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The role of centrality and market size in a 4-region asymmetric new economic geography model

Pasquale Commendatore · Ingrid Kubin ·
Pascal Mossay · Iryna Sushko

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Abstract In this paper, we put forward a 4-region new economic geography footloose entrepreneur model, in which regions are differentiated on the basis of their size and geographical position along a line. There are two distinct trade blocs, each of them consisting of a pair of regions. Direct and indirect trade between all regions is allowed, whereas factor mobility can occur only between regions of the same bloc. Given this more general geographical structure, compared to previous studies, we are able to disentangle two manifestations of the market access effect: firms can take advantage of locating both in a more central region (centrality effect) and/or in a bigger region (local market size effect). The model is able to generate a plethora of long-term outcomes,

P. Commendatore
University of Naples ‘Federico II’
Dept of Law
via Mezzocannone 16, I-80134 Naples, Italy
Tel.: +39 081 253 7447
E-mail: commenda@unina.it

I. Kubin
Vienna University of Economics and Business Administration
Dept of Economics
Welthandelsplatz 1, Building D4, 1020 Vienna, Austria
Tel.: +43 1 31336 4456
E-mail: ingrid.kubin@wu.ac.at

P. Mossay
Newcastle University, UK, and CORE, Belgium
Business School
5 Barrack Road, Newcastle upon Tyne, NE1 4SE, UK
Tel.: +44 191 208 1668
E-mail: pascal.mossay@ncl.ac.uk

I. Sushko
Institute of Mathematics, National Academy of Sciences of Ukraine
3 Tereshchenkivska st., 01601 Kyiv, Ukraine
Tel.: +38 050 443 4824
E-mail: sushko@imath.kiev.ua

including four equilibria with full agglomeration in each trade bloc that can be ranked by factor owners. Equilibria where industry is dispersed or agglomerated in a bloc and dispersed in the other one, are also possible as well as more complex attractors. Finally, by allowing direct and indirect trade between regions, we are able to look at the effect of trade integration on transit traffic by evaluating in a preliminary analysis the consequences of policies aiming at limiting transport volumes in a model with shifting industry.

Keywords New Economic Geography · Market Access · Centrality · Transit Traffic · Industrial Agglomeration · Two-dimensional piecewise smooth map · Local and global dynamics

JEL classification: C62, F12, F2, R12

1 Introduction

Models of the New Economic Geography (NEG) are now widely used to assess the effect of various policy options. These models analyze how agglomerations (i.e. the clustering of productive factors and thus of industrial production in a limited number of regions) may occur endogenously and underpin endogenous cumulative processes of factor relocation.

Since the seminal paper by Krugman, 1991, a broad family of NEG models has emerged. A mechanism common to most of these models relies on the interplay between market access and competition effects: in monopolistically competitive commodity markets, a region with a better market access can offer a higher remuneration to its production factors, which in turn attracts them even further; on the other hand, the larger the local market, the more intense the local competition and thus the lower the factor remuneration.

It is notorious that NEG models use many simplifying assumptions; prominent among them is the two-country/region setup. A recent stream of papers shows that extending the two-region setup so as to encompass more regions may affect traditional agglomeration patterns (see Commendatore *et al.*, 2015a for a comprehensive review of this literature).

In addition, NEG models usually assume regions to be symmetric (see e.g. Fujita *et al.*, 1999; Ikeda *et al.*, 2012; or Castro *et al.*, 2012). While acknowledging the fact that endogenous processes may lead to second-nature asymmetries even with symmetric regions, studying the interaction between first-nature asymmetries and endogenous second-nature processes has proven to be fruitful (see, for example, Krugman, 1993a, b; Fujita and Mori, 1996; Matsuyama and Takahashi, 1998; Ago *et al.*, 2006; Roos, 2005; Venables, 2006; Ostbye, 2010; Agliari *et al.* 2011, 2015).

In this paper we develop a 4-region NEG model with first-nature asymmetries and try to shed additional light on the role of market access. In a two-region model, market access is primarily determined by local market size and the access to the other region. In a multiple-region framework, a better market access can result from a larger local market, but also from a more central geographical location that provides a better access to external markets.

In order to grasp the centrality aspect, we assume a specific first-nature geography: the four regions are located on a line providing two regions with a centrality advantage (while the two others are remote); in addition, the size of regions alternate along the line: small, big, small, big. The first two and the last two regions are well integrated with each other – thus we have two integrated blocs; whereas integration between the two blocs is lower. Factor mobility occurs only between well integrated regions, whereas commodities are traded between all regions, at different cost levels though, reflecting the degree of integration. This setup resembles the regional line-up encountered in real-world examples such as that of Southern Italy, Northern Italy, Austria, and Germany, or that of Canada, the US, Mexico and Latin America.

The interplay between centrality and local market size that is at the core of the present paper cannot be studied in other 4-region NEG models. Closest to our analysis are Montfort and Nicolini, 2000, Behrens, 2007, and Commendatore *et al.*, 2015b. In contrast to our paper, they assume regions of equal size. In addition, they consider a simpler geographical set-up than in the current paper: In Montfort and Nicolini, 2000, and Behrens, 2007, the distance from a region in a bloc to any other region in the other bloc is the same; Commendatore *et al.*, 2015b, assume that trade can only occur between neighbouring regions. Given the restrictive assumptions (i.e. symmetric regions and a specific simple geography), the main questions of the current paper cannot be addressed in the existing literature; and a more general framework had to be introduced.

The NEG framework we use is the footloose entrepreneur model; its tractability provides some scope for analytical results also in a 4-region model. As it relies on the mobility of entrepreneurs, the model is particularly suitable to study economies with a low labour mobility. The standard footloose entrepreneur (FE) model with two regions of different size (Baldwin *et al.*, 2003, p. 106-108) predicts that for low trade costs, industry will be agglomerated. For very low trade costs, it can agglomerate in either region – the two existing core-periphery equilibria are both possible outcomes; for somewhat higher trade costs, industry will locate only in the bigger region – the small region losing industry agglomeration because of its disadvantageous small local market size. Only for high trade costs, an internal equilibrium with industry distributed across both regions is stable.

Our model is more general and encompasses the two-region FE model as a special case (when trade costs between blocs are prohibitive). In contrast to many NEG studies, we set our model in discrete time, since this formulation better represents delays inherent to any firm relocation decisions.

From an analytic point of view the resulting model is a two-dimensional piecewise smooth noninvertible map depending on 8 parameters. It possesses very rich dynamics including attracting fixed points and cycles of any period, closed invariant attracting curves, chaotic attractors, and all such attractors can coexist. We investigate the bifurcation structure of equilibria by focusing on parameters related to trade costs and sizes of regions. In contrast to our previous contribution (Commendatore *et al.*, 2015b), here we focus on the

stationary equilibria. We study their existence and stability properties, basins of attraction, as well as the bifurcations that these fixed points can undergo. We show that coexistence is pervasive; moreover, basins of attraction can have quite complex structures. Given that the analytic representation of the map is very intricate (and no longer symmetric as in Commendatore *et al.*, 2015b), only a few results are obtained analytically while other findings are motivated by numerical simulations. The focus on the properties of fixed points allows us to address the economic questions related to centrality and market size.

Indeed, from an economic point of view, our analysis focuses on three issues:

First, we study the effects of deeper trade integration – i.e. of lower trade costs within a bloc – on regions of different size in the presence of an outside bloc. We show that in our framework coexistence of attracting core-periphery equilibria is also pervasive: we find that up to four different stable core-periphery industry allocations may coexist; that the pattern of coexistence changes when varying the trade cost within blocs; and that the sequence of coexistence patterns sheds some light on the economic forces at work. In particular, we find that for low trade costs industry agglomeration in a region with a small local market is possible if it is central, i.e. if it has at least some access to an outside bloc. At the same time, we find instances in which remoteness provides shelter from competition and facilitates the agglomeration of industry in a small remote region. Increasing trade costs favours agglomeration in the regions with bigger local markets. We characterize the core-periphery equilibria by its indirect utilities for workers and entrepreneurs. With a still higher trade cost, also (semi-) interior industry allocations appear to be stable and coexisting, now with complex basins of attraction. Finally, coexisting cyclical or complex attractors exist as well. We argue that this bifurcation sequence also nicely corresponds to the economic forces at work.

Second, our framework allows to study a new issue, namely the effects of trade costs between blocs. Reducing these trade costs increases the advantage of the central regions and we find that for low trade costs between the blocs the industry agglomerates in the two central regions. In addition, we find other coexisting stable core-periphery equilibria: even the small remote region may attract industry agglomeration for some range of trade costs, because it is sheltered from competition; the other remote region is able to attract the industry for a wider range of trade costs due to its advantageous large local market.

Third, the geographical line-up that we have chosen allows to shed light on another issue, namely transit traffic, i.e. the transport of commodities that passes by some intermediary region before reaching its final destination. Because of the involved environmental burden, transit traffic is a controversial issue debated not only in the popular press, but also in policy consulting reports and in academic studies. Since in our model the coexisting attracting equilibria involve distinct regional distributions of industry, they also imply different levels of transit traffic. Given this coexistence, a change in transport cost intended to reduce transit traffic may actually move the economy from one equilibrium to another, leading to an unexpected "jump" in transit traffic.

Fig. 1 Geographical representation of the 4-Region economy. Arrows represent trade flows. The size of R_1 and R_3 is smaller than the size of R_2 and R_4 .

The paper is structured as follows: Section 2 introduces the model framework and presents the short-run equilibrium solution. Section 3 presents the long-run dynamic process and studies core-periphery fixed points and their stability properties. Section 4 discusses economic properties of these core-periphery fixed points and reveals the underlying economic processes; in particular, we focus not only on industry allocations, but also on indirect utility of workers and entrepreneurs and on transit traffic. While Sections 3 and 4 can rely on analytic results, Section 5 demonstrates with simulations that other (coexisting) stable fixed points may also occur with increasingly complex basins of attraction and that even attracting cycles and chaotic attractors appear. We provide some economic interpretation to these dynamic phenomena. Section 6 concludes.

2 The model

The economy is composed of four regions, which are arranged along a line from 1 to 4 (see Fig. 1), two sectors – Agriculture and Manufacturing – and two factors of production – entrepreneurs and workers. The entrepreneurs are the mobile factor, whereas workers do not move. In line with our story, we assume that the four regions enjoy different degrees of integration, that is, region 1 (R_1) is more integrated with region 2 (R_2) and region 3 (R_3) with region 4 (R_4). One of the consequences is that entrepreneurial migration is only allowed between R_1 and R_2 and between R_3 and R_4 . The two integrated blocs (R_1 - R_2 and R_3 - R_4) could be such because they belong to the same country or because they enjoy some other type of proximity (sharing, for example, the same language). Regions may have different sizes: specifically, we consider that R_1 and R_3 could be smaller compared to R_2 and R_4 ; whereas, in order to keep matters simple, R_1 has the same size of R_3 and R_2 the same size of R_4 . This is represented by different endowments of the immobile factor, labor L , thus: $L_1 = L_3 = L_s \leq L_2 = L_4 = L_b$ (s = small region; b = big region; L_r = labor endowment of region r). Instead, the number of entrepreneurs located in each integrated bloc is the same and given by $E/2$.

Given the specific line-up of the regions, direct trade can only occur between some of the regions (i.e. those adjacent to each other; for example between R_1 and R_2 or between R_2 and R_3). However, differently from a previous paper of ours (see Commendatore *et al.*, 2015b), we allow for indirect trade (e.g. R_1 can export to R_3 , but only via R_2). This gives us a completely defined matrix of trade costs, which is based on the typical new economic geography

(NEG) assumption of iceberg – proportional to price – trade costs:

$$\mathbf{T} = \begin{pmatrix} 1 & T_C & T_C T_D & T_C^2 T_D \\ T_C & 1 & T_D & T_C T_D \\ T_C T_D & T_D & 1 & T_C \\ T_C^2 T_D & T_C T_D & T_C & 1 \end{pmatrix}$$

where, as shown in Fig. 1, C denotes low trade costs (i.e. those incurred by transporting commodities within the two trade blocs R_1 - R_2 or R_3 - R_4 , i.e. between “close” regions); and D high trade costs (i.e. those incurred by transporting commodities between R_2 and R_3 , i.e. between “distant” regions). Taking into account indirect trade costs, we are able to fill all the entries, where the entry T_{ij} represents the costs of trading between region i and region j : for example trade costs between R_2 and R_4 – which is same in both directions – correspond to $T_{24} = T_{42} = T_C T_D$. Moreover, trade costs are nil within each region, so that $T_{ii} = 1$, and increase proportionally with distance, so that, for example, trade costs between R_1 and R_2 are $T_{12} = T_C$, between R_1 and R_3 are $T_{13} = T_C T_D$ and the maximum possible costs are incurred when trading between the two more remote regions $T_{14} = T_C^2 T_D$ (thus, most of the times, the assumed natural geography gives to the two more central regions R_2 and R_3 an advantage in terms of lower trade costs). Using the typical NEG transformation of trade cost into trade freeness parameters, we can write:

$$\Phi = \begin{pmatrix} 1 & \phi_C & \phi_C \phi_D & \phi_C^2 \phi_D \\ \phi_C & 1 & \phi_D & \phi_C \phi_D \\ \phi_C \phi_D & \phi_D & 1 & \phi_C \\ \phi_C^2 \phi_D & \phi_C \phi_D & \phi_C & 1 \end{pmatrix},$$

where $\phi_{ij} = T_{ij}^{1-\sigma}$ and where σ is defined below.

The utility function of a representative consumer, worker or entrepreneur, is given by

$$U = C_M^\mu C_A^{1-\mu},$$

where $0 < \mu < 1$ is the share of manufacturing expenditure, C_A is the consumption of the agricultural good and C_M is the consumption of a CES composite of manufactured varieties given by

$$C_M = \left(\sum_{i=1}^n c_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where c_i is the consumption of variety i , n the number of available varieties and $\sigma > 1$ the elasticity of substitution across varieties.

The budget constraint faced by the representative consumer (worker or entrepreneur) is

$$\sum_{i=1}^n \tilde{p}_i c_i + p_A C_A = m,$$

where p_A is the agricultural price, $\tilde{p}_i$ is the price of variety i which includes trade costs and m is the consumer's income.

The agricultural sector is perfectly competitive. A unit of output requires solely 1 unit of unskilled labor.

The manufacturing sector is monopolistically competitive: identical firms produce differentiated varieties using a technology requiring a fixed component (an entrepreneur) and a variable component (workers), with β units of labor for each additional unit of output. The total cost of producing q_i units of variety i is given by

$$TC(q_i) = \pi_i + w\beta q_i,$$

where the entrepreneur remuneration π_i represents the fixed cost component and w the unitary wage.

The short-run equilibrium in period t is defined in terms of the spatial distribution of entrepreneurs across regions in that period. The share of entrepreneurs in R_1 - R_2 (resp. in R_3 - R_4) is denoted by x_t (resp. y_t).

Because of zero agricultural profits, and once the agricultural good is chosen as the numeraire, $w = p_A = 1$. Each manufacturing firm faces a demand curve with a constant elasticity $-\sigma$ and optimally sets the (mill) price of its variety as a fixed mark-up over marginal cost:

$$p = \frac{\sigma}{\sigma - 1}\beta.$$

The income of each region is given by:

$$Y_{1,t} = L_s + \pi_{1,t}x_t \frac{E}{2}, \quad Y_{2,t} = L_b + \pi_{2,t}(1 - x_t) \frac{E}{2}, \quad (1)$$

$$Y_{3,t} = L_s + \pi_{3,t}y_t \frac{E}{2}, \quad Y_{4,t} = L_b + \pi_{4,t}(1 - y_t) \frac{E}{2}. \quad (2)$$

The demand facing a firm located in each region is:

$$d_{1,t} = \mu \frac{Y_{1,t}P_{1,t}^{\sigma-1} + \phi_C Y_{2,t}P_{2,t}^{\sigma-1} + \phi_C \phi_D Y_{3,t}P_{3,t}^{\sigma-1} + \phi_C^2 \phi_D Y_{4,t}P_{4,t}^{\sigma-1}}{p^\sigma}, \quad (3)$$

$$d_{2,t} = \mu \frac{\phi_C Y_{1,t}P_{1,t}^{\sigma-1} + Y_{2,t}P_{2,t}^{\sigma-1} + \phi_D Y_{3,t}P_{3,t}^{\sigma-1} + \phi_C \phi_D Y_{4,t}P_{4,t}^{\sigma-1}}{p^\sigma}, \quad (4)$$

$$d_{3,t} = \mu \frac{\phi_C \phi_D Y_{1,t}P_{1,t}^{\sigma-1} + \phi_D Y_{2,t}P_{2,t}^{\sigma-1} + Y_{3,t}P_{3,t}^{\sigma-1} + \phi_C Y_{4,t}P_{4,t}^{\sigma-1}}{p^\sigma}, \quad (5)$$

$$d_{4,t} = \mu \frac{\phi_C^2 \phi_D Y_{1,t}P_{1,t}^{\sigma-1} + \phi_C \phi_D Y_{2,t}P_{2,t}^{\sigma-1} + \phi_C Y_{3,t}P_{3,t}^{\sigma-1} + Y_{4,t}P_{4,t}^{\sigma-1}}{p^\sigma}, \quad (6)$$

where $P_{r,t}$ denotes the manufacturing price index in region r :

$$P_{r,t} = p \left(\frac{E}{2} \right)^{\frac{1}{1-\sigma}} \Delta_{r,t}^{\frac{1}{1-\sigma}},$$

with $\Delta_{r,t}$ given by

$$\Delta_{1,t} = x_t + \phi_C(1 - x_t) + \phi_C\phi_D y_t + \phi_C^2\phi_D(1 - y_t), \quad (7)$$

$$\Delta_{2,t} = \phi_C x_t + 1 - x_t + \phi_D y_t + \phi_C\phi_D(1 - y_t), \quad (8)$$

$$\Delta_{3,t} = \phi_C\phi_D x_t + \phi_D(1 - x_t) + y_t + \phi_C(1 - y_t), \quad (9)$$

$$\Delta_{4,t} = \phi_C^2\phi_D x_t + \phi_C\phi_D(1 - x_t) + \phi_C y_t + (1 - y_t). \quad (10)$$

The entrepreneur remuneration in region r is given by

$$\pi_{r,t} = pq_{r,t} - \beta q_{r,t} = \frac{pq_{r,t}}{\sigma},$$

where $q_{r,t}$ is the supply of a firm in region r .

Since in equilibrium, $d_{r,t} = q_{r,t}$, the entrepreneur remuneration in each region can be rewritten by using expressions (3-6)

$$\pi_{1,t} = \frac{2\mu}{\sigma E} \left(\frac{Y_{1,t}}{\Delta_{1,t}} + \phi_C \frac{Y_{2,t}}{\Delta_{2,t}} + \phi_C\phi_D \frac{Y_{3,t}}{\Delta_{3,t}} + \phi_C^2\phi_D \frac{Y_{4,t}}{\Delta_{4,t}} \right), \quad (11)$$

$$\pi_{2,t} = \frac{2\mu}{\sigma E} \left(\phi_C \frac{Y_{1,t}}{\Delta_{1,t}} + \frac{Y_{2,t}}{\Delta_{2,t}} + \phi_D \frac{Y_{3,t}}{\Delta_{3,t}} + \phi_C\phi_D \frac{Y_{4,t}}{\Delta_{4,t}} \right), \quad (12)$$

$$\pi_{3,t} = \frac{2\mu}{\sigma E} \left(\phi_C\phi_D \frac{Y_{1,t}}{\Delta_{1,t}} + \phi_D \frac{Y_{2,t}}{\Delta_{2,t}} + \frac{Y_{3,t}}{\Delta_{3,t}} + \phi_C \frac{Y_{4,t}}{\Delta_{4,t}} \right), \quad (13)$$

$$\pi_{4,t} = \frac{2\mu}{\sigma E} \left(\phi_C^2\phi_D \frac{Y_{1,t}}{\Delta_{1,t}} + \phi_C\phi_D \frac{Y_{2,t}}{\Delta_{2,t}} + \phi_C \frac{Y_{3,t}}{\Delta_{3,t}} + \frac{Y_{4,t}}{\Delta_{4,t}} \right). \quad (14)$$

By substituting the income expressions (1-2) into the above expressions and after dropping the time subscript, the system can be solved for the profits as

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \left(\mathbf{I} - \frac{\mu}{\sigma} \mathbf{\Phi} \mathbf{C} \mathbf{G} \right)^{-1} \frac{2\mu}{\sigma E} (\mathbf{\Phi} \mathbf{C} \mathbf{L}), \quad (15)$$

where

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\Delta_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\Delta_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\Delta_4} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1-y \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} L_s \\ L_b \\ L_s \\ L_b \end{pmatrix}. \quad (16)$$

$\mathbf{C}$ is a matrix that shows how the interaction of enterprise location and trade costs shape local competition and the local price index; $\mathbf{G}$ is a matrix giving the geographical distribution of entrepreneurs; and $\mathbf{L}$ is the vector of regional labor endowments. The explicit values for π are presented in the Appendix A.

From the above expressions, the real profits of entrepreneurs $V_r = \pi_r / P_r^\mu$ can be formulated in terms of x and y .

3 Equilibria and their stability properties

Within each integrated area, entrepreneurs have an incentive to relocate to the region providing them with the highest indirect utility. The migration process resembles the evolutionary replicator dynamics leading to the following dynamic equations:

$$\begin{aligned} Z_h(x, y) &= x \left[1 + \gamma(1-x) \frac{V_1(x, y) - V_2(x, y)}{xV_1(x, y) + (1-x)V_2(x, y)} \right], \\ Z_f(x, y) &= y \left[1 + \gamma(1-y) \frac{V_3(x, y) - V_4(x, y)}{(1-y)V_4(x, y) + yV_3(x, y)} \right]. \end{aligned}$$

Taking into account the constraints $0 \leq x \leq 1$ and $0 \leq y \leq 1$, we are able to define the following family of 2D piecewise smooth maps $Z : R^2 \rightarrow I^2$, $I^2 = [0, 1] \times [0, 1]$:

$$Z : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \begin{cases} 0 & \text{if } Z_h(x, y) < 0, \\ Z_h(x, y) & \text{if } 0 \leq Z_h(x, y) \leq 1, \\ 1 & \text{if } Z_h(x, y) > 1, \end{cases} \\ \begin{cases} 0 & \text{if } Z_f(x, y) < 0, \\ Z_f(x, y) & \text{if } 0 \leq Z_f(x, y) \leq 1, \\ 1 & \text{if } Z_f(x, y) > 1, \end{cases} \end{pmatrix}, \quad (17)$$

where

$$\begin{aligned} Z_h(x, y) &= x \left[1 + \gamma(1-x) \frac{\Omega_h(x, y) - 1}{1 + x(\Omega_h(x, y) - 1)} \right], \\ Z_f(x, y) &= y \left[1 + \gamma(1-y) \frac{\Omega_f(x, y) - 1}{1 + x(\Omega_f(x, y) - 1)} \right], \end{aligned}$$

with

$$\Omega_h(x, y) = \frac{V_1(x, y)}{V_2(x, y)}, \quad \Omega_f(x, y) = \frac{V_3(x, y)}{V_4(x, y)}.$$

The map (17) depends on 8 parameters satisfying the following conditions:

$$0 < \phi_C, \phi_D, \mu < 1, \quad \sigma > 1, \gamma, E, L_b, L_s > 0.$$

3.1 Preliminaries

First note that the unit square I^2 of the phase plane is *invariant* under the map Z , as well as each of the borders of I^2 , which we denote as follows:

$$\begin{aligned} I_{x0} &= \{(x, y) : y = 0\}, \quad I_{x1} = \{(x, y) : y = 1\}, \\ I_{0y} &= \{(x, y) : x = 0\}, \quad I_{1y} = \{(x, y) : x = 1\}. \end{aligned}$$

On each of these borders, the 2D map Z is reduced to the corresponding 1D maps, listed in Appendix B. As we shall see, they are helpful to understand the overall dynamics of Z .

The following notations are used for the fixed points of Z :

- *Core-periphery (CP) fixed points* (related to the manufacturing activity agglomerated within both countries):

$$\begin{aligned} CP_{00} : (x, y) &= (0, 0), & CP_{11} : (x, y) &= (1, 1), \\ CP_{01} : (x, y) &= (0, 1), & CP_{10} : (x, y) &= (1, 0). \end{aligned}$$

- *Interior fixed points* (related to dispersion of the industry distributed within both countries):

$$IP_{ab} : (x, y) = (a, b),$$

with $a, b \neq 0, a, b \neq 1$.

- *Border fixed points* (related to agglomeration in a country and dispersion in the other one):

$$\begin{aligned} BP_{0a} : (x, y) &= (0, a), & BP_{1a} : (x, y) &= (1, a), \\ BP_{a0} : (x, y) &= (a, 0), & BP_{a1} : (x, y) &= (a, 1), \end{aligned}$$

with $a \neq 0, a \neq 1$.

Obviously, The CP fixed points always exist. Instead, the existence of the interior and border fixed points depends on the parameters of the model. It appears that any interior fixed point of the map Z – when it exists – is an intersection point of the curves

$$\Omega_h = \{(x, y) \in I^2 : \Omega_h(x, y) = 1\}, \quad (18)$$

and

$$\Omega_f = \{(x, y) \in I^2 : \Omega_f(x, y) = 1\}, \quad (19)$$

that is,

$$IP_{ab} \in \{\Omega_h \cap \Omega_f\}.$$

On the other hand, for border fixed points, the following conditions hold:

$$\begin{aligned} BP_{0a} &\in \{\Omega_f \cap I_{0y}\}, & BP_{1a} &\in \{\Omega_f \cap I_{1y}\}, \\ BP_{a0} &\in \{\Omega_h \cap I_{x0}\}, & BP_{a1} &\in \{\Omega_h \cap I_{x1}\}. \end{aligned}$$

This information about fixed points helps to visualize their locations in the phase plane. Note that we allow for the existence of several fixed points of the same kind.

The stability conditions of the CP fixed points¹ of Z can be derived either in a straightforward way using eigenvalues of the Jacobian matrix of Z (see Appendix C), or using 1D maps defined on the borders of I^2 (see Appendix B). In fact, $x = 0$ and $x = 1$ are obviously the fixed points of map z_{x0} defined in (28). Let $\lambda(0)$ be the multiplier (or eigenvalue) of the fixed point $x = 0$.

¹ Note that at the CP fixed points, as well as at the border fixed points, the map Z is not differentiable. Thus, rigorously speaking one can only discuss their one-side stability (on the other sides they are obviously always superstable). From now on, consider that the notion of stability that applies to these fixed points is one-side stability.

It holds $\lambda(0_-) = 0$, while $\lambda(0_+) = 1 + \gamma(\Omega_h(0,0) - 1)$. Due to the lower constraint of map z_{x0} it always holds $\lambda(0) \geq 0$ (in fact, if $\lambda(0_+) < 0$ then the flat branch $z_{x0}(x) = 0$ ‘enters’ the interval $[0, 1]$ so that the fixed point $x = 0$ becomes both-side superstable). Thus, for the stability of the fixed point $x = 0$ it must be that $\lambda(0_+) < 1$, which corresponds to the condition $\Omega_h(0,0) < 1$. Similarly, it can be shown that the fixed point $x = 1$ of the map z_{x0} is stable if the condition $\Omega_h(1,0) > 1$ is satisfied. The same reasoning applies to the fixed points of the 1D maps z_{x1} , z_{0y} and z_{1y} , defined in (29)-(31), which leads respectively to the following *stability conditions for the CP fixed points* of the map Z :

$$CP_{00} : \quad \{ \Omega_h(0,0) < 1, \Omega_f(0,0) < 1 \}, \quad (20)$$

$$CP_{11} : \quad \{ \Omega_h(1,1) > 1, \Omega_f(1,1) > 1 \}, \quad (21)$$

$$CP_{01} : \quad \{ \Omega_h(0,1) < 1, \Omega_f(0,1) > 1 \}, \quad (22)$$

$$CP_{10} : \quad \{ \Omega_h(1,0) > 1; \Omega_f(1,0) < 1 \}. \quad (23)$$

Consider now the border fixed point BP_{0a} . Recall that at BP_{0a} , $\Omega_f(0,a) = 1$ holds. The eigenvalues of BP_{0a} associated with the vertical and horizontal directions are denoted by $\lambda_v(0,a) = 1 + \gamma a(1-a)\Omega_{f'y}'(0,a)$ and $\lambda_h(0,a) = 1 + \gamma(\Omega_h(0,a) - 1)$ respectively (see Appendix C). In the horizontal direction, the fixed point BP_{0a} is stable if $\lambda_h(0,a) < 1$, that is, if $\Omega_h(0,a) < 1$ (note that, due to the constraints of the map Z , it always holds that $\lambda_h(0,a) \geq 0$). In the vertical direction, the fixed point BP_{0a} is stable if $|\lambda_f(0,a)| < 1$, that is, if $-\frac{2}{\gamma a(1-a)} < \Omega_{f'y}'(0,a) < 0$. Thus, the stability conditions of fixed point BP_{0a} are the following:

$$BP_{0a} : \quad \left\{ \Omega_h(0,a) < 1, \quad -\frac{2}{\gamma a(1-a)} < \Omega_{f'y}'(0,a) < 0 \right\}.$$

In a similar way, stability conditions of the other border fixed points can be obtained as follows:

$$BP_{1a} : \quad \left\{ \Omega_h(1,a) > 1, \quad -\frac{2}{\gamma a(1-a)} < \Omega_{f'y}'(1,a) < 0 \right\},$$

$$BP_{a0} : \quad \left\{ \Omega_f(a,0) < 1, \quad -\frac{2}{\gamma a(1-a)} < \Omega_{h'x}'(a,0) < 0 \right\},$$

$$BP_{a1} : \quad \left\{ \Omega_f(a,1) < 1, \quad -\frac{2}{\gamma a(1-a)} < \Omega_{h'x}'(a,1) < 0 \right\}.$$

Let us now describe bifurcations occurring when the fixed point BP_{0a} loses its stability. If $\lambda_v(0,a) = -1$, which holds for $\Omega_{f'y}'(0,a) = -\frac{2}{\gamma a(1-a)}$, a *flip bifurcation* of BP_{0a} occurs leading to the appearance of a 2-cycle on the border I_{0y} , while the condition $\lambda_v(0,a) = 1$ may be related to either a *border-transcritical* (BT for short), or a *fold bifurcation* of BP_{0a} . In fact, $\lambda_v(0,a) = 1$ if either $a = 0$, or $a = 1$, or $\Omega_{f'y}'(0,a) = 0$. The first two cases are associated to BT bifurcations of BP_{0a} at which this fixed point merges with the CP fixed point CP_{00} or CP_{01} , respectively. After the bifurcation, the fixed

point BP_{0a} disappears and the related CP fixed point changes its stability. The conditions $\Omega_{f_y}'(0, a) = 0$, $a \neq 0$, $a \neq 1$ correspond to a fold bifurcation leading to appearance/disappearance of two border fixed points (note that at the bifurcation, the curve Ω_f , defined in (19), is tangent to the y -axis at the point $(x, y) = (0, a)$). As for the horizontal stability of BP_{0a} , we have that $\lambda_h(0, a) = 1$ if $\Omega_h(0, a) = 1$, that is, at this bifurcation $\Omega_h(0, a) = \Omega_f(0, a) = 1$. So, a BT bifurcation occurs in the horizontal direction leading to the appearance of an interior fixed point.

Thus, in the model defined by the map Z any CP fixed point changes its stability due to a BT bifurcation, which is quite typical for NEG models. In a generic case this bifurcation results in the appearance/disappearance of a border fixed point, which in turn may undergo a flip bifurcation leading to a 2-cycle belonging to the related border, a fold bifurcation – in which case the border fixed point disappears in a pair with another border fixed point –, or a BT bifurcation leading to an interior fixed point.

3.2 Stability regions of CP fixed points in the (ϕ_C, ϕ_D) -parameter plane

In all the simulations of the paper, the values of the parameters μ , σ , E , γ and L_b are fixed as

$$\mu = 0.7, \sigma = 3, E = 100, \gamma = 20, L_b = 400, \quad (24)$$

while the parameter L_s can take the following values:

$$L_s = 200, 300, 400.$$

Particular attention will be paid to the case $L_s = L_b$, where all the regions have equal size.

We now investigate how the stability of the CP fixed points depends on the parameters ϕ_C and ϕ_D .

First, in Fig.2 we show the stability regions of the CP fixed points bounded by the BT bifurcation curves $L_s = 200$. In the curves of Fig.2a) ‘ vs ’ refers to ‘vertical stability’ associated with $\lambda_v(x, y) = 1$, ‘ hs ’ means ‘horizontal stability’ associated with $\lambda_h(x, y) = 1$, and the numbers indicate the related CP fixed point. For example, the curve marked ‘ $vs00$ ’ corresponds to the condition $\lambda_v(0, 0) = 1$ that holds for $\Omega_f(0, 0) = 1$. A CP fixed point is stable if the parameter point is located on the right-hand side of both bifurcation curves, vs and hs , where the stability conditions are satisfied (see (20)-(23)). For the considered parameter values there are four regions corresponding to different combinations of the stable CP fixed points. Namely, in light-blue area all four CP fixed points are stable; in the dark-gray area, the fixed points CP_{00} , CP_{01} and CP_{10} are stable; in the light-gray area CP_{00} and CP_{01} are stable; and, finally, in the green and the yellow areas, only one CP fixed point, CP_{01} and CP_{00} respectively, is stable. The one-dimensional bifurcation diagram ϕ_D versus (x, y) shown in Fig.2b) corresponds to the parameter path for $\phi_C = 0.8$,

Fig. 2 In *a*): stability regions of the CP fixed points in the (ϕ_C, ϕ_D) -parameter plane; In *b*): 1D bifurcation diagram ϕ_D versus (x, y) for $\phi_C = 0.8$ related to the parameter path indicated in *a*) by a red arrow. Here $L_s = 200$ while all the other parameters are fixed as in (24).

$0.1 < \phi_D < 0.8$, indicated by a red arrow in Fig.2*a*). In Fig.2*b*), numbered red points indicate BT bifurcations, tick solid lines stable fixed points, while dashed and dotted lines correspond to saddle and repelling fixed points, respectively.

An example of the basins of four coexisting attracting CP fixed points is shown in Fig.3*a*), where $L_s = 200$, $\phi_C = 0.8$, $\phi_D = 0.1$. These basins are separated by the stable invariant sets of the saddle border fixed points BP_{0a} , BP_{1b} , BP_{c0} and BP_{d1} , whose origin is the repelling interior fixed point IP_{ef} . In Fig.3*b*) we show the 1D maps z_{x0} , z_{x1} , z_{0y} and z_{1y} defined in Appendix B (see (28)-(31), respectively), to which map Z is reduced to on the related invariant borders of I^2 . In particular, one can see that the fixed points CP_{00} and CP_{01} are both-side superstable, while CP_{10} and CP_{11} are superstable on one side.

When increasing ϕ_D , the fixed point CP_{11} loses its stability and becomes a saddle via a BT bifurcation (the point BP_{x1} merges with CP_{11} and at the bifurcation $\lambda_h(1, 1) = 1$, see the point (1) in Fig.2). After this bifurcation three attracting CP fixed points coexist, CP_{00} , CP_{01} and CP_{10} (see an example in Fig.4 where $\phi_D = 0.19$). If ϕ_D keeps on increasing, the fixed point IP_{xy} undergoes a BT bifurcation merging with the border point BP_{1y} (at this bifurcation $\Omega_f \cap \Omega_h = IP_{xy} = BP_{1y}$ and $\lambda_h(1, y) = 1$), and then a BT bifurcation occurs for point BP_{x0} ; at the same time the fixed point CP_{10} becomes a saddle (see the point (2) in Fig.2). After this bifurcation only two attracting CP fixed points coexist, CP_{00} and CP_{01} . Then, the point CP_{00} loses stability and becomes a saddle via a BT bifurcation of the fixed point BP_{0y} (see point (3) in Fig.2), after which only one attracting fixed point is left, namely, CP_{01} . The bifurcation structure of the white parameter region in Fig.2 is investigated later.

Fig. 3 In *a*): basins of four coexisting attracting CP fixed points for $\phi_C = 0.8$, $\phi_D = 0.1$; in *b*): 1D maps defined on the invariant borders of I^2 . Here $L_s = 200$ while all the other parameters are fixed as in (24).

Fig. 4 In *a*): basins of coexisting attracting fixed points CP_{00} , CP_{01} and CP_{10} for $\phi_C = 0.8$, $\phi_D = 0.19$; in *b*): 1D maps defined on the invariant borders of I^2 . Here $L_s = 200$ while all the other parameters are fixed as in (24).

To see how the stability regions of CP fixed points in the (ϕ_C, ϕ_D) -parameter plane depend on the parameter L_s , we show this structure for $L_s = 300$ in Fig.5*a*), and $L_s = 400$ in Fig.5*b*).

Note that the bifurcation structure shown in Fig.5*b*) is associated with the case $L_s = L_b$, where all four regions have equal size. In such a situation there is an invariant line $\{y = 1 - x\}$ in the (x, y) -phase plane. Moreover, any invariant set A of Z must be itself symmetric with respect to this line, or there must exist another invariant set A' which is symmetric to A . This property implies that the complete phase portrait of Z has to be symmetric with respect to the line $\{y = 1 - x\}$. In particular, the ‘vertical’ and the ‘horizontal’ stability conditions of the fixed point CP_{01} must coincide. The same reasoning applies to the fixed point CP_{10} . For the fixed points CP_{00} and CP_{11} , the ‘vertical’ and ‘horizontal’ stability conditions coincide in pair. In fact, it can be seen in Fig.5*b*) that $vs_{01} = hs_{01}$, $vs_{10} = hs_{10}$, $vs_{00} = vs_{11}$ and $hs_{00} = hs_{11}$.

Fig. 5 Stability regions of the CP fixed points in the (ϕ_C, ϕ_D) -parameter plane for $L_s = 300$ in a) and $L_s = 400$ in b). All the other parameters are fixed as in (24).

4 Economic Interpretation

4.1 Stability of CP equilibria and economic forces at work

In this section, we provide an economic interpretation to the stability results of the CP equilibria obtained in the previous section. Fig.2a) and Fig.5a), plotted for $L_s = 200$ and $L_s = 300$, show that starting from the right bottom corner and moving in the upward (indicated by the red arrow in Fig. 2a)) or in the leftward direction, the stability properties of CP equilibria change when varying ϕ_C – the trade freeness within integrated blocs – or ϕ_D – the trade freeness between integrated blocs. The corresponding patterns depend on the interplay between three effects:

- i) centrality: regions R_2 and R_3 enjoy a central position in the line-up of regions;
- ii) local market size: regions R_2 and R_4 may take advantage of a higher local demand. The first two effects represent two manifestations of the so-called “market access effect” – with firms taking advantage of locating closer to other markets or simply benefiting from a larger local demand –; and
- iii) competition: firms may benefit from locating away from competitors. In our context this latter effect is related to remoteness: regions R_1 and R_4 could be preferred by firms to regions R_2 and R_3 as a way to shelter from competition.²

Looking at Fig.2a) and Fig.5a), we notice that:

- 1) in the light-blue area at the right bottom corner in the (x, y) -phase plane – with a weak integration between blocs (low values of ϕ_D) and strong integration within them (high values of ϕ_C) –, all CP equilibria are stable;

² Centrality is mainly determined by ϕ_D but it is also affected by ϕ_C (by a joint effect with ϕ_D). The higher ϕ_D , the closer are regions R_2 and R_3 to each other and to regions to R_1 and R_4 as well. The smaller ϕ_C , the more remote regions R_3 and R_4 so that the relative centrality of regions R_2 and R_3 increases. The connection between the trade freeness and the competition effect is much less direct.

2) as we move in the dark-grey area by increasing ϕ_D or decreasing ϕ_C , CP_{11} loses stability so that simultaneous agglomeration in the two small regions becomes impossible. The intuition for this is that as we reduce ϕ_C , analogously to what happens in the two-region asymmetric footloose entrepreneur (FE) model,³ agglomeration in the small regions is more difficult, the industrial activity located in R_1 moves to R_2 (since stability is lost horizontally – i.e. in the direction of the share x –, the basin of attraction of CP_{11} becomes part of that of CP_{01} , as can be seen by comparing Fig.3 and Fig.4a), and the competition effect loses relevance. For region R_3 this effect is compensated by centrality: an increase in ϕ_D makes regions R_2 and R_3 closer to each other while R_3 does not lose the industrial sector;

3) as we further decrease ϕ_C or increase ϕ_D , moving in the light-grey region, CP_{10} also loses stability: agglomeration in the small remote region is no more possible. Market size still allows for agglomeration in the large remote region but not in the small remote region (notice that CP_{10} loses stability horizontally, i.e. in the direction of the share x , with the basin of attraction of CP_{00} incorporating that of CP_{10} : entrepreneurs move from R_1 to R_2 , the latter region being more central and larger than the former and the need to shelter from competition becoming less relevant);

4) when leaving the light-grey area, the direction in which we move (leftward or upwards) matters. More precisely, a substantial change in the qualitative behaviour of the system already occurs before leaving the light-gray area, at the value of ϕ_D corresponding to the intersection between the stability curves vs_{01} and vs_{00} . Below that value of ϕ_D any reduction of ϕ_C leads to the yellow area – where the only stable fixed point entails agglomeration in the bigger regions R_2 and R_4 – and to the prevailing of the local market size effect, as it occurs in the two-region asymmetric FE model. Instead, above that threshold, the prevailing force is centrality. Any movement leftward or upwards leads to the green area – where the only stable fixed point entails agglomeration in the central regions R_2 and R_3 . Interesting is the case of R_3 : for low values of ϕ_D , lowering ϕ_C as well could determine the loss of the industry. On the other hand, for sufficiently high values of ϕ_D , R_3 keeps the industrial sector notwithstanding the small size of the local market. In summary, there are two transition paths: 4a) for a sufficiently small value of ϕ_D , if we reduce ϕ_C , only CP_{00} is stable, therefore only agglomeration in the two large regions is possible: the size effect prevails (CP_{01} loses stability in the vertical direction and its basin becomes part of the basin of CP_{00} : entrepreneurs move from R_3 to R_4); 4b) whereas for a sufficiently high value of ϕ_D , if we reduce ϕ_C or increase ϕ_D , only CP_{01} is stable, therefore only agglomeration in the two central regions is possible: the centrality effect prevails (CP_{00} loses stability in the vertical

³ Indeed, with a small ϕ_D the two countries are (almost) separated, so that the standard results of the 2-R asymmetric FE model (see Baldwin et al., 2003, pp. 106-108) hold: high trade freeness within countries – a higher ϕ_C – implies that all CP equilibria are locally stable. As we reduce this trade freeness parameter, size gains relevance and the only stable CP equilibrium is the one characterized by agglomeration in the largest region.

direction and its basin becomes part of the basin of CP_{01} : entrepreneurs move from R_3 to R_4).

Notice that one can also envisage transition paths going in the opposite direction of those just described. For example a reduction in ϕ_D – as a consequence of, let’s say, a trade restriction policy – could determine a movement from the green to the light-grey area in Fig.2a) or in Fig.5a). In this case, the CP_{00} equilibrium – characterized by agglomeration in the two big regions R_2 and R_4 – gains local stability and its basin of attraction has a positive measure. Therefore, a relocation of industry from region R_3 to region R_4 is possible. The likelihood of this event increases as ϕ_D is further lowered. Indeed, moving in the dark-grey area the basin of attraction of CP_{01} – characterized by agglomeration in the two central regions R_2 and R_3 – shrinks further and that of CP_{00} enlarges. Moreover, the CP_{10} equilibrium – characterized by agglomeration in the two remote regions R_1 and R_4 – gains stability and its basin of attraction has a positive measure (see Fig.4a). Once reached, the equilibrium CP_{10} also implies a relocation of industry from region R_3 to R_4 . Finally, lowering even further ϕ_D , moving in the blue region, the basin of attraction of CP_{01} shrinks even more and those of CP_{00} and CP_{10} become larger: for R_3 the likelihood of losing the industry increases. In the blue region, also the equilibrium CP_{11} is stable and its basin of attraction has a positive measure (see Fig.3a). This equilibrium is characterized by agglomeration in the two small regions, R_1 and R_3 : a switch from CP_{01} to CP_{11} has no effect on the industrial concentration in R_3 , determining the relocation of industry from R_2 to R_1 .

In order to focus on the impact of centrality as the only dimension of market accessibility, we now briefly consider the case of regions of equal size. In Fig.5b), we set $L_s = L_b$, so that the size effect is nil. We see that centrality itself can be affected in two different ways: by altering the access to the “close market” (i.e. the market composed of the two regions belonging to the same integrated bloc) or to the “distant market” (i.e. the market composed of the two regions belonging to the other integrated bloc). Comparing Fig.5b) with Fig.2a) and Fig.5a) three major differences emerge: first, the yellow area disappears. This is a direct consequence of the assumption of regions of equal size so that no region can take advantage of a larger local market; second, the transition paths leading immediately outside the light-blue area are different:

i) moving leftward in the dark-grey region, the equilibrium CP_{10} loses stability⁴ so that simultaneous agglomeration in both remote regions is not possible. Due to the trade costs structure, as ϕ_C is reduced, regions R_1 and R_4 become more remote thus improving the relative centrality of regions R_2 and R_3 . This induces simultaneous entrepreneurial migration from remote to central regions. CP_{00} and CP_{11} remain stable longer as they involve a remote region in one integrated bloc being sufficiently close to the core of the other integrated bloc to sustain agglomeration;

⁴ This occurs both horizontally and vertically – i.e both in the x and in the y directions, so that the basin of attraction of CP_{01} is incorporated in the basin of CP_{10} : entrepreneurs move from R_1 to R_2 and from R_4 to R_3 .

ii) instead, moving upwards (or leftward, if ϕ_D is sufficiently high) in the light-grey area, the equilibria CP_{01} and CP_{10} are locally stable, so that agglomeration in the two central regions or in the two remote regions are possible outcomes. The increase in ϕ_D reduces the trade distance between central regions (leaving unaffected the distance between regions within integrated blocs). Since the reduction in ϕ_D is not too large, agglomeration in R_1 and R_4 – each region being far away from the distant market – shelters firms from competition while allowing them an easy access to the market within the integrated bloc where they are located – as ϕ_C is quite high. Instead CP_{00} and CP_{11} lose stability because of the stronger competition effect due to the proximity to a region – in the other integrated bloc – with agglomerated industry.⁵

iii) Finally both transition paths end up in the green region where only CP_{01} is stable due to the strength of the centrality effect.

Concluding this section, observe that, in all Figures, going further on the left in correspondence of low trade freeness (especially within integrated blocs), the interior equilibrium becomes stable and no CP equilibria (except in the Milnor sense) is an attractor (see the discussion below).

4.2 Factors remunerations

Above, we argued that the bifurcation scenario between multiple coexisting CP equilibria obtained in the simulations could be explained by economic forces. We now study factor remunerations (nominal/real wages and nominal/real profits) related to CP equilibria. In particular, we are interested in how these factor remunerations change with lower transport costs and how they change during transitions between two different CP equilibria. In what follows we introduce the analytical propositions concerning the main properties of these remunerations evaluated at the CP equilibria (see Propositions 1-3 below). We then interpret the results with the help of numerical simulations that are also used to highlight additional effects of ϕ_D and ϕ_C on price indices, nominal and real profits (see Figs 6 and 7 below).

Real wages of workers are given by $W_r = 1/P_r^\mu$. As price indices can be conveniently rewritten in terms of Δ_r (as given by the expression $P_{r,t} = p(E/2)^{\frac{1}{1-\sigma}} \Delta_{r,t}^{\frac{1}{1-\sigma}}$), it appears that any change in trade cost is channeled through Δ_r : an increase in Δ_r reduces the price index and thus increases real wages (indirect utility) for workers and real profits for entrepreneurs. For CP equilibria, analytical expressions of Δ_r can be obtained as follows:

First we look at how the various CP equilibria affect the real wage of workers.

⁵ With no size effect CP_{00} and CP_{11} are symmetric and lose stability simultaneously as follows: the basin of attraction of CP_{00} and CP_{11} are simultaneously incorporated in the basin of attraction of CP_{01} as CP_{00} loses vertical stability (in the y direction) and CP_{00} horizontal stability (in the x direction): entrepreneurs move from R_1 to R_2 and from R_4 to R_3 .

	CP_{00}	CP_{01}	CP_{10}	CP_{11}
Δ_1	$\phi_C (\phi_C \phi_D + 1)$	$\phi_C (\phi_D + 1)$	$\phi_C^2 \phi_D + 1$	$\phi_C \phi_D + 1$
Δ_2	$\phi_C \phi_D + 1$	$\phi_D + 1$	$\phi_C (\phi_D + 1)$	$\phi_C + \phi_D$
Δ_3	$\phi_C + \phi_D$	$\phi_D + 1$	$\phi_C (\phi_D + 1)$	$\phi_C \phi_D + 1$
Δ_4	$\phi_C \phi_D + 1$	$\phi_C (\phi_D + 1)$	$\phi_C^2 \phi_D + 1$	$\phi_C (\phi_C \phi_D + 1)$

Table 1 Expressions of Δ_r for CP equilibria

Proposition 1 (i) CP equilibria are ranked by workers.

(ii) In a CP equilibrium, an increase in ϕ_C or ϕ_D leads to higher workers' real wages in all regions.

Proof (i) Since $\phi_C (\phi_C \phi_D + 1) < \phi_C (\phi_D + 1) < \phi_C^2 \phi_D + 1 < \phi_C \phi_D + 1$, in region 1 $P_1^{CP_{11}} < P_1^{CP_{10}} < P_1^{CP_{01}} < P_1^{CP_{00}}$ holds, and in region 4 $P_4^{CP_{00}} < P_4^{CP_{10}} < P_4^{CP_{01}} < P_4^{CP_{11}}$. Since $\phi_C (\phi_D + 1) < \phi_C + \phi_D < \phi_C \phi_D + 1 < \phi_D + 1$, in region 2 $P_2^{CP_{01}} < P_2^{CP_{00}} < P_2^{CP_{11}} < P_2^{CP_{10}}$ holds, and in region 3, $P_3^{CP_{01}} < P_3^{CP_{11}} < P_3^{CP_{00}} < P_3^{CP_{10}}$. (ii) As all Δ_r depend positively upon trade freeness, a reduction in either trade cost ϕ_C or ϕ_D unambiguously decreases price indices, thus increasing real wages W_r in all regions.

We interpret the ranking of CP fixed points by workers in Proposition 1 as follows. According to Part (i), the price index is the lowest, if the region under consideration has the industry agglomeration and if there is industry in the closest region of the other bloc as well; the price index increases, if the region under consideration keeps the agglomeration, but industry is no longer in the closest region of the other bloc, but in the more distant one; the price index further increases if the region under consideration loses the industry, but industry is located in the closest region of the other bloc; finally, the price index is the highest, if neither the region under consideration nor the closest region in the other bloc has industry. Part (ii) of the Proposition means that in a CP fixed point, trade liberalization, both within and between integrated blocs, is beneficial for all workers.⁶

Entrepreneurs also enjoy a lower price index. However, their real remuneration depends upon both nominal profits and the price index. Analytical expressions for nominal profits can be derived. We now summarize in the following Proposition how trade openness affect nominal and real profits depending on the agglomeration pattern.

Proposition 2 (i) When agglomerations occur in both central regions (i.e. in the CP_{01} equilibrium), an increase in either freeness of trade ϕ_C or ϕ_D leads to higher real profits of entrepreneurs in both central regions.

(ii) When agglomerations occur in a central region and a remote region (i.e. in the CP_{00} and CP_{11} equilibria), an increase in ϕ_D leads to higher nominal and real profits of entrepreneurs in the central region and to lower nominal profits in the remote region if $\phi_D < \sqrt{(\sigma - \mu)/(\sigma + \mu)}$.

⁶ In CP_{01} , an increase in ϕ_C has no effect on price indices in R_2 and R_3 . This is the only exception.

(iii) When agglomerations occur in both remote regions (i.e. in the CP_{10} equilibrium), an increase in ϕ_D leads to higher nominal and real profits of entrepreneurs in the large remote region when $\phi_D < \sqrt{(\sigma - \mu)/(\sigma + \mu)}/\phi_C$.

(iv) When agglomeration occurs in the small remote region (i.e. in the CP_{10} and CP_{11} equilibria), an increase in ϕ_C leads to higher nominal and real profits in that region. When agglomeration occurs in the large remote region (e.g. as in CP_{00} or CP_{10}), an increase in ϕ_C leads to higher nominal and real profits in that region if firms in the other country agglomerate in the large central region. Otherwise (that is if firms in the other country agglomerate in the small remote region), nominal profits in the large remote region decrease with ϕ_C .

(v) When agglomeration occurs in only one central region (e.g. in CP_{00} or CP_{11}), an increase in ϕ_C leads to lower nominal profits in that region. When agglomeration occurs in both central regions (i.e. in CP_{01}), nominal profits are insensitive to changes in ϕ_C or ϕ_D .

Proof (i) It can be shown that $\pi_3^{CP01} = \pi_2^{CP01} = 2\mu(L_b + L_s)/E/(\sigma - \mu)$. Therefore, lower price indices lead to higher real profits.

(ii) In the CP_{00} equilibrium, nominal profits in region 2 are given by

$$\pi_2^{CP00} = \frac{2\mu S_2^{CP00}}{E(\sigma - \mu)(\phi_C + \phi_D)(\sigma - \mu + (\sigma + \mu)\phi_C\phi_D)}$$

where

$$\begin{aligned} S_2^{CP00} = & \phi_C(\sigma + \mu)(L_b + L_s)\phi_D^2 + \\ & + (L_b(\sigma - \mu + (\sigma + \mu)\phi_C^2) + 2L_s(\mu\phi_C^2 - \mu + \sigma))\phi_D + \\ & + \phi_C(\sigma - \mu)(L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_2^{CP00}}{\partial \phi_C} < 0$, and $\frac{\partial \pi_2^{CP00}}{\partial \phi_D} > 0$ for $\phi_D < \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$. Moreover, we have

$$\pi_4^{CP00} = \frac{2\mu S_4^{CP00}}{E(\sigma - \mu)(\phi_C + \phi_D)(\sigma - \mu + (\sigma + \mu)\phi_C\phi_D)}$$

where

$$\begin{aligned} S_4^{CP00} = & \phi_C(\sigma + \mu)(L_b + L_s)\phi_D^2 + \\ & + (L_b(\sigma - \mu + (\sigma + \mu)\phi_C^2) + 2L_s\sigma\phi_C^2)\phi_D + \\ & + \phi_C(\sigma - \mu)(L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_4^{CP00}}{\partial \phi_C} > 0$, and $\frac{\partial \pi_4^{CP00}}{\partial \phi_D} < 0$ for $\phi_D < \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$.

Similarly, in CP_{11} , nominal profits in region 3 are given by

$$\pi_3^{CP11} = \frac{2\mu S_3^{CP11}}{E(\sigma - \mu)(\phi_C + \phi_D)(\sigma - \mu + (\sigma + \mu)\phi_C\phi_D)}$$

where

$$\begin{aligned} S_3^{CP_{11}} = & \phi_C (\sigma + \mu) (L_b + L_s) \phi_D^2 + \\ & + (L_s (\sigma - \mu + (\sigma + \mu) \phi_C^2) + 2L_b (\mu \phi_C^2 - \mu + \sigma)) \phi_D + \\ & + \phi_C (\sigma - \mu) (L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_3^{CP_{11}}}{\partial \phi_C} < 0$, and $\frac{\partial \pi_3^{CP_{11}}}{\partial \phi_D} > 0$ for $\phi_D < \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$. Finally, we also have

$$\pi_1^{CP_{11}} = \frac{2\mu S_1^{CP_{11}}}{E (\sigma - \mu) (\phi_C + \phi_D) (\sigma - \mu + (\sigma + \mu) \phi_C \phi_D)}$$

where

$$\begin{aligned} S_1^{CP_{11}} = & \phi_C (\sigma + \mu) (L_b + L_s) \phi_D^2 + \\ & + (L_s (\sigma - \mu + (\sigma + \mu) \phi_C^2) + 2L_b \sigma \phi_C^2) \phi_D + \\ & + \phi_C (\sigma - \mu) (L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_1^{CP_{11}}}{\partial \phi_C} > 0$, and $\frac{\partial \pi_1^{CP_{11}}}{\partial \phi_D} < 0$ for $\phi_D < \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$.

(iii) In the CP_{10} equilibrium, nominal profits in region 4 are given by

$$\pi_4^{CP_{10}} = \frac{2\mu S_4^{CP_{10}}}{E (\sigma - \mu) (\phi_D + 1) (\sigma - \mu + (\sigma + \mu) \phi_C^2 \phi_D)}$$

where

$$\begin{aligned} S_4^{CP_{10}} = & \phi_C^2 (\sigma + \mu) (L_b + L_s) \phi_D^2 + \\ & + 2 (L_b (\sigma + \mu (\phi_C^2 - 1)) + L_s \sigma \phi_C^2) \phi_D + \\ & + (\sigma - \mu) (L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_4^{CP_{10}}}{\partial \phi_C} < 0$, and $\frac{\partial \pi_4^{CP_{10}}}{\partial \phi_D} > 0$ for $\phi_D < \frac{1}{\phi_C} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$

(iv) In the CP_{10} equilibrium, nominal profits in region 1 are given by

$$\pi_1^{CP_{10}} = \frac{2\mu S_1^{CP_{10}}}{E (\sigma - \mu) (\phi_D + 1) (\sigma - \mu + (\sigma + \mu) \phi_C^2 \phi_D)}$$

where

$$\begin{aligned} S_1^{CP_{10}} = & \phi_C^2 (\sigma + \mu) (L_b + L_s) \phi_D^2 + \\ & + (2L_s (\sigma + \mu (\phi_C^2 - 1)) + 2L_b \sigma \phi_C^2) \phi_D + \\ & + (\sigma - \mu) (L_b + L_s) \end{aligned}$$

with $\frac{\partial \pi_1^{CP_{10}}}{\partial \phi_C} > 0$, and $\frac{\partial \pi_1^{CP_{10}}}{\partial \phi_D} < 0$ for $\phi_D < \frac{1}{\phi_C} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$. In CP_{11} , nominal profits in region 1 are such that $\frac{\partial \pi_1^{CP_{11}}}{\partial \phi_C} > 0$ as shown already in part (ii).

(v) From parts (i) and (ii), we have $\frac{\partial \pi_2^{CP_{00}}}{\partial \phi_C} < 0$, $\frac{\partial \pi_3^{CP_{11}}}{\partial \phi_C} < 0$, and $\pi_3^{CP_{01}} = \pi_2^{CP_{01}}$ depend neither on ϕ_D nor on ϕ_C .

We now compare how nominal profits in a region change during a transition from one CP equilibrium to another.

Proposition 3 (i) *The nominal profit in a central region or in the large remote region is larger when firms in the other country agglomerate further away.*

(ii) *The nominal profit in the small remote region is larger when firms in the other country agglomerate further away if ϕ_C is sufficiently large or if ϕ_D is sufficiently small.*

Proof (i) It can be shown that $\pi_2^{CP00} > \pi_2^{CP01}$, $\pi_3^{CP11} > \pi_3^{CP01}$, and $\pi_4^{CP10} > \pi_4^{CP00}$. (ii) $\pi_1^{CP10} > \pi_1^{CP11}$ as long as $A\phi_D^2 + B\phi_D + C > 0$ where $A = \phi_C(\sigma + \mu)[(1 - \phi_C)L_b - L_s]$, $B = -L_s[\sigma - \mu + \phi_C^2(\sigma + \mu)] < 0$ and $C = -(\sigma - \mu)[L_b(1 - \phi_C) + L_s\phi_C] < 0$. The sign of A is indeterminate. If $\phi_C > \frac{L_b - L_s}{L_b}$, A is negative, then $\pi_1^{CP11} - \pi_1^{CP10} < 0$ always. If $\phi_C < \frac{L_b - L_s}{L_b}$ then A is positive and $\pi_1^{CP11} - \pi_1^{CP10} < 0$ for $\phi_D < \widetilde{\phi}_D = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$ where $\widetilde{\phi}_D$ is real, positive and the larger root in absolute value of the equation $A\phi_D^2 + B\phi_D + C = 0$. $\phi_D < \widetilde{\phi}_D$ is a necessary and sufficient condition in order to have $\pi_1^{CP11} - \pi_1^{CP10} < 0$. Maybe, it could be of some interest to show that $\phi_D < \sqrt{\frac{1}{\phi_C}} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$ is a sufficient condition. We proceed as follows: notice first that for $B = 0$, $\widetilde{\phi}_D = \frac{\sqrt{-4AC}}{2A}$ and for $B < 0$, $\widetilde{\phi}_D = \frac{-B + \sqrt{B^2 - 4AC}}{2A} > \frac{\sqrt{-4AC}}{2A}$. Then consider that $\frac{\sqrt{-4AC}}{2A}$ decreases continuously with L_s and at $L_s = 0$ $\frac{\sqrt{-4AC}}{2A} = \sqrt{\frac{1}{\phi_C}} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$.⁷ Therefore for $L_s > 0$ (that gives $B < 0$) $\widetilde{\phi}_D$ is always larger than $\sqrt{\frac{1}{\phi_C}} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$ and it is equal to it when $L_s > 0$ (that gives $B < 0$). In summary, $\pi_1^{CP11} - \pi_1^{CP10} < 0$ if and only if $\phi_C > \frac{L_b - L_s}{L_b}$ or $\phi_C < \frac{L_b - L_s}{L_b}$ and $\phi_D < \widetilde{\phi}_D$ or if $\phi_D < \sqrt{\frac{1}{\phi_C}} \sqrt{\frac{\sigma - \mu}{\sigma + \mu}}$.

The effects outlined in Propositions 1-3 are now illustrated by numerical simulations, see Figs 6 and 7. In these Figures, for each region, the first two panels show price indices and nominal profits as a function of ϕ_D (in Fig.6) or ϕ_C (in Fig.7). In the Figures, these variables are plotted as long as the corresponding equilibrium is locally stable. While the real wage of workers corresponds to the inverse of the price index, the real profit of entrepreneurs can be obtained as the ratio of the nominal profit and the price index. Specific colors refer to the various CP equilibria (red relates to CP_{00} , blue to CP_{01} , green to CP_{10} , and pink to CP_{11}). We consider $L_s = 200$ and set the other parameters as in the previous simulations.⁸

⁷ We could have checked directly the sign of the derivative $\partial\widetilde{\phi}_D/\partial L_s$, however that was not possible to evaluate given the length of the corresponding expression.

⁸ These are: $L_b = 400$, $E = 100$, $\beta = 1$, $\mu = 0.7$ and $\sigma = 3$. The value of $\sigma = 3$ implies a mark-up $\frac{\sigma}{\sigma-1} = 1.5$. In Fig.6 the value of ϕ_C is set to 0.9 and ϕ_D varies from 0.1 to 0.8.

With these parameters, the trade costs are around the following values: $T_C = \phi_C^{\frac{1}{1-\sigma}} = 1.118$ for $\phi_C = 0.8$, $T_D = \phi_D^{\frac{1}{1-\sigma}} = 3.162$ for $\phi_D = 0.1$ and $T_D = 1.118$ for $\phi_D = 0.8$; and in Fig.7

Fig. 6 Abscissa: ϕ_D is varied from 0.1 to 0.8. Ordinate: three different welfare indicators. Different colours refer to different distribution of entrepreneurs: Red CP_{00} ; blue CP_{01} , green CP_{10} , pink CP_{11} . Parameters setting: $L_s = 200$, $L_b = 400$, $\mu = 0.7$, $\sigma = 3$, $\phi_C = 0.8$, $\beta = 1$, $E = 100$.

Proposition 1 can be illustrated as follows. Considering region 1, the first frame in Figs 6 and 7 show that: the price index is the lowest along the pink line, where the CP_{11} equilibrium prevails (with agglomeration in R_1 and R_3); the second lowest along the green line, where the CP_{10} equilibrium prevails (with agglomeration in R_1 and R_4); the second highest along the blue line, where the CP_{01} equilibrium prevails (with agglomeration in R_2 and R_3); and, finally, the highest along the red line, where the CP_{00} equilibrium prevails (with agglomeration in R_2 and R_4). Corresponding results for the other regions are illustrated in Figs 6 and 7. We conclude that workers closer to the industry – located in both integrated blocs – incur smaller trade costs, thus enjoying smaller price indices and in turn higher real wages. The opposite occurs to workers who are more distant to the industry: because of higher commodity prices due to higher trade costs, their real wages are lower. The beneficial

the value of ϕ_D is set to 0.1 and ϕ_C varies from 0.2 to 0.8. With these parameters the trade costs are: $T_D = 3.162$ for $\phi_D = 0.1$, $T_C = 2.236$ for $\phi_C = 0.2$ and $T_C = 1.118$ for $\phi_C = 0.8$. We consider reasonable these values.

Fig. 7 Abscissa: ϕ_C is varied from 0.2 to 0.8. Ordinate: three different welfare indicators. Different colours refer to different distribution of entrepreneurs: Red CP_{00} ; blue CP_{01} , green CP_{10} , pink CP_{11} . Parameters setting: $L_s = 200$, $L_b = 400$, $\mu = 0.7$, $\sigma = 3$, $\phi_D = 0.1$, $\beta = 1$, $E = 100$.

effect of lower price indices – due to the proximity to manufactured commodity markets – on workers is magnified by the increase in either trade freeness.

Proposition 2 refers to the effect of both types of trade freeness on regional profits in each CP equilibria. Parts (i)-(iii) are mainly concerned with the effects of ϕ_D on nominal profits: these are illustrated in Fig.6. Consider the case where industry is agglomerated in central regions (as in the equilibria CP_{00} , CP_{01} and CP_{10} – represented in Fig.6 by the red, blue and green lines): if ϕ_D is not too high, by increasing this trade freeness, nominal profits are stationary or rise. This result illustrates the effect of centrality. On the other hand, when industry agglomeration occurs in remote regions (as in CP_{00} , CP_{10} and CP_{11} – represented by the red, green and pink lines in Fig.6), nominal profits only increase in the large remote region – under the effect of a bigger market size – when industry is agglomerated in both remote regions (that is, in the CP_{11} equilibrium – represented by the pink line in Fig.6) and decrease in all other cases. When ϕ_D is sufficiently high, especially referring to the equilibria CP_{00} , CP_{01} and CP_{11} , some of these effects could be reversed. However, as

confirmed by numerical simulations (see Fig.6 as an example), this occurs when these equilibria are no longer stable.

Parts (iv)-(iv) deal with the effects of ϕ_C on nominal profits: these are illustrated in Fig.7. A change in ϕ_C has a non ambiguous effect on nominal profits in each region. First consider the CP_{11} equilibrium with agglomeration in R_1 and in R_3 – the two small regions –, as represented by the pink line in Fig.7. Here, when ϕ_C increases, profits increase in R_1 and decrease in R_3 . This is due to the fact that R_1 is becoming less remote and R_3 more subject to competition originating from R_1 . Consider now the CP_{10} equilibrium with agglomeration in R_1 and R_4 – the two remote regions –, as represented by the green line in Fig.7. In this equilibrium, profits are increasing in ϕ_C in R_1 and are decreasing in R_4 : the size effect loses relevance with a stronger integration within trade blocs. Consider now the CP_{01} equilibrium with agglomeration in R_2 and R_3 – the two central regions –, as represented by the blue line in Fig.7. Here (as shown in the proof of Proposition 2), profits do not depend on trade costs. Obviously, centrality does not play any role (both regions are equally central), whereas market size affects profits only via the sum $(L_b + L_s)$, that is, via the relative size of the workers population, the local immobile demand.⁹ Finally consider the CP_{00} equilibrium with agglomeration in R_2 and R_4 – the two large regions –, as represented by the red line in Fig.7. Analogously to the case of agglomeration in the two small regions, market size does not play any role. An increase in ϕ_C favours R_4 relative to R_3 because the former region becomes less remote and the latter one more subject to competition.

A consequence of Proposition 1 Part (ii) on the effects ϕ_D and ϕ_C on price indices and of Proposition 2 Parts (i-v) on the effects of ϕ_D and ϕ_C on nominal profits is that when nominal profits rise due to an increase in either trade freeness, real profits increase as well. In addition when nominal profits decrease with ϕ_D in some region, numerical simulations suggest that real profits still increase with ϕ_D .¹⁰ In other words, the price index effect always dominates any decrease in nominal profits so that real profits do increase with trade liberalization between integrated blocs.¹¹

According to Proposition 3, nominal profits are higher in a region where industry agglomerates if industry in the other bloc agglomerates in the more distant region. Profits in a region are lower when the industry in the other bloc agglomerates in the closer region as a consequence of stronger competition. This result always holds except when industry is agglomerated in the

⁹ The size effect, via a change in the trade freeness within integrated blocs (ϕ_C), could still play a role by affecting the local stability properties of the CP_{01} equilibrium. Indeed, a sufficient reduction of ϕ_C , if ϕ_D is not too high, could determine a loss of stability of CP_{01} in the vertical direction, i.e. in the direction of the share y (see Figs 2a) and 5a)), and a relocation of the industrial sector from R_3 to R_4 , as CP_{00} becomes the only stable equilibrium.

¹⁰ These simulations, not presented here, are used in substitution of analytical results that cannot be obtained for these cases.

¹¹ Regarding the effect of ϕ_C , we found some numerical simulations where trade liberalization within trade blocs induce nominal and real profits to move in the same direction notwithstanding the counterbalancing of the price index effect.

small remote region and ϕ_C is sufficiently small and ϕ_D is sufficiently large.¹² However, as confirmed by numerical simulations (see for example Fig.7), this latter scenario does not occur while the respective equilibrium is still stable.

4.3 Effect on transit traffic

The geographical line-up that we have chosen allows to shed light on another issue, namely transit traffic, i.e. the transport of commodities that passes by some intermediary region before reaching its final destination. Because of the often involved environmental burden, transit traffic is a hotly debated issue not only in the popular press, but also in policy consulting reports and in academic studies (see e.g. Marletto, 2010; Sutton, 2013; Enchelmaier, 2013).

Very often, a CGE framework is used (see e.g. Steininger, 2002 and for surveys Steininger, 1999; Böhringer and Löschel, 2006) to analyze whether policy instruments that change transport costs can affect the modal split (i.e. the transport shift to less polluting modes of transportation) and whether these instruments can reduce trade and transport volumes. However, less attention is paid to the location choice of individuals and firms that can potentially affect traffic patterns (see Grazi and van den Bergh, 2008; see also the interesting study by Bednar-Freidl *et al.*, 2011, that focuses on residential location choice and commuting traffic, but not on firm location choices or commodity transport). In this modeling framework, the response of transport volumes to policy instruments is smooth.

In contrast, the NEG approach allows for explicit firm location choices. As such, the NEG may naturally complement the result of CGE studies. Here we provide first insights into these issues.

Our model allows to analyze the impact of economic integration that affects transport costs and in turn transit traffic. For this purpose, trade flows are derived by using the matrices Φ and G introduced in Section 2:

$$D = \frac{\mu}{p} G \Phi Y, \quad (25)$$

where the element D_{rs} denotes the quantity produced in region r and sold in region s and Y denotes the regional income levels

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = L + \frac{E}{2} G \pi. \quad (26)$$

¹² A possible explanation is that as ϕ_D becomes smaller competition from the central region in the other bloc reduces its intensity; whereas as ϕ_C becomes larger the competition from the larger region in the other bloc increases its intensity.

Given the specific geographical line-up of regions, transit traffic generated by indirect trade can be written as

$$\mathbf{Trans} = \begin{pmatrix} 0 \\ D_{13} + D_{14} + D_{31} + D_{41} \\ D_{14} + D_{24} + D_{41} + D_{42} \\ 0 \end{pmatrix}. \quad (27)$$

The first and fourth entries of $\mathbf{Trans}$ are necessarily equal to zero because no traffic passes through regions 1 and 4 given the line-up of regions. The second and third entries account for the traffic passing through region 2 and 3 respectively.

Given the complex structure of our model, no analytical results regarding transit traffic are available. Figs 6 and 7 illustrate the effect of ϕ_D and ϕ_C on transit traffic by relying on numerical simulations. Our findings suggest the following regularities:

(i) In every CP equilibrium, reducing the trade freeness (i.e. increasing transport costs) increases price indices and thus reduces trade volumes and transit traffic. The response of transit traffic is continuous: an increase in trade costs smoothly reduces transit traffic.

(ii) There is a clear ordering of CP equilibria: A region (R_2 or R_3) is less exposed to transit traffic, if it has attracted all industry within the bloc and the industry in the other bloc is distant. If a region has no industry, it is still better off, when the industry in the other bloc is distant. The worst possible case is when the region has no industry and is located in between two Cores.

(iii) Finally, given the different parameter ranges for which each CP equilibrium is stable, discontinuous jumps in transit traffic are possible: Once trade freeness crosses a threshold, a sudden change in the long run regional distribution of industrial activity may happen and a region may lose its industrial agglomeration. In that case, transit traffic jumps upwards in a discontinuous way. This kind of result is only possible in models exhibiting multiple equilibria such as those of the NEG.

This discussion shows that neglecting firm location choice may actually mislead policy advice: An increase in trade costs may initially smoothly reduce transit traffic; however, if a threshold is crossed, a further increase in trade costs may trigger industry relocation and in turn upward jumps – as opposed to smooth reductions – in transit traffic.

5 Complex dynamics

Let us investigate now the complete bifurcation structure of (ϕ_C, ϕ_D) -parameter plane. The parameter region which is left uncolored in Figs 2a) and 5 is associated with more complex dynamics. As we show below, for parameter values belonging to this region map Z can have various attractors, such as a border fixed point, an interior fixed point, a cycle of any period (belonging to a border

or an interior cycle), cyclic closed invariant curves, as well as chaotic attractors, moreover, some of these attractors can coexist (see, e.g. Mira *et al.*, 1996, where properties of 2D noninvertible maps are studied, and Wiggins, 2003, for a general bifurcation theory).

A peculiarity of map Z is related to the fact that besides standard attractors this map can have attractors in Milnor sense (see Milnor, 1985), which are caused by the ‘flat’ branches of Z . Recall that (1) according to the most spread definition, an attractor is a closed invariant set with a dense orbit, which has a *neighborhood each point of which is attracted to the attractor*¹³, while (2) an attractor in Milnor sense does not require existence of such a neighborhood, but only a *positive measure set of points attracted to the attractor*.¹⁴ In fact, definition (2) of an attractor obviously includes also attractors with an attracting neighborhood, as in definition (1). For short we say that a set is *M-attractor* if it is attracting in Milnor sense, *but not in a sense of the definition (1)*. In such a way, an *M-attractor* has no an attracting neighborhood, but it has a basin of attraction of positive measure. For example, a CP fixed point which is locally repelling can be an *M-attractor*, with quite a large basin of attraction.

In Fig.8a) we present an enlargement of the window indicated in Fig.5a) where regions related to attracting cycles of different periods $n \leq 23$ are shown by different colors (periods of the largest regions are indicated by numbers), white region is associated either with cycles of higher periods, or with chaotic attractors, pink and red regions correspond to attracting border and interior fixed points, respectively, and dark gray region is related to *M-attracting* CP fixed points denoted CP_M . To clarify which bifurcations occur at the transitions from one area to another we show in Fig.8b) a 1D bifurcation diagram ϕ_C versus (x, y) (note that values of ϕ_C are shown decreasing) for fixed $\phi_D = 0.1$, which corresponds to the parameter path indicated by the horizontal arrow in Fig.8a). Given that only one initial point is used to produce Fig.8a), coexisting attractors cannot be seen in this figures. To clarify this issue we show below several examples of basins of coexisting attractors.

Let us first comment on the bifurcations occurring for parameter values indicated by numbers (1)-(6) in Fig.8. For decreasing ϕ_C first, at $\phi_C \approx 0.2295$, a BT bifurcation of the saddle fixed point CP_{10} occurs in the vertical direction (the related parameter point is indicated by (1)) at which CP_{10} becomes repelling merging with a repelling border fixed point BP_{1y} which is born a bit before, at $\phi_C \approx 0.2389$, due to a fold bifurcation (see the black circle in Fig.8b)) in a pair with a saddle border fixed point BP'_{1y} . Then at $\phi_C \approx 0.1962$ the attracting fixed point CP_{00} becomes locally a saddle via a BT bifurca-

¹³ More precisely, an *attracting set* A of a map f is a closed invariant set for which a neighborhood $U(A)$ exists such that $f(U(A)) \subset U(A)$ and $f^n(x) \rightarrow A$ as $n \rightarrow \infty$ for *any* $x \in U(A)$. An *attractor* A is an attracting set with a dense orbit.

¹⁴ More precisely, a *Milnor attractor* is a closed invariant set $A \subset J$ such that the set $\rho(A)$, consisting of all the points $x \in J$ for which ω -limit set $\omega(x) \subset A$, has *strictly positive measure*, and there is no strictly smaller closed subset A' of A such that $\rho(A')$ coincides with $\rho(A)$ up to a set of measure zero.

Fig. 8 In *a*): bifurcation structure of window indicated in Fig.5*a*); In *b*): bifurcation diagram ϕ_C versus (x, y) for $\phi_D = 0.1$ related to the parameter path indicated in *a*) by an arrow. Note that values of ϕ_C are shown decreasing. Here $L_s = 300$ while all the other parameters are fixed as in (24).

tion in vertical direction (in Fig.8 this bifurcation is marked by (2)) that leads to the appearance of an attracting border fixed point BP_{0y} . If we continue to decrease ϕ_C , at $\phi_C \approx 0.173$ the saddle fixed point CP_{01} undergoes a BT bifurcation in the horizontal direction (see the point (3) in Fig.8) and becomes repelling leading to the appearance of a saddle border fixed point BP_{x1} . Then at $\phi_C \approx 0.1659$ the attracting border fixed point BT_{0y} undergoes a BT bifurcation in the horizontal direction (see the point (4)) leading to an attracting interior fixed point IP_{xy} . A BT bifurcation of CP_{00} in horizontal direction occurring at $\phi_C \approx 0.1534$ (see the point (5)) transforms the saddle fixed point CP_{00} into a repelling one and gives birth to a saddle border fixed point BP_{x0} . First period-doubling bifurcations of the border and interior fixed points are indicated in Fig.8 by gray circles. Period-doubling bifurcation of the interior fixed point occurs at $\phi_C \approx 0.1481$. Each border 2-cycle undergoes a cascade of period-doubling bifurcations leading to chaos, while an interior 2-cycle undergoes a Neimark-Sacker bifurcation resulting in appearance of 2-cyclic attracting closed invariant curves (we show below an example of such an attractor). Orange circles in Fig.8 indicate homoclinic bifurcations leading to disappearance of chaotic attractors on the related borders of I^2 , as well as disappearance of an interior chaotic attractor.

Let us clarify the described above bifurcation sequence presenting several examples of basins of coexisting attractors.

For $0.1962 \lesssim \phi_C \lesssim 0.2295$ (see the range of ϕ_C between the points (2) and (1) in Fig.8) map Z has a unique attractor which is fixed point CP_{00} , while for $0.1659 \lesssim \phi_C \lesssim 0.1962$ (see the range between the point (4) and (2)) map Z has coexisting attracting fixed point BP_{0a} and M -attracting fixed point CP_{00} : Fig.9*a*) shows basins of attraction of these fixed points at $\phi_C = 0.196$, and in Fig.9*b*) the 1D maps defined on the related borders of I^2 are presented. It becomes clear why fixed point CP_{00} is an M -attractor if one looks at the graph of map z_{0y} defined on border I_{0y} : even if locally the fixed point $y = 0$ is

Fig. 9 In *a*): coexisting attracting fixed point BP_{0a} (with blue basin) and M -attracting fixed point CP_{00} (with red basin) for $\phi_C = 0.196$, $\phi_D = 0.1$; in *b*): 1D maps defined on the invariant borders of I^2 . Here $L_s = 300$ while all the other parameters are fixed as in (24).

Fig. 10 In *a*): coexisting attracting fixed point IP_{ef} (with orange basin), M -attracting fixed points CP_{00} and BP_{0a} (with red and blue basins, respectively) for $\phi_C = 0.159$, $\phi_D = 0.1$; in *b*): 1D maps defined on the invariant borders of I^2 . Here $L_s = 300$ while all the other parameters are fixed as in (24).

already repelling and initial points from its right neighborhood are attracted to fixed point $y = a$, it has quite a large basin of attraction shown in Fig.9b) in red. Indeed, any point of the interval associated with the flat branch of z_{0y} , as well as any its preimage, is mapped into $y = 0$ in a finite number of steps. Respectively, initial points belonging to the red islands Fig.9a) are mapped into CP_{00} in a finite number of steps (in particular, points from the largest red island are mapped into CP_{00} in one step).

For $0.1534 \lesssim \phi_C \lesssim 0.1659$ (see the range between the point (5) and (4) in Fig.8) map Z has coexisting attracting fixed point IP_{ef} and two M -attracting fixed points, CP_{00} and BP_{0a} . An example of their basins is shown in Fig.10 where $\phi_C = 0.159$. Note that map z_{1y} has an attracting 2-cycle which for map Z is a period-2 saddle cycle.

For $0.1481 \lesssim \phi_C \lesssim 0.1534$ (see the range between the point (6) and (5) in Fig.8) map Z has coexisting attracting fixed point IP_{ef} and three M -

attracting fixed points, CP_{00} , BP_{0a} and BP_{c0} . An example of their basins is shown in Fig.11a) where $\phi_C = 0.15$.

As already mentioned, for decreasing ϕ_C the fixed point IP_{ef} undergoes a period-doubling bifurcation leading to an interior 2-cycle which then undergoes the Neimark-Sacker bifurcation. As a result of this bifurcation 2-cyclic attracting closed invariant curves are born. Fig.11b) presents an enlarged part of the basin of such an attractor for $\phi_C = 0.1398$, as well as basins of coexisting M -attracting fixed points CP_{00} , BP_{c0} and M -attracting 2-cycle $\{C_{0a1}, C_{0a2}\}$ belonging to the border I_{0y} . If we continue to decrease ϕ_C 2-cyclic closed invariant curves are transformed in a chaotic attractor. An example of 2-cyclic chaotic attractor is shown in Fig.12 for $\phi_C = 0.1375$, which is near its final bifurcation occurring due to a contact of the attractor with its basin boundary.

Also the bifurcation sequence detected in Fig.8 allows for a neat economic interpretation that complements the discussion on the role of centrality and local market size initiated in Section 4. In that section, for low values of ϕ_D , we detected a sequence starting with the simultaneous stability of all four possible CP equilibria for high values of ϕ_C and ending with CP_{00} (i.e. agglomeration in the big regions), as unique stable equilibrium. Here, when reducing ϕ_C further, CP_{00} loses vertically local stability, but it gains global stability as an M -attractor, and the coexisting stable equilibrium BP_{0a} appears, involving the spreading of industry between regions 3 and 4, while industry is still agglomerated in region 2; at still lower values of ϕ_C , the stable equilibrium IE_{ef} is born, involving dispersion of the industry across all regions, coexisting with the other two attractors. This sequence reflects the fact that region 3, though having a small local market, enjoys the advantage of centrality; therefore, industry from the big region 4 partly moves to the small region 3; region 2 keeps the core for a larger range of ϕ_C values because it enjoys both, centrality and a large local market.

Interestingly enough, for still lower values of ϕ_C , BP_{c0} appears, involving an equilibrium in which industry is spread between regions 1 and 2, and agglomerated in region 4. Industry in region 4 takes advantage of the big local market and of the shelter from competition.

In contrast to Section 4, the basins of attraction exhibit a quite complex structure. Therefore, abrupt transitions between the different types of equilibria are more likely to occur. Since the equilibria involve quite distinct patterns of industry location, abrupt changes in transit traffic are also expected. This reinforces our argument developed above according to which policy advice neglecting side effects upon industry location might be seriously misleading.

Finally, our analysis reveals that long run equilibria exhibit not only stationary industry allocations, but also cyclical and/or chaotic attractors with a continuous movement of industry between the regions. Thus, continuous movements between regions may not only be driven by stochastic shocks, but by deterministic economic forces at work in our non-linear NEG model, as discussed above.

Fig. 11 In *a*): coexisting M -attracting fixed points CP_{00} , BP_{0a} and BP_{c0} (with red, blue and dark-green basins, respectively) and attracting fixed point IP_{ef} (with orange basin) for $\phi_C = 0.15$; In *b*): an enlarged part of the phase plane with coexisting M -attracting fixed points CP_{00} and BP_{c0} (with red and dark-green basins, respectively), M -attracting 2-cycle $\{C_{0a1}, C_{0a2}\}$ (with blue basin) and 2-cyclic closed invariant curves (with orange basin) for $\phi_C = 0.1398$. Here $L_s = 300$ while all the other parameters are fixed as in (24).

Fig. 12 An enlarged part of the phase plane with coexisting M -attracting fixed points CP_{00} and BP_{c0} (with red and dark-green basins, respectively), M -attracting 2-cycle $\{C_{0a1}, C_{0a2}\}$ (with blue basin) and 2-cyclic chaotic attractor (with orange basin) for $\phi_C = 0.1375$. Here $L_s = 300$ while all the other parameters are fixed as in (24).

6 Conclusions

In this paper, we developed a 4-region new economic geography (NEG) model – by extending the standard 2-region Footloose Entrepreneur model (FE) variant. We assumed specific first nature differences: the four regions are aligned,

the first and the third regions are of smaller size and the second and the fourth ones of a larger size; regions one and two (and regions three and four) are geographically and/or culturally closer to each other than to the outside neighbors – as a stylized representation of some real world cases (South Italy - North Italy - Austria - Germany or Canada - USA - Mexico - Latin America). Our objective was to explore: i) the effect of regional integration within an integrated area in the presence of an outside integrated bloc; ii) the effect of trade liberalization between integrated blocs; and finally, iii) the impact of changing transport costs on transit traffic.

The first nature characteristics of the model allowed us to disentangle two different aspects of the market access effect: the size of the local market – a feature already captured by the standard 2-region asymmetric FE model – and the centrality of some regions. Moreover, our extended setup has shown that, depending on parameter values, the model is able to generate a large variety of long-run outcomes. We have obtained four possible core-periphery equilibria: agglomeration can occur in the two central or the two remote regions, in the two small or in the two large regions. We determined analytically (when possible) the ranking of the core-periphery equilibria for each region in terms of real wages, the effect of trade costs on the remuneration of (immobile) workers and the nominal/real profits of (mobile) entrepreneurs. The main conclusion is that workers are better off in regions with a better access to commodity markets (of both blocs). In most cases, trade integration (between and within blocs) increases the real profits of entrepreneurs. Also, nominal profits in central regions or in the large remote region are higher when the industry agglomeration in the other bloc is far away. More generally, the degree of trade integration (between or within blocs) may alter in a non trivial way entrepreneurial remunerations as they result from the effects of centrality, market size, and competition. When the latter effect dominates, industry agglomerates in the regions that are at the fringe in our geographical line-up.

Moreover, our results extend those of the 2-region FE model in two other directions. First, the coexistence of feasible long-run outcomes goes beyond the coexistence of stable core-periphery equilibria. We have been able to detect situations in which industry is agglomerated in a bloc but dispersed in the other one due to the interplay between the agglomeration and dispersion forces mentioned above. These results cannot emerge in simpler setups. Second, instances of multistability arise. This allows us to focus the study on cumulative causation, positive feed-back processes, locational hysteresis, and path dependence, which provides an evolutionary perspective to our results.

Finally we have been able to derive implications of the model on transit traffic for which multistability is of crucial relevance. With its explicit focus on location choice (and trade volumes), the NEG approach provides a complementary approach to other studies that focus on trade volume and the so-called modal split. Typically, an increase in trade costs reduces transit traffic - this is also the case in our model in each of the equilibria. However, due to the pervasiveness of multistability in NEG models, an increase in trade costs might actually induce upward jumps in transit traffic as a consequence of industry

relocation across regions which arises when the economy switches from one to a different core-periphery equilibrium.

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Appendix A

In (15) profits π_i for $i = \overline{1, 4}$, are defined as follows:

$$\pi_1 = \frac{A_1(1 - C_2) + A_2C_1}{(1 - B_1)(1 - C_2) - B_2C_1}, \quad \pi_2 = \frac{A_2(1 - B_1) + A_1B_2}{(1 - B_1)(1 - C_2) - B_2C_1},$$

$$\pi_3 = \frac{A_3(1 - B_4) + A_4B_3}{(1 - C_3)(1 - B_4) - B_3C_4}, \quad \pi_4 = \frac{A_4(1 - C_3) + A_3C_4}{(1 - C_3)(1 - B_4) - B_3C_4},$$

where

$$\begin{aligned} A_1 &= (\Theta_1 + \phi_C \phi_D h_b \beta_3 Q_3 \Theta_3) Q_1, \quad A_2 = (\Theta_2 + \phi_D h_b \beta_3 Q_3 \Theta_3) Q_2, \\ A_3 &= \left(\Omega_3 + \phi_D \tilde{h}_b \beta_2 M_2 \Omega_2 \right) M_3, \quad A_4 = \left(\Omega_4 + \phi_C \phi_D \tilde{h}_b \beta_2 M_2 \Omega_2 \right) M_4, \\ B_1 &= (\phi_C^2 \phi_D^2 h_b^2 \beta_1 \beta_3 Q_3) Q_1, \quad B_2 = \phi_C \beta_1 (h_a + \phi_D^2 h_b^2 \beta_3 Q_3) Q_2, \\ B_3 &= \phi_C \beta_4 (\tilde{h}_a + \phi_D^2 \tilde{h}_b^2 \beta_2 M_2) M_3, \quad B_4 = \left(\phi_C^2 \phi_D^2 \tilde{h}_b^2 \beta_2 \beta_4 M_2 \right) M_4, \\ C_1 &= \phi_C \beta_2 (h_a + \phi_D^2 h_b^2 \beta_3 Q_3) Q_1, \quad C_2 = \phi_D^2 h_b^2 \beta_2 \beta_3 Q_2 Q_3, \\ C_3 &= \phi_D^2 \tilde{h}_b^2 \beta_2 \beta_3 M_2 M_3, \quad C_4 = \phi_C \beta_3 (\tilde{h}_a + \phi_D^2 \tilde{h}_b^2 \beta_2 M_3) M_4, \\ \Theta_1 &= \alpha_1 z_1 + \phi_C \alpha_2 z_2 + \phi_C \phi_D \alpha_3 z_3 + \phi_C^2 \phi_D \alpha_4 z_4, \\ \Theta_2 &= (\phi_C \alpha_1 + \alpha_2) z_2 + \phi_D \alpha_3 z_3 + \phi_C \phi_D \alpha_4 z_4, \\ \Theta_3 &= (\phi_C \phi_D \alpha_1 + \phi_D \alpha_2 + \alpha_3) z_3 + \phi_C \alpha_4 z_4, \\ \Omega_2 &= \phi_C \alpha_1 \tilde{z}_1 + (\phi_C \phi_D \alpha_4 + \phi_D \alpha_3 + \alpha_2) \tilde{z}_2, \\ \Omega_3 &= \phi_C \phi_D \alpha_1 \tilde{z}_1 + \phi_D \alpha_2 \tilde{z}_2 + (\alpha_3 + \phi_C \alpha_4) \tilde{z}_3, \\ \Omega_4 &= \phi_C^2 \phi_D \alpha_1 \tilde{z}_1 + \phi_C \phi_D \alpha_2 \tilde{z}_2 + \phi_C \alpha_3 \tilde{z}_3 + \alpha_4 \tilde{z}_4, \end{aligned}$$

$$\begin{aligned}
Q_1 &= \frac{1}{\frac{1}{\kappa} - \beta_1 (1 + \phi_C^4 \phi_D^2 z_4 \kappa \beta_4)}, \quad Q_2 = \frac{1}{\frac{1}{\kappa} - \beta_2 h_a}, \quad Q_3 = \frac{1}{\frac{1}{\kappa} - \beta_3 h_b}, \\
M_2 &= \frac{1}{\frac{1}{\kappa} - \beta_2 \tilde{h}_b}, \quad M_3 = \frac{1}{\frac{1}{\kappa} - \beta_3 \tilde{h}_a}, \quad M_4 = \frac{1}{\frac{1}{\kappa} - \beta_4 (1 + \phi_C^4 \phi_D^2 \tilde{z}_1 \kappa \beta_1)}, \\
\alpha_1 &= \frac{L_s}{\Delta_1}, \quad \alpha_2 = \frac{L_b}{\Delta_2}, \quad \alpha_3 = \frac{L_s}{\Delta_3}, \quad \alpha_4 = \frac{L_b}{\Delta_4}, \\
\beta_1 &= \frac{x}{\Delta_1} \frac{E}{2}, \quad \beta_2 = \frac{1-x}{\Delta_2} \frac{E}{2}, \quad \beta_3 = \frac{y}{\Delta_3} \frac{E}{2}, \quad \beta_4 = \frac{1-y}{\Delta_4} \frac{E}{2}, \\
\kappa &= \frac{2\mu}{\sigma E},
\end{aligned}$$

$$\begin{aligned}
z_1 &= z_4 (1 - \kappa \beta_4 (1 - \phi_C^4 \phi_D^2)), \quad z_2 = z_4 (1 - \kappa \beta_4 (1 - \phi_C^2 \phi_D^2)), \\
z_3 &= z_4 (1 - \kappa \beta_4 (1 - \phi_C^2)), \quad z_4 = \frac{1}{1 - \kappa \beta_4},
\end{aligned}$$

$$\begin{aligned}
\tilde{z}_1 &= \frac{1}{1 - \kappa \beta_1}, \quad \tilde{z}_2 = \tilde{z}_1 (1 - \kappa \beta_1 (1 - \phi_C^2)), \\
\tilde{z}_3 &= \tilde{z}_1 (1 - \kappa \beta_1 (1 - \phi_C^2 \phi_D^2)), \quad \tilde{z}_4 = \tilde{z}_1 (1 - \kappa \beta_1 (1 - \phi_C^4 \phi_D^2)), \\
h_a &= 1 + \phi_C^2 \phi_D^2 z_4 \kappa \beta_4, \quad h_b = 1 + \phi_C^2 z_4 \kappa \beta_4, \\
\tilde{h}_a &= 1 + \phi_C^2 \phi_D^2 \tilde{z}_1 \kappa \beta_1, \quad \tilde{h}_b = 1 + \phi_C^2 \tilde{z}_1 \kappa \beta_1.
\end{aligned}$$

Appendix B

The borders of the unit square I^2 are denoted as follows:

$$\begin{aligned}
I_{x0} &= \{(x, y) : y = 0\}, \quad I_{x1} = \{(x, y) : y = 1\}, \\
I_{0y} &= \{(x, y) : x = 0\}, \quad I_{1y} = \{(x, y) : x = 1\}.
\end{aligned}$$

It is easy to see that these borders are invariant under map Z . Moreover, on each of these borders 2D map Z is reduced to the corresponding 1D map. Namely,

- on border I_{x0} map Z is reduced to a 1D map, denoted z_{x0} , which is defined as

$$z_{x0} : x \mapsto z_{x0}(x) = \begin{cases} 0 & \text{if } Z_{h0}(x) < 0, \\ Z_{h0}(x) & \text{if } 0 \leq Z_{h0}(x) \leq 1, \\ 1 & \text{if } Z_{h0}(x) > 1, \end{cases}$$

where

$$Z_{h0}(x) = x \left[1 + \gamma(1-x) \frac{\Omega_h(x, 0) - 1}{1 + x(\Omega_h(x, 0) - 1)} \right]; \quad (28)$$

- on border I_{x1} map Z is reduced to a 1D map, denoted z_{x1} , which is defined as

$$z_{x1} : x \mapsto z_{x1}(x) = \begin{cases} 0 & \text{if } Z_{h1}(x) < 0, \\ Z_{h1}(x) & \text{if } 0 \leq Z_{h1}(x) \leq 1, \\ 1 & \text{if } Z_{h1}(x) > 1, \end{cases}$$

where

$$Z_{h1}(x) = x \left[1 + \gamma(1-x) \frac{\Omega_h(x, 1) - 1}{1 + x(\Omega_h(x, 1) - 1)} \right]; \quad (29)$$

- on border I_{0y} map Z is reduced to a 1D map

$$z_{0y} : y \mapsto z_{0y}(y) = \begin{cases} 0 & \text{if } Z_{f0}(y) < 0, \\ Z_{f0}(y) & \text{if } 0 \leq Z_{f0}(y) \leq 1, \\ 1 & \text{if } Z_{f0}(y) > 1, \end{cases}$$

where

$$Z_{f0}(y) = y \left[1 + \gamma(1-y) \frac{\Omega_f(0, y) - 1}{1 + y(\Omega_f(0, y) - 1)} \right]; \quad (30)$$

- and, finally, on border I_{1y} the map Z is reduced to a 1D map

$$z_{1y} : y \mapsto z_{1y}(y) = \begin{cases} 0 & \text{if } Z_{f1}(y) < 0, \\ Z_{f1}(y) & \text{if } 0 \leq Z_{f1}(y) \leq 1, \\ 1 & \text{if } Z_{f1}(y) > 1, \end{cases}$$

where

$$Z_{f1}(y) = y \left[1 + \gamma(1-y) \frac{\Omega_f(1, y) - 1}{1 + y(\Omega_f(1, y) - 1)} \right]. \quad (31)$$

Appendix C

Jacobian matrix of map Z (in its central part, without constraints) is given by

$$DZ = \begin{pmatrix} Z'_{hx}(x, y) & Z'_{hy}(x, y) \\ Z'_{fx}(x, y) & Z'_{fy}(x, y) \end{pmatrix},$$

where

$$Z'_{hx}(x, y) = 1 + \gamma \left((1-2x) \frac{\Omega_h(x, y) - 1}{1 + x(\Omega_h(x, y) - 1)} + x(1-x) \frac{\Omega'_{hx}(x, y) - (\Omega_h(x, y) - 1)^2}{(1 + x(\Omega_h(x, y) - 1))^2} \right),$$

$$Z'_{hy}(x, y) = \gamma x(1-x) \frac{\Omega'_{hy}(x, y)}{(1 + x(\Omega_h(x, y) - 1))^2},$$

$$Z'_{fx}(x, y) = \gamma y(1-y) \frac{\Omega'_{fx}(x, y)}{(1 + y(\Omega_f(x, y) - 1))^2},$$

$$Z'_{fy}(x, y) = 1 + \gamma \left((1-2y) \frac{\Omega_f(x, y) - 1}{1 + y(\Omega_f(x, y) - 1)} + y(1-y) \frac{\Omega'_{fy}(x, y) - (\Omega_f(x, y) - 1)^2}{(1 + y(\Omega_f(x, y) - 1))^2} \right).$$

The eigenvalues are

$$\lambda_{1,2}(x, y) = 0.5 \left(Z'_{hx}(x, y) + Z'_{fy}(x, y) \pm \sqrt{(Z'_{hx}(x, y) - Z'_{fy}(x, y))^2 + 4Z'_{hy}(x, y)Z'_{fx}(x, y)} \right).$$

We use the following notations: $\lambda_1(x, y) = \lambda_h(x, y)$ and $\lambda_2(x, y) = \lambda_v(x, y)$. Eigenvalues of the fixed points of map Z are as follows:

$$CP_{00} : \quad \lambda_h(0, 0) = 1 + \gamma(\Omega_h(0, 0) - 1), \quad \lambda_v(0, 0) = 1 + \gamma(\Omega_f(0, 0) - 1),$$

$$CP_{11} : \quad \lambda_h(1,1) = 1 - \gamma \left(1 - \frac{1}{\Omega_h(0,0)} \right), \quad \lambda_v(1,1) = 1 - \gamma \left(1 - \frac{1}{\Omega_f(0,0)} \right),$$

$$CP_{01} : \quad \lambda_h(0,1) = 1 + \gamma(\Omega_h(0,1) - 1), \quad \lambda_v(0,1) = 1 - \gamma \left(1 - \frac{1}{\Omega_f(0,1)} \right),$$

$$CP_{10} : \quad \lambda_h(1,0) = 1 - \gamma \left(1 - \frac{1}{\Omega_h(1,0)} \right), \quad \lambda_v(1,0) = 1 + \gamma(\Omega_f(1,0) - 1),$$

$$BP_{0a} : \quad \lambda_h(0,a) = 1 + \gamma(\Omega_h(0,a) - 1), \quad \lambda_v(0,a) = 1 + \gamma a(1-a)\Omega_{fy}'(0,a),$$

$$BP_{1a} : \quad \lambda_h(1,a) = 1 - \gamma \left(1 - \frac{1}{\Omega_h(1,a)} \right), \quad \lambda_v(1,a) = 1 + \gamma a(1-a)\Omega_{fy}'(1,a),$$

$$BP_{a0} : \quad \lambda_h(a,0) = 1 + \gamma a(1-a)\Omega_{hx}'(a,0), \quad \lambda_v(a,0) = 1 + \gamma(\Omega_f(a,0) - 1),$$

$$BP_{a1} : \quad \lambda_h(a,1) = 1 + \gamma a(1-a)\Omega_{hx}'(a,1), \quad \lambda_v(a,1) = 1 - \gamma \left(1 - \frac{1}{\Omega_f(a,1)} \right),$$

$$IP_{ab} : \lambda_{h,v}(a,b) = 1 + 0.5\gamma(a(1-a)\Omega_{hx}'(a,b) + b(1-b)\Omega_{fy}'(a,b) \pm \sqrt{(a(1-a)\Omega_{hx}'(a,b) - b(1-b)\Omega_{fy}'(a,b))^2 + 4a(1-a)b(1-b)\Omega_{hy}'(a,b)\Omega_{fx}'(a,b)}).$$