics* **229**, 7–15.
42. Gerhold S. (2011). Moment explosion in the LIBOR market model. *Statistics & Probability Letters* **81**, 560–562.
43. Glasserman P., Kim K.-K. (2010). Moment explosions and stationary distributions in affine diffusion models. *Mathematical Finance* **20**, 1–33.
44. Grundy B.D. (1991). Option prices and the underlying asset’s return distribution. *Journal of Finance* **46**, 1045–1069.
45. Gulisashvili A. (2012). *Analytically Tractable Stock Price Models*. Springer, New York.

46. Gotoh J., Konno H. (2002). Bounding option prices by semidefinite programming: a cutting plane algorithm. *Management Science* **48**, 665–678.
47. Hamza K., Klebaner F.C. (2007). A family of non-Gaussian martingales with Gaussian marginals. *Journal of Applied Mathematics & Stochastic Analysis* **2007**, article ID 92723, 19 pp.
48. Hobson D. (1998). Robust hedging of the lookback option. *Finance & Stochastics* **2**, 329–347.
49. Hobson D. (2015). Mimicking martingales. Available: arXiv:1505.03709v1 [math.PR] 14 May 2015
50. Jarrow R., Rudd A. (1982). Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics* **10**, 347–369.
51. Jeablang M., Yor M., Chesney M. (2009). *Mathematical Methods for Financial Markets*. Springer, London.
52. Kagan A., Nagaev S. (2001). How many moments can be estimated from a large sample? *Statistics & Probability Letters* **55**, 99–105.
53. Keller-Ressel M., Papapantoleon A. (2013). The affine LIBOR models. *Mathematical Finance* **23**, 627–658.
54. Kijima, M. (2012). *Stochastic Processes with Financial Applications*. 2nd ed. Chapman & Hall/CRC, London.
55. Kijima M., Muromachi Y. (2008). An extension of the Wang transform derived from Bühlmanns economic premium principle for insurance risk. *Insurance: Mathematics and Economics* **42**, 887–896.
56. Kleiber C., Stoyanov, J. (2013). Multivariate distributions and the moment problem. *Journal of Multivariate Analysis* **113**, 7–17.
57. Koutras V.M., Drakos K., Koutras M.V. (2014). A polynomial logistic distribution and its applications in finance. *Communications in Statistics: Theory & Methods* **43**, 2045–2065.
58. Küchler U., Neumann K. (1999). Stock returns and hyperbolic distributions. *Mathematical & Computer Modelling* **29**, 1–15.
59. Kulik R., Soulier Ph. (2013). Estimation of limiting conditional distributions for heavy tailed long memory stochastic volatility process. *Extremes* **16**, 203–239.
60. Lasserre J.B., Prieto-Rumeau T., Zervos M. (2006). Pricing a class of exotic options via moments and SDP relaxations. *Mathematical Finance* **16**, 469–494.
61. Lee R.W. (2004). The moment formula for implied volatility at extreme strikes. *Mathematical Finance* **14**, 469–480.
62. Lee R.W. (2004). Option pricing by transform methods: extensions, unification and error control. *Journal of Computational Finance* **7**, 51–86.
63. Levy H. (1985). Upper and lower bounds of put and call option value: stochastic dominance approach. *Journal of Finance* **40**, 1197–1217.
64. Li H., Melnikov A. (2014). Polynomial extensions of distributions and their applications in actuarial and financial modeling. *Insurance: Mathematics and Economics* **55**, 250–260.
65. Lin G.D. (1997). On the moment problem. *Statistics & Probability Letters* **35**, 85–90. Addendum: *ibid* **50**(2000), 205.

66. Lin G.D., Stoyanov J. (2009). The logarithmic skew-normal distributions are moment indeterminate. *Journal of Applied Probability* **46**, 909–916.
67. Lin G.D., Stoyanov J. (2014). Moment determinacy of powers and products of nonnegative random variables. *Journal of Theoretical Probability*. In press. Available online: doi 10.1007/s10959-014-0546-z
68. Lin G.D., Stoyanov J. (2015). On the moment determinacy of products of non-identically random variables. *Probability and Mathematical Statistics*. In press. Available online: arXiv:1406.1654v2 [math.PR] 12 Nov 2014.
69. Lions P.-L., Musiela M. (2007). Correlations and bounds for stochastic volatility models. *Annales Institut H. Poincaré. Analysis* **24**, 1–16.
70. Lishang J., Youshan T. (2001). Identifying the volatility of underlying assets from option prices. *Inverse Problems* **17**, 137–155.
71. Lo A.W. (1987). Semi-parametric upper bounds for option prices and expected payoffs. *Journal of Financial Economics* **19**, 373–387.
72. Lykov K.V. (2012). New uniqueness conditions for the classical moment problem. *Mathematical Notes* **92**, 797–806.
73. Madan D., Yor M. (2002). Making Markov martingales meet marginals: with explicit constructions. *Bernoulli* **8**, 509–536.
74. Malevergne Y., Pisarenko V., Sornette D. (2005). Empirical distributions of stock returns: between the stretched exponential and the power law? *Quantitative Finance* **5**, 379–401.
75. Maslov S., Zhang Y.-C. (1999). Probability distribution of drawdowns in risky investments. *Physics A* **262**, 232–241.
76. Merton R C, Samuelson P A (1974). Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. *Journal of Financial Economics* **1**, 67–94.
77. Mnatsakanov R., Hakobyan A. (2009). Recovery of distributions via moments. *IMS Lecture Notes - Monograph Series* **57**, 252–265.
78. Ninomiya S., Victoir N. (2008). Weak approximation of SDEs and application to derivative pricing. *Applied Mathematical Finance* **15**, 107–121.
79. Ostrovska S. (2014). Constructing Stieltjes classes for M-indeterminate absolutely continuous probability distributions. *ALEA Latin American Journal of Probability & Mathematical Statistics* **11**, 253–258.
80. Ostrovska S., Stoyanov J. (2005). Stieltjes classes for M-indeterminate powers of inverse Gaussian distributions. *Statistics & Probability Letters* **71**, 165–171.
81. Pakes A.G. (2001). Remarks on converse Carleman and Krein criteria for the classical moment problem. *Journal of the Australian Mathematical Society* **71**, 81–104.
82. Pakes A.G. (2007). Structure of Stieltjes classes of moment-equivalent probability laws. *Journal of Mathematical Analysis & Applications* **326**, 1268–1290.
83. Rolski T., Schmidli H., Schmidt V., Teugels J. (1999). *Stochastic Processes for Insurance and Finance*. Wiley, New York.
84. Rasz S., Tari A., Telek M. (2006). A moments based distribution bounding method. *Mathematical & Computational Modelling* **43**, 1367–1382.

85. Rodrigo M.R., Mamon R.S. (2007). Recovery of time-dependent parameters of a Black-Scholes-type equation: an inverse Stieltjes moment approach. *Journal of Applied Mathematics* **2007**, Article ID 62098, 8 pp.
86. Satchell S., Knight J. (2000). *Return Distributions in Finance*. Elsevier, Amsterdam.
87. Shephard N., Andersen T.G. (2009). *Stochastic Volatility: Origins and Overview*. Springer, London.
88. Stein M.S., Stein C.S. (1991). Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies* **4**, 727–752.
89. Stoyanov J. (2000). Krein condition in probabilistic moment problems. *Bernoulli* **6**, 939–949.
90. Stoyanov J. (2002). Moment problems related to the solutions of SDEs. *Lecture Notes in Control and Information Sciences* (Springer) **280**, 459–469.
91. Stoyanov J. (2004). Stieltjes classes for moment indeterminate probability distributions. *Journal of Applied Probability* **41A**, 669–685.
92. Stoyanov J. (2012). Inference problems involving determinacy of distributions. *Communications in Statistics: Theory & Methods* **41**, 2864–2878.
93. Stoyanov J. (2013). *Counterexamples in Probability*, 3rd edn. Dover, New York.
94. Stoyanov J., Lin G.D. (2012). Hardy’s condition in probabilistic moment problems. *Theory of Probability and Its Applications* **57**, 811–820. [SIAM edition **57**(2013), 699–708.]
95. Stoyanov J., Lin G.D., DasGupta A. (2014). Hamburger moment problem for powers and products of random variables. *Journal of Statistical Planning and Inference* **154**, 166–177.
96. Tagliani A., Milev M. (2013). Laplace transform and finite difference method for the Black-Scholes equation. *Applied Mathematics & Computations* **220**, 649–658.
97. Tehranchi M.R. (2013). On the uniqueness of martingales with certain prescribed marginals. *Journal of Applied Probability* **50**, 557–575.
98. Tian R. (2008). Moment Problems with Applications to Value-At-Risk and Portfolio Management. Georgia State University. Dissertations, Paper 21.