Towards an understanding of variations in the buckling of tailored variable angle tow composite plates

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Abstract

In this paper, variable angle tow (VAT) composite plates tailored to enhance buckling performance are studied with the use of stochastic finite element method to quantify uncertainties in buckling measures arising from variations in material properties and fibre tow path. Detailed formulations for predicting buckling statistics in terms of mean value and standard deviation are derived to enable a perturbation-based stochastic finite element analysis. The derivations are built on a linear variation formula for fibre tow path and plate element based on the first order shear deformation theory. They are integrated with Taylor series expansion to propagate uncertainties from inputs to buckling performance measures, including buckling eigenvalues, critical buckling coefficients, etc. A twelve-layer VAT composite plate, with optimally designed fibre tow paths under various boundary conditions, has been investigated to illustrate the uncertainty quantification procedure. The performance of the perturbation-based stochastic finite element method has

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been validated using Monte Carlo simulation. Influences of variations in material properties and fibre tow path are thoroughly examined to understand the variability of buckling statistics of VAT composites.

Keywords: Buckling tailoring; Variable angle tow (VAT); Laminated composite plates; Uncertainty quantification; Stochastic finite element method.

1. Introduction

The freedom to obtain tailored properties is one of the most attractive features of fibre reinforced polymer composites for developing lightweight structures, achieved using well established techniques, e.g. the specific selection of new constituent materials or different compositions, designing alternative material architectures, etc. Recently, more advanced composite manufacturing techniques have emerged, such as variable angle tow (VAT) or tow steering composites, in which the fibre path is continuously varied to tailor the local stiffness properties of laminates, further enhancing their structural performance. VAT composites are receiving significant attentions from several industry sectors such as aerospace, where it has shown great potential in structures with high demands on buckling, aeroelasticity, and vibration with restricted weight allowance. However, this additional design freedom is at the expense of complexity in maintaining uniform fibre paths during manufacturing and increased computational cost. One of the main additional computational costs comes from finding the optimal fibre tow path (Hyer and Charette, 1991; Setoodeh et al., 2009; Ijsselmuiden et al., 2010; Wu et al., 2012; Sabido et al., 2017). In addition, the nature of its curved

19 fibre courses means that they are inevitably more vulnerable to manufactur-
20 ing defects, which have been recognized as the main sources of variations in
21 structural performances, compared with unidirectional (UD) or straight fibre
22 composites. It is thus desirable to understand the uncertainties of this type
23 of advanced composite to provide confidences in their applications.

24 VAT composites are relatively new, meaning that reports on quantifying
25 uncertainties in either material properties or structural performance are sel-
26 dom available. Variations in fibre tow path have, however, been reported in
27 (Kim et al., 2014). Due to the hierarchical nature of composites, relatively
28 complex manufacturing technologies are required, with the manufacturing
29 processes becoming the primary source for introducing material and geomet-
30 ric uncertainties. As a type of product using Automated Fibre Placement
31 (AFP) technologies, VAT composites may share many manufacturing defects
32 and variations in their properties with conventional composites. In general,
33 variability exists in the constituent material properties arising from defects,
34 such as fibre misalignment, waviness, etc. (Potter et al., 2008). It has been
35 widely accepted that defects and their inherent variations are some of the
36 main sources of damage initiation (Hörrmann et al., 2016). The formations
37 of defects were investigated in (Belnoue et al., 2017; Lukaszewicz et al., 2012).
38 For instance, one of the processes known to influence the formation of fibre
39 path defects is the consolidation of laminates manufactured by AFP. More-
40 over, these defects are random in nature. Uncertainty in the geometry of
41 fibre preforms manufactured with Automated Dry Fibre Placement and its
42 effects on permeability were studied in (Matveev et al., 2017). The influence
43 of manufacturing uncertainties on mechanical and/or physical properties of

44 conventional composites, e.g. UD or textile, has been continuously inves-
45 tigated through numerical and experimental approaches, e.g. ([Endruweit](#)
46 [et al., 2013](#); [Bednarczyk et al., 2014](#); [Zhou et al., 2017](#); [Zhou and Gosling,](#)
47 [2018](#)). In the present study, the influences on buckling performance are of
48 interest.

49 Uncertainty-driven probabilistic buckling analysis has been historically
50 researched for composite structures. [Lin et al. \(1998\)](#) investigated the buck-
51 ling failure probability evaluation of laminated composite plates subjected to
52 different in-plane random loads using the stochastic finite element method,
53 with the feasibility and accuracy of the results validated using results ob-
54 tained from Monte Carlo Simulations (MCS). Using alternative methods,
55 reliability predictions of laminated composite plates with random system
56 parameters subjected to transverse loads have also been made ([Lin, 2000](#)).
57 [Singh et al. \(2001\)](#) studied the effects of random material properties on the
58 buckling of composite plates. [Orifici and Bisagni \(2013\)](#) used a perturbation-
59 based technique to investigate the impacts of imperfections on the buckling
60 of composite cylindrical shells. [Chen and Guedes Soares \(2007\)](#) conducted
61 a reliability analysis of post-buckling performance. In addition to mechani-
62 cal load induced buckling, other actions, e.g. temperature-induced buckling
63 has also been reported. [Lal et al. \(2009\)](#) investigated the effects of ran-
64 dom system properties on the thermal buckling of composites. [Lal et al.](#)
65 [\(2011\)](#) further conducted a stochastic post-buckling analysis of laminated
66 composites subjected to hygro-thermal-mechanical loading. The effect of
67 random system properties, plate geometry, stacking sequences, supporting
68 conditions, and fibre volume fractions on hygro-thermal-mechanical buckling

load of the laminated shells and panels were investigated using a first order
perturbation-based stochastic finite element method (PSFE). Most recently,
[Li et al. \(2016\)](#) conducted a stochastic thermal buckling analysis using a per-
turbation technique to propagate uncertainties from the constituent material
scale to the structural scale. It is noted that the perturbation-based finite
element method is the most commonly adopted technique for probabilistic
buckling analysis. However, all the aforementioned studies have only inves-
tigated conventional unidirectional fibre reinforced polymer composites. No
equivalent studies appear to have been published for VAT composites.

In this study, the perturbation-based stochastic finite element method is
adopted to develop an uncertainty quantification approach to investigate the
buckling characteristics of VAT composite plates subject to mechanical loads
e.g. uniform end shortening displacement. The first-order shear deformation
theory, which enables to consider transverse shear strain that can provide
sufficiently accurate prediction of the structural response for the medium
thickness laminated plates, is used as the basis to develop the stochastic fi-
nite element method. As the primary objective of the present study is to
understand the variations in buckling performance, relatively simple VAT
composite plates with a linear variation to define fibre tow path (see Figure
[1b](#) based on the formula proposed by [Gürdal et al. \(2008\)](#)) are considered
to investigate the mechanical characteristics of such structures. However,
the method described in this paper can be straightforwardly applied to VAT
composites with more complicated fibre path functions, such as Lagrangian
polynomials (see Figure [1c](#)) proposed by [Wu et al. \(2012\)](#). In the present
study, the formulation for propagating uncertainties from material properties

94 e.g. moduli and Poisson's ratios and fibre tow path e.g. rotation angle in the
 95 linear variation formula to buckling performance measures, i.e. eigenvalue
 96 and critical buckling coefficients, is derived and given in the form of a finite
 97 element analysis. Statistics, namely mean values and standard deviations,
 98 of the buckling measures are given. The proposed uncertainty quantification
 99 procedure is demonstrated through studies on a VAT composite plate tailored
 100 for maximum buckling performance, under alternative boundary conditions,
 101 and subjected to uniform displacement shortening. Monte Carlo simulation
 102 is used to validate the performance of the method in terms of accuracy and
 103 efficiency. Further studies are conducted to investigate the stochastic me-
 104 chanical behaviours of VAT composite plates in the presence of uncertainties
 105 in material properties and fibre tow path directions.

106 **2. Methodology**

107 *2.1. Fibre path formulation*

Whilst nonlinear tow angle descriptions have been shown to replacedpro-
 vide an extended design space be advantageous when endeavouring to opti-
 mize the performance of VAT composite plates, a linear tow angle variation
 proposed by [Gürdal et al. \(2008\)](#) is adopted here as the primary purpose of
 the current study is to quantify the uncertainties in buckling performance of
 VAT composite structures. The simpler description of fibre tow path can be
 easier to evaluate the variations in mechanical behaviour of VAT composites,
 while the stochastic analysis procedure developed in the following sections
 is straightforward to incorporate with other fibre path functions. The lin-
 ear variation of fibre tow path adopted in the present study is illustrated in

Figure 1a and expressed as:

$$\theta(x) = \phi + T_0 + (T_1 - T_0) \frac{|x'|}{d} \quad (1)$$

where T_0 and T_1 are the angles at two control points, A and B , respectively, d is the distance between these two points, ϕ is the direction moving from A to B , and $x' = x \cos(\phi) + y \sin(\phi)$. With this formula, a single VAT composite layer may be represented by $\phi < T_0 | T_1 >$, and laminates with multiple VAT composite layers can be specified by different combinations of the angle ϕ and stacks of $\pm < T_0 | T_1 >$.

2.2. Linear buckling analysis of variable angle tow laminated composite plates

For the structural analysis of VAT laminated composite plates, one of the essential differences comparing with conventional composite laminates, is that the stiffness properties now vary as a function of spatial location. Therefore, in the finite element analysis context, the discretized domain will produce locally straight fibre paths; each element will have a local fibre angle associated with it (see Figure 2).

For a linear buckling analysis, the buckling load is determined through the following eigenvalue problem:

$$(\mathbf{K} - \lambda \mathbf{K}_g) \mathbf{V} = \mathbf{0} \quad (2)$$

where $\mathbf{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is the right-hand eigenvector corresponding to eigenvalues of $\lambda = \{\lambda_1, \dots, \lambda_n\}$ with n for the total number of eigenmodes, $\mathbf{K}$ is the global stiffness matrix, and $\mathbf{K}_g$ is the global geometric stiffness matrix due to an applied mechanical load.

In the present study, the above equation is solved using four-node quadrilateral plate elements based on the first order shear deformation theory. Each node has five degrees of freedom, including three translations u , v , w and two rotations θ_x , θ_y . Briefly, to solve it, a prebuckling analysis is conducted first to find structural responses under a given loading condition to assemble K_g , and the equation is solved with an eigensolver. This is conducted with an in-house program developed in MATLAB. In what follows, we briefly present this computational scheme, following the notations adopted by (Oñate, 2013). The global stiffness matrix is thus written as:

$$\mathbf{K} = \sum_{e=1}^{NE} \mathbf{k}_e = \sum_{e=1}^{NE} \left[\mathbf{B}^T \hat{\mathbf{D}}_e \mathbf{B} \right] = \sum_{e=1}^{NE} \sum_{k=1}^{NP} \left[\mathbf{B}^T \mathbf{L}_{ek}^T \mathbf{D}_{ek} \mathbf{L}_{ek} \mathbf{B} \right] \quad (3)$$

where the superscript T denotes a vector or matrix transposition; $\mathbf{k}_e$ is the element stiffness matrix with the dimension of 20×20 ; $\mathbf{B}$ is the strain-displacement transformation matrix with the dimension of 8×20 (see e.g. (Oñate, 2013)); $\hat{\mathbf{D}}_e$ is the generalized constitutive matrix with the dimension of 8×8 ; NE and NP are the numbers of elements and plies, respectively; $\mathbf{D}_{ek}$ is the constitutive matrix of the k -th ply for the e -th element given by:

$$\mathbf{D}_{ek} = \begin{bmatrix} t_k \mathbf{C}_{1k} & -t_k \bar{z}_k \mathbf{C}_{1k} & \mathbf{0} \\ -t_k \bar{z}_k \mathbf{C}_{1k} & \frac{1}{3} (z_{k+1}^3 - z_k^3) \mathbf{C}_{1k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \kappa \mathbf{C}_{2k} \end{bmatrix} \quad (4)$$

where $t_k = z_{k+1} - z_k$ is the thickness of k -th ply, $\bar{z}_k = \frac{1}{2}(z_k + z_{k+1})$ is the mid-plane of the k -th ply, z_{k+1} and z_k are the through-thickness coordinate of the top and bottom surface of the k -th ply, and κ is the shear correction factor, 5/6 used here; $\mathbf{C}_{1k}$ and $\mathbf{C}_{2k}$ are the in-plane constitutive and out-plane shear constitutive matrices of the k -th ply. For example, $\mathbf{C}_{1k}$ and $\mathbf{C}_{2k}$ for a lamina

made of a transversely isotropic material, such as the carbon fibre reinforced polymer composite investigated the following numerical study section, are:

$$\mathbf{C}_{1k}^{3 \times 3} = \begin{bmatrix} \frac{E_{1k}^2}{E_{1k} - E_{2k}\nu_{12k}^2} & \frac{E_{1k}E_{2k}\nu_{12k}}{E_{1k} - E_{2k}\nu_{12k}^2} & 0 \\ \frac{E_{1k}E_{2k}\nu_{12k}}{E_{1k} - E_{2k}\nu_{12k}^2} & \frac{E_{1k}E_{2k}}{E_{1k} - E_{2k}\nu_{12k}^2} & 0 \\ 0 & 0 & G_{12k} \end{bmatrix}, \quad \mathbf{C}_{2k}^{2 \times 2} = \begin{bmatrix} G_{23k} & 0 \\ 0 & G_{13k} \end{bmatrix}. \quad (5)$$

where E_{1k} and E_{2k} are the elastic moduli of the k -th ply along the fibre direction and the through-thickness direction, respectively; ν_{12k} is the in-plane Poisson's ratio of the k -th ply; and $G_{12k} = G_{13k}$, and G_{23k} are the shear moduli of the k -th ply in material planes 12, 13 and 23, respectively.

$\mathbf{L}_{ek}$ is the coordinate transformation matrix for the contribution from the k -th ply to the e -th element. It is expressed as:

$$\mathbf{L}_{ek}^{8 \times 8} = \begin{bmatrix} \mathbf{T}_{1ek} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{1ek} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{T}_{2ek} \end{bmatrix} \quad (6)$$

with

$$\mathbf{T}_{1ek}^{3 \times 3} = \begin{bmatrix} \cos^2 \theta_{ek} & \sin^2 \theta_{ek} & \sin \theta_{ek} \cos \theta_{ek} \\ \sin^2 \theta_{ek} & \cos^2 \theta_{ek} & -\sin \theta_{ek} \cos \theta_{ek} \\ -2 \sin \theta_{ek} \cos \theta_{ek} & 2 \sin \theta_{ek} \cos \theta_{ek} & \cos^2 \theta_{ek} - \sin^2 \theta_{ek} \end{bmatrix},$$

$$\mathbf{T}_{2ek}^{2 \times 2} = \begin{bmatrix} \cos \theta_{ek} & -\sin \theta_{ek} \\ -\sin \theta_{ek} & \cos \theta_{ek} \end{bmatrix} \quad (7)$$

where θ_{ek} is the corresponding fibre tow orientation (see Figure 2).

The geometric stiffness matrix $\mathbf{K}_g$ consists of contributions from both bending, $\mathbf{K}_{gb}$, and shear, $\mathbf{K}_{gs}$, and is written as (Ferreira, 2009):

$$\mathbf{K}_g = \mathbf{K}_{gb} + \mathbf{K}_{gs} \quad (8)$$

with

$$\begin{aligned}\mathbf{K}_{gb} &= \sum_{e=1}^{NE} (\mathbf{B}_{gb1}^T \mathbf{S}_e \mathbf{B}_{gb1} + \mathbf{B}_{gb2}^T \mathbf{S}_e \mathbf{B}_{gb2} + \mathbf{B}_{gb3}^T \mathbf{S}_e \mathbf{B}_{gb3}) \\ \mathbf{K}_{gs} &= \sum_{e=1}^{NE} (\mathbf{B}_{gs1}^T \mathbf{S}_e \mathbf{B}_{gs1} + \mathbf{B}_{gs2}^T \mathbf{S}_e \mathbf{B}_{gs2})\end{aligned}\quad (9)$$

where $\mathbf{B}_{gb}$, $\mathbf{B}_{gs1}$ and $\mathbf{B}_{gs2}$ are the strain-displacement matrices for geometric stiffness matrix, and for a give node i :

$$\mathbf{B}_{gb1_i} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{gb2_i} = \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{gb3_i} = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} & 0 & 0 \end{bmatrix} \quad (10)$$

and

$$\mathbf{B}_{gs1_i} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & 0 \end{bmatrix}, \quad \mathbf{B}_{gs2_i} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y} \end{bmatrix}. \quad (11)$$

$\mathbf{S}_e$ is the rearranged in-plane stress resultant matrix in the e -th element written as:

$$\mathbf{S}_e = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}, \quad (12)$$

130 with N_x , N_y , and N_{xy} for the in-plane stress resultants, which are obtained
 131 from the standard static analysis or so-called prebuckling analysis. In the
 132 present study, they are structural responses of the studied laminated plates
 133 subject to uniform two end shortening displacements, which will be explained
 134 further in the numerical study section.

135 2.3. Predicting variations in buckling performance

With the equations for a deterministic analysis at hand, it is possible now to derive the predictive formulation for buckling uncertainty quantification.

We initially define an m -dimensional random vector $\mathbf{b} = \{b_1, b_2, \dots, b_m\}^T$ to represent the set of random variables, e.g. material properties and fibre path parameters in the present study. As the buckling eigenvalue is a function of the constituent material properties and fibre angle as indicated in Eq. (2), the presence of variations in these input parameters eventually leads the buckling eigenvalue to be a stochastic function. It is well known that an arbitrary stochastic function, $\varphi(\mathbf{b})$, can be approximated via a Taylor series expansion as

$$\varphi(\mathbf{b}) = \varphi(\bar{\mathbf{b}}) + \epsilon \sum_{r=1}^m \frac{\partial \varphi(\bar{\mathbf{b}})}{\partial b_r} \delta b_r + \epsilon^2 \frac{1}{2} \sum_{r=1}^m \sum_{s=1}^m \frac{\partial^2 \varphi(\bar{\mathbf{b}})}{\partial b_r \partial b_s} \delta b_r \delta b_s + \dots \quad (13)$$

where $\bar{\mathbf{b}}$ is the mean value of the random vector $\mathbf{b}$, δb_r denotes the variation around mean value of the r -th random variable, $\frac{\partial \varphi}{\partial b_r}$ and $\frac{\partial^2 \varphi}{\partial b_r \partial b_s}$ denote the first- and second-order partial derivatives of $\varphi(\mathbf{b})$ with respect to the r - or/and s -th variables, respectively, and ϵ is a scalar representing a given small perturbation. In addition, given the approximation for $\varphi(\mathbf{b})$ in Eq. (13), its statistics in terms of mean value and variance can be easily calculated from the following equations for the second-order approximation. According to the definition of the mean value, $\int_{-\infty}^{+\infty} \varphi(\mathbf{b}) p(\mathbf{b}) d\mathbf{b}$, substituting Eq. (13) and using the definitions $\int_{-\infty}^{+\infty} p(b_r) db_r = 1$, $\int_{-\infty}^{+\infty} \delta b_r p(b_r) db_r = 0$ and $\int_{-\infty}^{+\infty} \delta b_r \delta b_s p(b_r) p(b_s) db_r db_s = \text{COV}(b_r, b_s)$, then the mean value of $\varphi(\mathbf{b})$ is,

$$E[\varphi(\mathbf{b})] = [\varphi(\bar{\mathbf{b}})] + \frac{1}{2} \sum_r^m \sum_s^m \left[\frac{\partial^2 \varphi(\bar{\mathbf{b}})}{\partial b_r \partial b_s} \right] \cdot \text{COV}(b_r, b_s). \quad (14)$$

Similarly, based on its definition $\int_{-\infty}^{+\infty} \{\varphi(\mathbf{b}_r) - E[\varphi(\mathbf{b})]\} \{\varphi(\mathbf{b}_s) - E[\varphi(\mathbf{b})]\} p(\mathbf{b}) d\mathbf{b}$, the variance is found:

$$\begin{aligned} \text{COV}([\varphi(\mathbf{b})], [\varphi(\mathbf{b})]) &\approx \sum_{r,s}^m \left[\frac{\partial \varphi(\bar{\mathbf{b}})}{\partial b_r} \right] \left[\frac{\partial \varphi(\bar{\mathbf{b}})}{\partial b_s} \right] \cdot \text{COV}(b_r, b_s) \\ &+ \frac{1}{4} \sum_{r,s,t,w}^m \left[\frac{\partial^2 \varphi(\bar{\mathbf{b}})}{\partial b_r \partial b_s} \right] \left[\frac{\partial^2 \varphi(\bar{\mathbf{b}})}{\partial b_t \partial b_w} \right] E[b_r b_s b_t b_w]. \end{aligned} \quad (15)$$

with r, s, t, w denote the corresponding random variable in $\mathbf{b}$. Therefore, it is crucial to obtain the first- and second-order partial derivatives of the stochastic function with respect to the random variables to estimate its mean value and covariance from Eqs. (14) and (15), respectively.

2.3.1. Statistics of critical buckling load or coefficient

When combining Eq. (13) with a finite element representation of the linear buckling analysis in Eq. (2), a stochastic form of the eigenvalue problem can be defined, which is a popular uncertainty quantification technique featured with relatively small computational cost and promising accuracy. In the present study, the second-order approximation has been found to be sufficient, whilst higher-order approximations may be used to describe stochastic functions that are highly nonlinear functions of the underlying random variables.

Taking the partial derivative of Eq. (2) with respect to the r -th parameter yields the relation,

$$\left(\frac{\partial \mathbf{K}}{\partial b_r} - \frac{\partial \lambda_i}{\partial b_r} \mathbf{K}_g - \lambda_i \frac{\partial \mathbf{K}_g}{\partial b_r} \right) \mathbf{v}_i + (\mathbf{K} - \lambda_i \mathbf{K}_g) \frac{\partial \mathbf{v}_i}{\partial b_r} = \mathbf{0}. \quad (16)$$

The first partial derivative of λ_i may now be found by premultiplying Eq. (16) by the transpose of the left-hand eigenvector $\mathbf{v}_i$, then substituting Eq.

(2) into the results and finally solving for the partial derivative, such that,

$$\frac{\partial \lambda_i}{\partial b_r} = \left[\mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial b_r} - \lambda_i \frac{\partial \mathbf{K}_g}{\partial b_r} \right) \mathbf{v}_i \right] / (\mathbf{v}_i^T \mathbf{K}_g \mathbf{v}_i) \quad (17)$$

Similarly, the second order partial derivative of the eigenvalue λ_i may be found by differentiating Eq. (16) with respect to the s -th parameter, pre-multiplying by the transpose of the left-hand eigenvector $\mathbf{v}_i$, making use of Eq. (2) and then solving for the second partial derivative:

$$\begin{aligned} \frac{\partial^2 \lambda_i}{\partial b_r \partial b_s} = & \left[\mathbf{v}_i^T \left(\frac{\partial^2 \mathbf{K}}{\partial b_r \partial b_s} - \frac{\partial \lambda_i}{\partial b_r} \frac{\partial \mathbf{K}_g}{\partial b_s} - \frac{\partial \lambda_i}{\partial b_s} \frac{\partial \mathbf{K}_g}{\partial b_r} - \lambda_i \frac{\partial^2 \mathbf{K}_g}{\partial b_r \partial b_s} \right) \mathbf{v}_i \right. \\ & + \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial b_r} - \frac{\partial \lambda_i}{\partial b_r} \mathbf{K}_g - \lambda_i \frac{\partial \mathbf{K}_g}{\partial b_r} \right) \frac{\partial \mathbf{v}_i}{\partial b_s} \\ & \left. + \mathbf{v}_i^T \left(\frac{\partial \mathbf{K}}{\partial b_s} - \frac{\partial \lambda_i}{\partial b_s} \mathbf{K}_g - \lambda_i \frac{\partial \mathbf{K}_g}{\partial b_s} \right) \frac{\partial \mathbf{v}_i}{\partial b_r} \right] / (\mathbf{v}_i^T \mathbf{K}_g \mathbf{v}_i) \quad (18) \end{aligned}$$

To compute the second order derivative of the eigenvalue, λ_i , several additional elements are required, including the second order derivatives of $\mathbf{K}$ and $\mathbf{K}_g$. The first order derivative of the eigenvector, while the calculation of $\partial \mathbf{v}_i / \partial b_r$, is also needed, and is less straightforward to compute than $\frac{\partial^2 \mathbf{K}}{\partial b_r \partial b_s}$ and $\frac{\partial^2 \mathbf{K}_g}{\partial b_r \partial b_s}$. Several methods, including the Nelson method (Nelson, 1976), the modal method (Fox and Kapoor, 1968), and the modified modal method (Wang, 1991), are available to calculate $\partial \mathbf{v}_i / \partial b_r$. In the present study, the modified modal method is adopted as a compromise between accuracy and computational efficiency (Sutter et al., 1988). It uses a linear combination of a given number of eigenvectors as shown in Eq. (19).

$$\frac{\partial \mathbf{v}_i}{\partial b_r} = \mathbf{K}^{-1} \left(\frac{\partial \lambda_k}{\partial b_r} \mathbf{K}_g - \frac{\partial \mathbf{K}}{\partial b_r} + \lambda_k \frac{\partial \mathbf{K}_g}{\partial b_r} \right) \mathbf{v}_k + \sum_{j=1} c_{kj} \mathbf{v}_j \quad (19)$$

where the coefficients c_{kj} are defined as:

$$c_{kj} = \lambda_k \left[\mathbf{v}_j^T \left(\frac{\partial \mathbf{K}}{\partial b_r} - \lambda_k \frac{\partial \mathbf{K}_g}{\partial b_r} \right) \mathbf{v}_k \right] / [\lambda_j (\lambda_k - \lambda_j) \mathbf{v}_j^T \mathbf{K}_g \mathbf{v}_j], \quad k \neq j \quad (20)$$

$$c_{kk} = -\frac{1}{2} \mathbf{v}_k^T \frac{\partial \mathbf{K}_g}{\partial b_r} \mathbf{v}_k \quad (21)$$

149 Note that $\mathbf{v}_k$ in Eq. (21) is normalized with $\mathbf{K}_g$, i.e. $\mathbf{v}_k^T \mathbf{K}_g \mathbf{v}_k = 1$.

In the present study, the uncertainties in material properties and fibre paths are of interest thereby treated as random variables. For the material properties, the basic variables are defined as, $\mathbf{b} = \{E_{1k}, E_{2k}, \nu_{12k}, \nu_{23k}, G_{13k}\}^T, k = 1, 2, \dots, NP$. The first- and second-order partial derivatives of $\mathbf{K}$ are then:

$$\begin{aligned} \frac{\partial \mathbf{K}}{\partial b_r} &= \sum_{e=1}^{NE} \mathbf{B}^T \frac{\partial \hat{\mathbf{D}}_e}{\partial b_r} \mathbf{B} = \sum_{e=1}^{NE} \sum_{k=1}^{NP} \mathbf{B}^T \mathbf{L}_{ek}^T \frac{\partial \hat{\mathbf{D}}_{ek}}{\partial b_r} \mathbf{L}_{ek} \mathbf{B} \\ \frac{\partial^2 \mathbf{K}}{\partial b_r \partial b_s} &= \sum_{e=1}^{NE} \mathbf{B}^T \frac{\partial^2 \hat{\mathbf{D}}_e}{\partial b_r \partial b_s} \mathbf{B} = \sum_{e=1}^{NE} \sum_{k=1}^{NP} \mathbf{B}^T \mathbf{L}_{ek}^T \frac{\partial^2 \hat{\mathbf{D}}_{ek}}{\partial b_r \partial b_s} \mathbf{L}_{ek} \mathbf{B} \end{aligned} \quad (22)$$

150 where for $\frac{\partial \hat{\mathbf{D}}_{ek}}{\partial b_r}$ it is required to calculate $\frac{\partial \mathbf{C}_{1k}}{\partial b_r}$ and $\frac{\partial \mathbf{C}_{2k}}{\partial b_r}$ as functions of the
 151 material properties as indicated in Eq. (5). In this way, uncertainties in the
 152 material properties can be propagated to the buckling eigenvalues.

It is also required to calculate the first- and second-order partial derivatives of $\mathbf{K}$ with respect to fibre tow path parameters in Eq. (1) to propagate uncertainties associated with variations in fibre paths. As the fibre path directions are assumed to vary linearly, uncertainty in $\mathbf{K}$ will propagate through the parameters $\mathbf{b} = \{\phi_k, T_{0k}, T_{1k}\}^T, k = 1, 2, \dots, NP$. The partial derivatives of $\mathbf{K}$ with respect to $\mathbf{b}$ are calculated, therefore, as:

$$\begin{aligned} \frac{\partial \mathbf{K}}{\partial b_r} &= \sum_{e=1}^{NE} \sum_{k=1}^{NP} \frac{\partial \mathbf{k}_e}{\partial \theta_{ek}} \frac{\partial \theta_{ek}}{\partial b_r} \\ &= \sum_{e=1}^{NE} \sum_{l=1}^{NP} \left[\mathbf{B}^T \left(\frac{\partial \mathbf{L}_{ek}^T}{\partial \theta_{ek}} \mathbf{D}_{ek} \mathbf{L}_{ek} + \mathbf{L}_{ek}^T \mathbf{D}_{ek} \frac{\partial \mathbf{L}_{ek}}{\partial \theta_{ek}} \right) \frac{\partial \theta_{ek}}{\partial b_r} \mathbf{B} \right] \end{aligned} \quad (23)$$

and

$$\begin{aligned}\frac{\partial^2 \mathbf{K}}{\partial b_r \partial b_s} &= \sum_{e=1}^{NE} \sum_{k=1}^{NP} \left[\frac{\partial^2 \mathbf{k}_e}{\partial \theta_{ek}^2} \frac{\partial \theta_{ek}}{\partial b_r} \frac{\partial \theta_{ek}}{\partial b_s} + \frac{\partial \mathbf{k}_e}{\partial \theta_{ek}} \frac{\partial^2 \theta_{ek}}{\partial b_r \partial b_s} \right] \\ &= \sum_{e=1}^{NE} \sum_{k=1}^{NP} \left[\mathbf{B}^T \left(\frac{\partial^2 \mathbf{L}_{ek}^T}{\partial \theta_{ek}^2} \mathbf{D}_{ek} \mathbf{L}_{ek} + \mathbf{L}_{ek}^T \mathbf{D}_{ek} \frac{\partial^2 \mathbf{L}_{ek}}{\partial \theta_{ek}^2} \right) \frac{\partial \theta_{ek}}{\partial b_r} \frac{\partial \theta_{ek}}{\partial b_s} \mathbf{B} \right] \quad (24)\end{aligned}$$

Once the partial derivatives of $\mathbf{K}$ are obtained, the first- and second-order partial derivatives of $\mathbf{K}_g$ can be calculated. As indicated in Eq. (8), K_g is an implicit function of $\mathbf{K}$. Because stress resultants, $\mathbf{S}$, result from solving a stationary problem of $\mathbf{K}\mathbf{u} = \mathbf{f}$, where $\mathbf{u}$ and $\mathbf{f}$ are the nodal displacement vector and the nodal force vector, respectively. Therefore, the derivatives of $\mathbf{K}_g$ are straightforwardly deduced following some matrix manipulations.

The above formulations are derived in a generalised form such that random variables for each layer are treated as independent parameters. In practice, layers in a laminate stack in a given pattern, e.g. balance or symmetry, and the same material may be used for all layers or multiple layers, requiring contributions from identical random variables to be grouped together to represent the uncertainty quantification. For uncertainties in material properties, contributions from layers with the same material are added. Uncertainties in fibre tow path are summed up with the use of its stacking sequence.

3. Results and discussion

In this research, a twelve-layer VAT composite laminate studied in (Gürdal et al., 2008) has been adopted to understand its stochastic mechanical behaviours. It is made up of AS4 carbon fibres and a 3501-6 epoxy matrix with

material properties: $E_1 = 181GPa$, $E_2 = 8.96GPa$, $\nu_{12} = 0.3$, $\nu_{23} = 0.28$ and $G_{13} = 7.2GPa$. It has a total thickness of $h = 1.524$ mm with all plies having an identical thickness of 0.127 mm, and it is square with length and width of $a = 0.254$ m. A uniform end shortening displacement is applied to the structure along x direction. Following (Gürdal et al., 2008), two types of boundary conditions recognized as typical ones are considered, which include one with translation constrained in the y -direction on the transverse edge (denoted as Case I shown in Figure 3) and another with the transverse edge free to deform (denoted as Case II shown in Figure 3). Several numerical studies have been conducted to understand the variations in VAT plates performances with the use of the proposed uncertainty quantification method. In addition to buckling eigenvalue, a further buckling measure, namely, the critical buckling coefficient, which is related to critical buckling strength in the form of load normalized by $a^2/(E_1 h^3)$ for better indicating of the enhancement comparing with a straight fibre laminate is used and calculated as:

$$K_{cr} = \lambda N_x^{cr} \frac{a^2}{E_1 h^3} \quad (25)$$

where N_x^{cr} is the critical loading in terms of axial stress resultant. For the Case I panel, N_x^{cr} is the same as N_x because it is constant over the panel, while N_x varies in y -direction and N_x^{cr} is an average critical load calculated by $\bar{N}_x^{cr} = \frac{1}{b} \int_{-h/2}^{h/2} N_x(y) dy$ (Gürdal et al., 2008) for Case II. Eq. (25) is approximated again using Taylor series expansion with its first- and second-order derivatives given in Eq. (26). Thus, the outcomes from PSFE are the

required basic elements to predict statistics of the critical buckling coefficient.

$$\begin{aligned}\frac{\partial K_{cr}}{\partial b_r} &= \left(\frac{\partial \lambda}{\partial b_r} N_x^{cr} + \lambda \frac{\partial N_x^{cr}}{\partial b_r} \right) \frac{a^2}{E_1 h^3} \\ \frac{\partial^2 K_{cr}}{\partial b_r \partial b_s} &= \left(\frac{\partial^2 \lambda}{\partial b_r \partial b_s} N_x^{cr} + \frac{\partial \lambda}{\partial b_r} \frac{\partial N_x^{cr}}{\partial b_s} + \frac{\partial \lambda}{\partial b_s} \frac{\partial N_x^{cr}}{\partial b_r} + \lambda \frac{\partial^2 N_x^{cr}}{\partial b_r \partial b_s} \right) \frac{a^2}{E_1 h^3}\end{aligned}\quad (26)$$

169 3.1. Model validation

170 3.1.1. Finite element analysis

171 Table 1 illustrates the first five buckling eigenvalues of the investigated
 172 VAT plates. The results obtained using the in-house finite element imple-
 173 mentation are compared with those from ANSYS. The laminate has been
 174 discretized into a mesh of 20 by 20 elements. In ANSYS, the shell181 ele-
 175 ment, which is a four node rectangular shell element and suitable for ana-
 176 lyzing thin to moderately-thick shell structures, is chosen as the underlying
 177 theory is the same as the one used in the present study. Elemental fibre
 178 angles from Eq. (1) are imported into the finite element model in ANSYS.
 179 For the two panels, two types of layup with one for single player and another
 180 with 12 layers have been considered. For a single layer laminate, the fibre
 181 path is defined by $0^\circ < 0^\circ|50^\circ >$ for Case I and $90^\circ < 0^\circ|75^\circ >$ for Case
 182 II. For the multilayer laminates, Case I panel features with fibre paths of
 183 $0^\circ \pm < 0^\circ|50^\circ >_{3s}$ and Case II of $90^\circ \pm < 0^\circ|75^\circ >_{3s}$. Eigenvalues under
 184 the applied displacement have been calculated by both ANSYS and in-house
 185 code, and results for the first five modes are listed in the table. The compar-
 186 isons show that the in-house code match excellently with those from ANSYS.
 187 In addition, the critical buckling loads for the two multilayer laminates from
 188 the in-house are compared with those reported in (Gürdal et al., 2008), with
 189 a load of 1.41 from the former, representing a prediction around 2% less than

that reported in (Gürdal et al., 2008) of 1.44 for Case I panel. For Case II panel, the maximum normalized value of the critical load is 3.07, which is only about 2% less than the reference value of 3.14 in (Gürdal et al., 2008). In all the cases, the critical buckling loads are slightly smaller than those found in Gürdal et al. (2008), where the buckling problem was solved by the Ritz method, which results in a slight overestimate of the actual buckling load as a finite set of terms are taken. Hence, the implementation is proven to provide promising predictions.

3.1.2. Uncertainty quantification

Before applying PSFE to study the probabilistic mechanical behaviours of VAT composites, its accuracy and efficiency are investigated through comparisons with outcomes from MCS. A simulation with 5000 samples has been performed to estimate the buckling statistics in the presence of uncertainties in material properties and fibre paths. Statistics of the buckling eigenvalues and the critical buckling coefficients are investigated. Although the perturbation method is independent of the type of probability distributions describing the input random variables, in principle, Normal distributions are usually considered to produce more accurate estimates due to their symmetry properties. For MCS, material properties are assumed to be Normally distributed variables, and fibre tow path parameters uniformly distributed. Two special cases are considered. In the first case, uncertainties are limited to variability in the material properties, with a coefficient of variation of 10%. In the second case, variations in the fibre path with its parameters distributed with $[-5^\circ, +5^\circ]$ are investigated. Results are presented in Tables 2 and 3. Both mean value and standard deviation predictions from

215 the proposed perturbation-based method are sufficiently close to those from
216 the MCS. It indicates that PSFE works well when having these practical
217 uncertainties.

218 It is well known that PSFE performs well for small variations, but its
219 accuracy is less assured for large variations when stochastic functions are
220 strongly nonlinear with respect to the input random variables. It is crucial
221 to understand the feasible regions to be confident to use PSFE for more
222 advanced studies. It is not straightforward to qualify the extent of nonlin-
223 earity of the buckling eigenvalues and the critical buckling coefficients with
224 respect to material properties and fibre tow path. Further numerical studies
225 have been conducted to determine the acceptable ranges of the coefficient
226 of variations in these two sets of random variables for accurately predicting
227 buckling statistics for VAT laminates. Outcomes of PSFE studies are given
228 in Figures 4 and 5. Considering a 'good' approximation as one giving less
229 than a 5% discrepancy from the exact result, i.e. from MCS here, it appears
230 that estimates of the coefficients of variation for the critical buckling loads
231 can be accepted as long as coefficients of variation in the material proper-
232 ties are less than 0.3 (or 30%), whilst smaller variations in fibre tow path
233 parameters, i.e. $\pm 7.5^\circ$, should be considered reasonable for PSFE. This can
234 be partially explained by the fact that the exact Taylor series expansion de-
235 scribing the critical buckling coefficient with respect to material properties
236 is predominantly controlled by constant and linear terms. Furthermore, it
237 may be explained from the constitutive model which is a linear function of
238 Young's modulus and the shear modulus, while a nonlinear function of the
239 Poisson's ratios. With the increase of variability in the Poisson's ratios, the

240 terms including $1/(1 - 2\nu)$ have the possibility (although improbable) to
 241 take values approaching infinity, with consequential loss of accuracy in pre-
 242 dicting the statistics of the buckling estimates. In contrast, nonlinear terms
 243 related to fibre tow paths have significant contributions in the terms omitted
 244 from the truncated Taylor series expansion describing the buckling measure,
 245 due to the nature of fibre tow angle's contribution through the terms $\cos \theta$
 246 and $\sin \theta$, which are infinitely differentiable, i.e. the existence of significant
 247 higher-order nonlinear terms. As a conclusion, it is seen in this example
 248 that the second-order perturbation method can provide sufficiently accurate
 249 estimates for buckling statistics given relatively large uncertainties in the
 250 material properties. Whilst the second-order perturbation method cannot
 251 necessarily provide satisfactory results for medium to large coefficients of
 252 variation in the input random variables for fibre angles, it can provide suf-
 253 ficiently accurate estimates when the fluctuations are in the range $\pm 7.5^\circ$,
 254 which is the most commonly reported variability in fibre misalignment made
 255 from AFP, e.g. (Potter et al., 2008). Therefore, it is hypothesised that PSFE
 256 can be confidently used to investigate the uncertainties in these two sets of
 257 random variables for understanding practical and realistic variations in VAT
 258 composite plates.

259 *3.2. Effect of material properties*

260 For the two cases, statistics of buckling measures have been calculated
 261 parametrically with various angles of T_1 from 0° to 90° while keeping the
 262 other two parameters ϕ and T_0 unchanged, i.e. $\phi = 0^\circ$ and $T_0 = 0^\circ$ for Case
 263 I panel, while $\phi = 90^\circ$ and $T_0 = 0^\circ$ for Case II. All the five independent
 264 material properties, i.e. E_1 , E_2 , ν_{12} , G_{12} and G_{23} , have been considered as

265 random variables with coefficient of variation of 0.1 or 10%. Eqs.(17) and (18)
 266 are used to obtain ingredients to calculate the mean value and the standard
 267 deviation of eigenvalue from Eqs (14) and (15), respectively, while Eq.(26)
 268 is called for the critical buckling coefficient. For each angle of T_1 , statistics
 269 of these two buckling measures have been calculated for the two cases. The
 270 probabilistic buckling analysis results are reported in terms of statistics of
 271 buckling eigenvalues and critical buckling coefficients in Figures 6 and 7 for
 272 Cases I and II, respectively.

273 For Case I, the peak (or the maximum) eigenvalue appears around $T_1 =$
 274 80° , which is different from the angle $T_1 = 50^\circ$ for the maximum of critical
 275 buckling coefficient, while the minima of the eigenvalue standard deviation
 276 and coefficient of variation occur when the critical buckling coefficient reaches
 277 its maximum. Figure 6a shows that the coefficients of variation for eigen-
 278 value vary from 0.2% to 4.5%, while the coefficients of variation for critical
 279 buckling coefficient are around 8% with slight dependence with respect to the
 280 variation of T_1 shown in Figure 6b. There is clear evidences that variation
 281 in the critical buckling coefficient increases when the structure shows better
 282 buckling performance, while no such a clear trend is shown for the variation
 283 in the corresponding eigenvalues. It is understood that the enhancement of
 284 buckling performance of Case I panel is largely due to a favourable distribu-
 285 tion of the transverse stress resultant $N_y = A_{11}/A_{12}N_x$ at the panel edges
 286 along the length of the panel, where A_{11} and A_{12} are terms of the membrane
 287 stiffness matrix and N_x is the x-direction in-plane stress resultant with con-
 288 stant value. Hence, the coefficient of variation in the eigenvalue becomes
 289 smaller when the structure becomes stiffer with the plate, therefore, achiev-

ing a higher buckling strength. In contrast, the coefficient of variation in the critical buckling factor, attains its maximum value at almost the same point at which the mean value reaches its maximum. This is partially due to the fact that the critical buckling coefficient is a function of λN_x which may be dominated by variations in N_x , as its mean value remains as a constant and its variation is only affected by uncertainties in material properties.

Similarly, statistics for the two buckling measures are given in Figure 7 for the Case II panel. It is observed that the variations in the eigenvalue and the critical buckling coefficient are linearly correlated to the input variations. For instance, the majority of the coefficients of variation of eigenvalue are around 10% with a 10% input variations in the material properties as illustrated in Figure 7a, while the coefficient of variation for the critical buckling coefficient gradually increases with T_1 before reaching its maximum shown in Figure 7b. Again, the peaks in the eigenvalue mean value and standard deviation occur when $T_1 \approx 55^\circ$ while the peak for the critical buckling coefficient appears when T_1 is around 75° . This is potentially caused by the fact that the eigenvalue only reflects the buckling occurrence in terms of the displacement, while the critical buckling coefficient measures the buckling strength by transforming the displacement based strength into the load. In addition, it is understood that the reason for the substantially improved buckling response characteristics is due to a favourable distribution of the applied load across the panel width as a function of the y coordinate, which is different from the one observed for the Case I panel. N_{xy} and N_y are zero over the panel, while $N_x(y) = 2u_0 [A_{11} - A_{12}^2/A_{22}] / a$ with A_{11} , A_{12} and A_{22} are the terms of in-plane stiffness matrix, u_0 is the applied uniform displacement,

315 and a is the panel length. The variation of the eigenvalue is directly linked
 316 to the variation in the stiffness, and the same magnitude of the coefficient
 317 of variation is thus observed in the eigenvalue as the input. In addition,
 318 the critical buckling load factor is calculated through $\lambda \int N_x(y)dy/b$, which
 319 indicates that its variation is around the square sum of the corresponding
 320 values for λ and N_x . For instance, the maximum coefficient of variation of
 321 the critical buckling coefficient is around $\sqrt{2} \times 10\% \approx 14.1\%$. In addition,
 322 the slight discontinuities in the curves for coefficient of variation is caused
 323 by the change of buckling mode shape when T_1 increases from 73° to 74°
 324 as shown in Figure 8. The first two buckling mode shapes swap their posi-
 325 tions, i.e. the first mode in $T_1 = 73^\circ$ becomes the second mode in $T_1 = 74^\circ$.
 326 Due to the fact that the eigenvalues of these two first modes are very close,
 327 there is no apparent discontinuity observed on the curves of mean values.
 328 However, the swapping of mode shape leads to a jump in the first-order par-
 329 tial derivatives with respect to material properties. For instance, Figure 9
 330 gives the first-order derivatives of the first two lowest eigenvalues and critical
 331 buckling coefficients with respect to the longitudinal Young's modulus, E_1 ,
 332 for T_1 changing from 69° and 79° . The derivatives are discontinued between
 333 73° and 74° due the swapping of mode shapes. It finally reflects on the
 334 discontinuities in the curves of coefficient of variation as they are functions
 335 of the first order partial derivatives.

336 3.3. *Effect of fibre tow angle*

337 In this section, statistics of buckling performance measures are inves-
 338 tigated with considerations of uncertainties in fibre tow paths for the two
 339 panels. Again, ϕ and T_0 keep unchanged while T_1 varies from 0° to 90° . The

uncertainties in fibre tow paths are considered through the three parameters ϕ , T_0 and T_1 . In this study, they are treated as uniform distribution variables with fluctuations of $\pm 5^\circ$. For each T_1 , mean values and standard deviations for the two buckling measures are calculated from Eqs. (14) and (15), respectively. They are then shown in Figures 10 and 12 for Cases I and II, respectively.

For the eigenvalue curves of Case I, the standard deviations in the upper panel of Figure 10 show almost the same trend as the mean value with increasing T_1 . However, the standard deviation reaches its maximum at $T_1 \approx 65^\circ$, which is different to the mean value. In addition, as changes in the standard deviations differ from those of the mean values, the coefficients of variation vary from 0 to 4% with the increase of T_1 . In general, the coefficient of variation follows a similar pattern to the standard deviation. According to Eq. (15), the first order derivatives are major contributors to the standard deviation. Hence, these observations are confirmed by the plots of the first order derivatives of the eigenvalue with respect to the three parameters, ϕ , T_0 and T_1 as given in Figure 11. The change of T_1 almost linearly affects the positive contributions of these three parameters before reaching its maximum for upper panel figure for eigenvalues. For the statistics describing the critical buckling coefficient shown in the lower panel of Figure 10, the plot of the standard deviation shows a more complicated trend. Figure 10b indicates that the standard deviation reaches a local maximum around $T_1 = 25^\circ$ before it reaches its local minimum around $T_1 = 50^\circ$ when the mean value attains its maximum, but the standard deviation then increases till $T_1 \approx 80^\circ$. A close inspection of the first order derivatives of the critical buckling coefficient

365 shown in Figure 11b explains the fluctuation as the three parameters switch
 366 from positive contributions to the standard deviation to negative contribu-
 367 tions with the increasing of T_1 . The coefficient of variation shows a similar
 368 trend as the standard deviation. Again, it is not straightforward to link the
 369 variation of coefficient of variation with the change of fibre angle path in
 370 terms of T_1 here. The transverse stress resultant $N_y = A_{11}/A_{12}N_x$ indicates
 371 that the variation in eigenvalue is closely linked with changes in the stiffness
 372 as reflected in N_x but slightly affected by A_{11}/A_{12} . Hence, the eigenvalue
 373 coefficients of variation and critical buckling coefficient become smaller when
 374 the structure becomes stiffer, thereby displaying a higher buckling strength.

375 For the eigenvalue plots of Case II panel, the standard deviations in Fig-
 376 ure 12a show that they become larger before reducing in magnitude to reach
 377 a local minimum when the eigenvalue displays a maximum value with the
 378 change of T_1 from 0 to 90° . This trend appears on the plot of the coeffi-
 379 cient of variation for the buckling eigenvalue in the lower diagram in Figure
 380 12a. Similarly, the stochastic analysis confirms that the buckling eigenvalue
 381 reaches its maximum before swapping buckling mode shape as a jump on the
 382 critical buckling coefficient curve is found, which shows similar phenomenon
 383 as the one with uncertainties in material properties. This is further confirmed
 384 by the plot of the first order derivatives of eigenvalue and critical buckling
 385 coefficient in Figure 13. To show clearly the discontinuities of the buckling
 386 statistics caused by mode switching, the first order partial derivatives of the
 387 eigenvalues and critical buckling coefficients for the first two modes are given
 388 in Figure 14, where it can be seen that the first order partial derivatives of
 389 the lowest eigenvalues are swapped. According to Eq. (15), the first order

derivatives are major contributors of the standard deviation. For the statistics of critical buckling coefficient shown in the lower panel of Figure 12, the curve of standard deviation shows almost the same trend as the mean value with the increase of T_1 . This leads to a plateau for the curve of the coefficient of variation. Although there is no evident link between the increase of T_1 and the variation of the coefficient of variation of the buckling measures, the varying of transverse stress resultant $N_x(y) = 2u_0 [A_{11} - A_{12}^2/A_{22}] / a$ in y -direction indicates that the variation in eigenvalue is originated from the changes in the stiffness as A_{11} , A_{12} and A_{22} are functions of fibre tow path.

3.4. *Effect of boundary conditions*

As it has been well recognized, boundary conditions have a significant influence on the buckling performance of VAT composite panels. For example, Case I panel with restricted translation in the y -direction may have a maximum critical buckling coefficient of around 1.41 with fibre tow path defined by $0^\circ \pm < 0^\circ | 50^\circ >_{3s}$, whilst the Case II panel with free deformation on the transverse edges has a higher critical buckling coefficient of 3.07 with a fibre tow path of $90^\circ \pm < 0^\circ | 75^\circ >_{3s}$. Their buckling statistics with uncertainties in either material properties or fibre tow path also display clear differences as shown in Figures 6 and 7, and 10 and 12, respectively. For instance, in the presence of uncertainties in material properties, the variations in eigenvalue in Case I have completely different trends and also magnitudes of coefficients of variation when the angle T_1 is varied. Figure 15 shows the sensitivities of the critical buckling coefficients with respect to variation in each material property for the two panels with their optimal design given in (Gürdal et al., 2008). The buckling statistics of Case I panel are mostly affected by ν_{23} ,

415 while Case II panel is most sensitive to ν_{12} . In terms of the influence of each
 416 material property, it can be seen that the Case I panel is positively linked
 417 with E_1 , E_2 , ν_{23} , and G_{13} , but ν_{12} has a negative contribution to its variation.
 418 While E_1 , ν_{12} , and G_{13} have positive effects on the buckling statics of Case
 419 II panel, E_2 and ν_{23} show negative correlations. It appears that these char-
 420 acteristics are due to changes in the boundary conditions. For variations in
 421 buckling performance measures arising from uncertainties in fibre tow path,
 422 no significant difference between the two boundary conditions is observed.

423 **4. Conclusions**

424 In this paper, the widely used uncertainty quantification method, namely
 425 perturbation-based stochastic finite element method, is adopted to investi-
 426 gate the variability in mechanical performance of variable angle tow com-
 427 posite plates, which are tailored to have an enhanced buckling performance.
 428 Variations in buckling performance arising from uncertainties in material
 429 properties and fibre tow path are investigated and measured by its first two
 430 moments, namely mean value and standard deviation. Detailed formula-
 431 tions for predicting buckling statistics are derived and given to conduct a
 432 perturbation-based stochastic finite element analysis. As the primary objec-
 433 tive of this work is to understand the variations in buckling performance,
 434 relatively simple VAT composite plates with the use of a linear variation of
 435 the fibre tow path are studied. However, the formulae can be easily extended
 436 to analyse VAT plates with more complicated fibre tow paths.

437 The use of PSFE to quantify uncertainties in buckling performance mea-
 438 sures including eigenvalue and critical buckling coefficient has been verified

439 first through comparing statistics against corresponding values from MCS.
 440 The comparisons show that PSFE can predict the variations in buckling mea-
 441 sures with sufficient accuracy, while the computational cost can be signifi-
 442 cantly reduced. PSFE can successfully propagate uncertainties in material
 443 properties with a coefficient of variation of 0.3, and it is also applicable to
 444 quantify uncertainties in fibre tow path, which have fluctuations as large as
 445 about $\pm 10^\circ$. Therefore, PSFE can deal with these two types of uncertainties
 446 with a sufficiently wide range. With this confidence, the method is further
 447 applied to understand the stochastic mechanical behaviour of VAT compos-
 448 ite plates. Results show that buckling statistics in terms of mean value and
 449 standard deviation or coefficient of variation are significant functions of un-
 450 certainties in fibre tow paths, and should not be ignored. Uncertainties in
 451 buckling measures arising from variations in material properties are posi-
 452 tively correlated as statistics reaches their maxima while maximal buckling
 453 performances are achieved. In contrast, statistics of buckling measures have
 454 their minima while optimal buckling performance attains. These findings
 455 imply the importance to take variations in material properties into account
 456 to find robust design of VAT composite structures.

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Table 1: Comparisons of buckling eigenvalues obtained from in-house program and ANSYS

Layup	Mode	Case I		Case II	
		ANSYS	Inhouse	ANSYS	Inhouse
Single	1	2.7181e-7	2.7148e-7	8.4566e-7	8.4864e-7
	2	5.5561e-7	5.4834e-7	8.5250e-7	8.5304e-7
	3	6.8151e-7	6.7786e-7	1.1386e-6	1.1384e-6
	4	7.8559e-7	7.8298e-7	1.4290e-6	1.4267e-6
	5	1.1699e-6	1.1493e-6	1.8525e-6	1.8588e-6
12-layer	1	2.2802e-5	2.2798e-5	1.3608e-4	1.3579e-4
	2	4.3446e-5	4.3443e-5	1.3708e-4	1.3689e-4
	3	5.5295e-5	5.5239e-5	1.8683e-4	1.8669e-4
	4	7.0165e-5	7.0110e-5	2.6192e-4	2.6180e-4
	5	8.5142e-5	8.5163e-5	2.7680e-4	2.7565e-4

Table 2: Mean values (MV) and standard deviations (STD) of buckling eigenvalues and critical buckling coefficients for VAT plate due to variations in material properties

Case	Method	λ		K_{cr}	
		MV	STD	MV	STD
I	MCS	2.2295e-5	8.7518e-8	1.3834	0.1265
	PSFE	2.2294e-5	8.6747e-8	1.3824	0.1252
II	MCS	1.3502e-4	1.1473e-5	3.0273	0.3594
	PSFE	1.3479e-4	1.1426e-5	3.0315	0.3533

Table 3: Mean values and standard deviations of buckling eigenvalues and critical buckling coefficients for VAT plate due to variations in fibre tow path

Case	Method	λ		K_{cr}	
		MV	STD	MV	STD
I	MCS	2.2208e-5	5.0548e-8	1.3771	0.0055
	PSFE	2.2207e-5	5.6706e-8	1.3769	0.0057
II	MCS	1.3238e-4	4.5574e-6	2.9988	0.0346
	PSFE	1.3258e-4	5.0503e-6	2.9979	0.0382

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(a) Fibre path functions

(b) Linear variation

(c) Nonlinear variation

Figure 1: Configuration of the variable angle tow composite laminate.

Figure 2: Discretised fibre path with linear variation formula: $0^\circ < 0^\circ|50^\circ >$.

Figure 3: In-plane deformation profile of Case I and Case II panels under uniform end shortening.

Figure 4: Coefficients of variation of normalized buckling factors from MCS and PSFE with uncertainties in material properties with normal distributions

Figure 5: Coefficients of variation of normalized buckling factors from MCS and PSFE with uncertainties in fibre tow path with uniform distributions

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 6: Buckling statistics of Case I panel under variations in material properties

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 7: Buckling statistics of Case II panel under variations in material properties

(a) 1st mode shape, $K_{cr} = 3.0240$

(b) 2nd mode shape, $K_{cr} = 3.02594$

(c) 1st mode shape, $K_{cr} = 3.0673$

(d) 2nd mode shape, $K_{cr} = 3.0679$

Figure 8: The first two buckling mode shapes under different T_1 : (a) and (b) for $T_1 = 73^\circ$; (c) and (d) for $T_1 = 74^\circ$.

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 9: The first order partial derivative of the first two lowest λ and K_{cr} with respect to E_1 .

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 10: Buckling statistics of Case I panel under variations in fibre tow paths

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 11: The first order partial derivatives of λ and K_{cr} with respect to fibre tow path parameters for Case I panel

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 12: Buckling statistics of Case II panel under variations in fibre tow paths

(a) Eigenvalues

(b) Critical buckling coefficients

Figure 13: The first order partial derivatives of λ and K_{cr} with respect to fibre tow path parameters for Case II panel

(a) $\partial\lambda/\partial\phi$

(b) $\partial K_{cr}/\partial\phi$

(c) $\partial\lambda/\partial T_0$

(d) $\partial K_{cr}/\partial T_0$

(e) $\partial\lambda/\partial T_1$

(f) $\partial K_{cr}/\partial T_1$

Figure 14: Discontinuous region of the first order partial derivatives of λ and K_{cr}

Figure 15: Sensitivity factors of material properties for buckling statistics.

Figure 16: Sensitivity factors of fibre tow path parameters for buckling statistics.