Application of Wake Oscillators to Two-Dimensional Vortex-Induced Vibrations of Circular Cylinders in Oscillatory Flows

Pierre-Adrien Opinel & Narakorn Srinil[†]

Marine, Offshore & Subsea Technology Group, School of Engineering, Newcastle University,
Newcastle upon Tyne, NE1 7RU, United Kingdom

Abstract

A nonlinear time-domain simulation model for predicting two-dimensional vortex-induced vibration (VIV) of a flexibly mounted circular cylinder in planar and oscillatory flow is presented. This model is based on the utilization of van der Pol wake oscillators, being unconventional since wake oscillators have typically been applied to steady flow VIV predictions. The time-varying relative flow-cylinder velocities and accelerations are accounted for in deriving the coupled hydrodynamic lift, drag and inertia forces leading to the cylinder cross-flow and in-line oscillations. The system fluid-structure interaction equations explicitly contain the time-dependent and hybrid trigonometric terms. Depending on the Keulegan-Carpenter number (KC) incorporating the flow maximum velocity and excitation frequency, the model calibration is performed, entailing a set of empirical coefficients and expressions as a function of KC and mass ratio. Parametric investigations in the case of varying KC, reduced flow velocity, cylinder-to-flow frequency ratio and mass ratio are carried out, capturing some qualitative features of oscillatory flow VIV and exploring the effects of system parameters on response prediction characteristics. The model dependence of hydrodynamic coefficients on the Reynolds number is studied. Discrepancies and limitations versus advantages of the present model with different feasible solution scenarios are illuminated to inform the implementation of wake oscillators as a computationally efficient prediction model for VIV in oscillatory flows.

Keywords: vortex-induced vibration (VIV), oscillatory flow, circular cylinder, wake oscillator, Keulegan-Carpenter number.

[†] Corresponding author: narakorn.srinil@newcastle.ac.uk
Tel. +44 191 208 6499; Fax +44 191 208 5491

Nomenclature

A_s	Cylinder outer cross-sectional area
$A_x/D, A_y/D$	In-line and cross-flow amplitudes per diameter
C_a	Added mass coefficient of oscillating cylinder in still water
$C_D (C_{D0})$	Unsteady drag force coefficient of oscillating (stationary) cylinder
$C_L (C_{L0})$	Unsteady lift force coefficient of oscillating (stationary) cylinder
C_{DM}	Morison drag force coefficient of stationary cylinder in oscillatory flow
C_M	Inertia coefficient of stationary cylinder in oscillatory flow
C_s	Cylinder viscous damping coefficient
D	Cylinder outer diameter
F_{cx}, F_{cy}	Cylinder feedback terms in dimensional form
F_D, F_L	Dimensional total drag and lift hydrodynamic forces
F_I	Dimensional inertia force
f_{cx}, f_{cy}	Cylinder feedback terms in dimensionless form
f_h	Higher-order nonlinear effect due to relative flow-cylinder velocities
f_n	Cylinder natural frequency in still water
f_r	Cylinder-to-flow frequency ratio
$f_v(\omega_v)$	Fundamental or dominant lift force frequency (angular frequency)
$f_w(\omega_w)$	Oscillatory flow frequency (angular frequency)
f_{wx}, f_{wy}	Total in-line and cross-flow hydrodynamic forces in dimensionless form
f_x, f_y	Dimensional in-line and cross-flow oscillation frequencies
KC	Keulegan-Carpenter number
M_L, M_D, M_{DM}	Combined hydrodynamic coefficient and mass ratio parameters
m_s, K_s	Cylinder mass and stiffness in dimensional system
n	Number of oscillations in the lift force per flow cycle
$N_x (N_y)$	Number of cylinder vibrations per flow cycle in in-line (cross-flow) direction
p, q	Wake oscillator variables modelling the fluctuating drag and lift forces
Re	Reynolds number
St	Strouhal number
$t (\tilde{t})$	Dimensionless (dimensional) time
U	Oscillatory flow velocity
U_m	Oscillatory flow maximum velocity
V_r	Reduced flow velocity parameter
V_{rel}	Oscillatory flow-cylinder relative velocities
$x, y (X, Y)$	Dimensionless (dimensional) in-line and cross-flow displacements
$\alpha_x, \alpha_y, \beta_x, \beta_y$	Geometrically nonlinear coefficients in dimensionless form
$\alpha_x^*, \alpha_y^*, \beta_x^*, \beta_y^*$	Geometrically nonlinear coefficients in dimensional form
β	Stokes, viscous or frequency parameter
$\varepsilon_x, \varepsilon_y (A_x, A_y)$	Empirical wake damping (wake-cylinder coupling) coefficients
θ	Instantaneous angle of lift and drag forces acting on oscillating cylinder
λ	Combined fluid-structural damping terms
μ, m^*	Mass parameter and mass ratio
ζ	Cylinder damping ratio in still water
ρ	Fluid density
ν	Fluid kinematic viscosity

1. Introduction

In coastal, offshore and ocean engineering applications, a cylindrical structure such as a floating spar platform or fixed-foundation monopile supporting wind turbines, an underwater cable, subsea pipeline, marine riser or mooring line may experience vortex-induced vibration (VIV) in oscillatory flow caused by wave or support motion. Although the dynamic features of cylinder VIV in steady flows have been extensively investigated, much less attention has been paid to the modelling and prediction of unsteady and oscillatory flow VIV in the literature. Therefore, practical offshore design in waves is subject to such lacking knowledge and consequent conservatism. With a recent progress in offshore energy development across shallow to deep waters where effects of waves and platform motion become predominant, there is a growing industrial demand for VIV research involving oscillatory flows and a computationally efficient tool for use in a preliminary analysis.

Several theoretical, numerical and experimental studies have contributed to an improved understanding of fluid mechanics, hydrodynamics and wake vortex-shedding visualizations associated with oscillatory flows past stationary cylinders (Justesen, 1991; Obasaju et al., 1988; Sarpkaya, 1976, 1979; Williamson, 1985). One of the key features of a circular cylinder exposed to an oscillatory flow, distinguishing itself from that of a steady flow, is the occurrence of periodic flow reversals causing a return of the shed vortices from being downstream to upstream and a sudden change in the associated hydrodynamic forces with multiple excitation frequencies. These occur whenever the flow velocity changes in sign or direction. As the cylinder is free to move transversely, cross-flow VIV may take place with multiple lock-in events leading to multi-peak amplitude response characteristics (Sumer and Fredsoe, 2006). The oscillatory flow VIV behaviour of an elastically mounted circular cylinder is governed by some dimensionless fluid-structure parameters. With a maximum oscillatory flow velocity U_m , key parameters include the Keulegan-Carpenter number (KC), Reynolds number (Re), reduced flow velocity (V_r) and Strouhal number (St), whose expressions read

$$KC = \frac{U_m D}{f_w D}, \quad Re = \frac{U_m D}{\nu}, \quad V_r = \frac{U_m}{f_n D}, \quad St = \frac{f_v D}{U_m}, \quad (1)$$

in which f_w is the flow frequency, D the cylinder diameter, ν the fluid kinematic viscosity, f_v the fundamental or dominant lift force frequency as a result of the vortex shedding and reversal, and f_n the structural natural frequency in still water. Equation (1) enables additional relationships:

$$\frac{KC}{V_r} = \frac{f_n}{f_w} = f_r, \quad StKC = \frac{f_v}{f_w} = n, \quad StV_r = \frac{f_v}{f_n}, \quad \beta = \frac{Re}{KC} = \frac{D^2 f_w}{\nu}, \quad (2)$$

in which f_r is the cylinder-to-flow frequency ratio, n is the number of lift force oscillations which may be equivalent to the number of vortex pairs shed from the cylinder plus a unity (i.e. a flow reversal) within each oscillation cycle, and β is the so-called Stokes, viscous or frequency parameter (Sarpkaya, 2010). In general, n increases with KC. For stationary cylinders, the oscillatory flow in the $1.8 \times 10^3 < Re < 10^4$ range was found to produce one, two, three and four pairs of vortices for $7 < KC < 15$, $15 < KC < 24$, $24 < KC < 32$ and $32 < KC < 40$, respectively, by Williamson (1985). Obasaju et al. (1988) showed that one additional vortex is shed per half flow oscillation cycle

each time KC is increased by about 8. This is a consequence of the Strouhal law applicable to an oscillatory flow (Sumer and Fredsoe, 2006).

For a one-degree-of-freedom (1-DOF) cross-flow VIV response, Sumer and Fredsoe (1988) carried out experimental tests using a carriage technique and reported some interesting VIV features of an elastically mounted rigid cylinder in oscillatory flows with $KC = 5, 10, 20, 30, 40$ and 100 versus a steady flow case ($KC = \infty$), at $Re = 2 \times 10^4 - 10^5$. Further, Sumer and Fredsoe (1989) accounted for the Re effect by varying U_m and V_r from 0 to 16. Therein, f_w was also varied to keep KC fixed. For a dominant oscillation frequency of the cylinder f_i , it may be written that

$$\frac{f_i}{f_n} = \frac{f_i}{f_w} \left(\frac{f_w}{f_n} \right) = \frac{f_i}{f_w} \left(\frac{V_r}{KC} \right) = N_i \left(\frac{V_r}{KC} \right) = \frac{N_i}{f_r} \quad (3)$$

where N_i represents the number of cylinder vibrations per flow cycle ($N_i = f_i/f_w$) and index $i = x$ or y denotes in-line or cross-flow direction. From Eqs. (2) and (3), $N_y = n$ if $f_y = f_v$ due to a lock-in or resonance. Sumer and Fredsoe (1988) reported that, for $KC=10$, $N_y = 2$ throughout the V_r range and the cross-flow amplitude reaches a resonance peak at $f_y/f_n \approx 1$ at $V_r = 6$. For $KC=20$, the response begins with $N_y = 4$ in a lower V_r range, dropping to $N_y = 3$ and $N_y = 2$ at $V_r = 5.5$ and $V_r = 9$, respectively. This decreased N_y is due to the increased f_w (for keeping KC fixed) which, in turn, reduces the number of vortices generated per flow cycle (Kozakiewicz et al., 1997). Such a variation of N_y entails a multi-peak response occurrence as V_r is varied. Similarly, for $KC=40$, the response starts with $N_y = 8$, exhibiting a zigzagging trend in the f_y/f_n plot and consecutively decreasing N . This unique multi-peak behaviour is different from a single upper-branch response found in a steady flow VIV (Williamson and Govardhan, 2004). In all KC cases, $N_y = 2$ is the absolute minimum number of oscillation cycles. These results suggest that the cylinder cross-flow oscillation pattern follows the fundamental vortex-shedding frequency as V_r is increased until reaching a first lock-in point where $N_y \approx n$ ($f_y \approx f_v$).

For a 1-DOF inline response prediction, the so-called Morison's equation (Morison et al., 1950) may be used to model an in-line hydrodynamic force acting on a stationary or oscillating cylinder in oscillatory flow. For a stationary cylinder, this semi-empirical equation has two components comprising an inertia force in phase with the local flow acceleration and a drag force proportional to the signed and square of the instantaneous flow velocity. For an oscillating cylinder, the relative velocities and accelerations between the flow and the cylinder are accounted for. The associated inertia and drag coefficients are determined based on experimental data typically depending on KC , Re and surface roughness (Sarpkaya, 2010; Sarpkaya and Isaacson, 1981). Williamson (1985) performed a U-shaped oscillating flow tube test of an elastically mounted cylinder with KC up to 35 and compared experimental results with those predicted by an extended Morison's equation accounting for the cylinder motion. Near a primary resonance ($f_n/f_w \approx 1$), he suggested that a relative flow-cylinder velocity formulation may be a reasonable assumption for a prediction model. For a fixed $Re = 200$ and $2 < KC < 20$, Anagnostopoulos and Iliadis (1998) performed numerical simulations of in-line vibrations of a circular cylinder, capturing the cylinder oscillation effect on the flow pattern and hydrodynamic forces. By varying f_w/f_n (V_r/KC) and changing f_n (V_r) for a

fixed $KC=10$ and 20 , they reported that the in-line force contains f_w and the odd multiples of f_w components amplifying the cylinder response near resonance.

Experimental and numerical results for 2-DOF in-line and cross-flow responses are very sparse. Bearman and Mackwood (1991) conducted experiments in a U-tube water tank generating oscillatory flows in a range of $0 < KC < 50$ and maximum β of 750 ($Re = 3.75 \times 10^4$). Their results emphasize the cross-flow VIV dependence on KC and V_r (i.e. f_n/f_w), justifying the applicability of Eq. (3). The in-line response increases monotonically with KC such that $f_x \approx f_w$, regardless of the cross-flow motion being restrained or allowed. In some cases, the in-line response may be modified due to the cross-flow oscillation. This two-directional fluid-cylinder interaction and coupling were further experimentally investigated by Lipsett and Williamson (1994) who measured cross-flow (Y) and in-line (X) cylinder responses in oscillatory flows with $2 < KC < 60$ and $1 < f_n/f_w < 9$. At low $f_n/f_w < 4$, repeatable X - Y trajectories with variable phase differences were captured depending on both KC and f_n/f_w , with X responses being greater than Y responses. At higher f_n/f_w , the trajectories become more irregularly complex due to the increasing transverse cycles of each in-line periodic oscillation. While the in-line maximum response increases with KC , the cross-flow response remains comparable to that in a lower f_n/f_w case. More recently, VIV phenomena in oscillatory flows have further been captured through two-dimensional numerical simulations by Zhao (2013) who considered a flexibly mounted circular cylinder in oscillatory flows with a fixed $KC = 10, 20$ and 40 , $308 < Re < 9240$ and $0 < V_r < 30$. For a given KC , both U_m and f_w were simultaneously varied. Zhao (2013) showed that cross-flow VIV exhibits a multi-peak response subject to multiple excitation frequencies with a primary component decreasing as V_r is increased. When the response is regular and repeatable, N_y is found to be equal to the number of vortex pairs shed per each cycle plus 1 associated with a flow reversal event. This implies that, for a lock-in, $N_y \approx n$ as in 1-DOF cross-flow VIV studies. Nevertheless, at a transition boundary between two different response regimes, the vibration trajectories become irregular. In agreement with Williamson (1985), Bearman and Mackwood (1991), the primary frequency of in-line response is equal to f_w for all KC and V_r . Zhao (2013) also reported that the in-line amplitude becomes significantly greater than the cross-flow one as KC and/or V_r is increased, with the maximum value of about $6D$ for $KC=40$. This is much larger than cross-flow and in-line VIV amplitudes in steady flows for the same mass-damping cylinder. It is important to remark that the multi-frequency feature of cross-flow VIV in oscillatory flows would play a meaningful role in fatigue life analysis of offshore structures in waves.

Recently, a few researchers have investigated VIV of long flexible cylinders in oscillatory flows. Fu et al. (2014) performed experimental tests and reported cross-flow VIV of a horizontal cylinder whose ends were forced to harmonically move in still water for $26 < KC < 178$ and for specific $V_r = 4, 6.5$ and 7.9 . Depending on KC , they suggested three regions including the building-up, lock-in and dying-out cross-flow VIV. Key features of amplitude modulation, modal transition, hysteresis and intermittent response were remarked. For $KC=84$, Fu et al. (2018) carried out numerical simulations using a strip theory for the tested cylinder model of Fu et al. (2014). They reported the occurrence of such a three-stage VIV development process and hysteresis between flow decelerating and accelerating events in oscillatory flows. Similar time-varying characteristics were reported by Wang et al.

(2018) for VIV of a free-hanging catenary riser subject to a vessel motion-induced oscillatory flow. Nevertheless, these experimental and numerical results are based on an imposed end excitation. These observations might not be directly applicable to the oscillatory flow-induced VIV of flexible cylinders with fixed ends.

From a modelling and prediction viewpoint, there have been just a few attempts in the literature to semi-empirically model coupled in-line and cross-flow hydrodynamic forces acting on cylinders in oscillatory flows. The Morison's equation is commonly used to describe an in-line force (Lipsett and Williamson, 1994; Williamson, 1985). Bearman et al. (1984) proposed a quasi-steady model to predict the transverse forces on cylinders in waves and oscillatory flows. This model assumes $St=0.2$ and a constant amplitude of the lift coefficient over a half flow cycle, providing a good prediction when compared with experimental results for $15 < KC < 53$. Thorsen et al. (2016) presented a transverse force model including the excitation, damping and added mass terms. The lift excitation coefficient depends on the response amplitude, and the force fluctuation depends on the time-varying phase difference between the cylinder motion and force excitation. The damping term is nonlinear, depending on the cylinder velocity and the drag coefficient as a function of cylinder response. Some empirical coefficients were proposed and limited to the available low-amplitude data range. This model has recently been extended to account for in-line VIV in oscillatory flows by Ulveseter et al. (2018).

As an alternative approach, Hayashi (1984) was perhaps the first who attempted to apply a van der Pol wake oscillator, developed by Hartlen and Currie (1970) for steady flows, to model the fluctuating lift forces in oscillatory flows in predicting cross-flow VIV in waves. The amplification of the lift force around a resonance was reproduced, but the model was unable to predict the cylinder response in a certain frequency ratio range. In addition, the effect of in-line motion was neglected. To the present authors' knowledge, the application of nonlinear wake oscillators to predict oscillatory flow VIV is still lacking in the literature. Therefore, it is the main aim of the present study to investigate the feasibility and limitation of using wake oscillators for modelling combined cross-flow/in-line VIV in oscillatory flows, by extending previous model developments and applications in steady flow VIV cases for both rigid (Bai and Qin, 2014; Facchinetti et al., 2004; Srinil et al., 2018; Zanganeh and Srinil, 2014) and long flexible (Gao et al., 2019; Low and Srinil, 2016; Srinil, 2010, 2011; Srinil et al., 2009; Xu et al., 2008; Yang et al., 2018; Zanganeh and Srinil, 2016) cylinders.

This paper is structured as follows. In Section 2, the mathematical equations of coupled cross-flow and in-line motions describing the nonlinear fluid-cylinder interaction of a 2-DOF VIV in oscillatory flow are presented. The forcing terms, a set of dimensionless parameters and empirical coefficients are defined. The model calibration and sensitivity studies are carried out in Section 3 introducing a new set of empirical coefficients and functions depending on KC and mass ratio. In Section 4, parametric studies are performed with different model solution scenarios to highlight several 2-DOF VIV prediction characteristics, maximum response amplitudes, oscillation frequencies and effects of system parameters including KC , V_r , system frequency ratios and mass ratio. The paper ends with conclusions in Section 5.

2. Oscillatory Flow VIV Prediction Model

A mathematical model to predict the fluid-structure interaction for VIV of a rigid circular cylinder in a planar and oscillatory flow is presented by utilizing a nonlinear wake oscillator model recently advanced in Srinil et al. (2018) following an original concept in Srinil and Zanganeh (2012) to account for three main aspects. These include (i) the effect of nonlinear coupling of cross-flow and in-line responses as well as lift and drag hydrodynamic forces, (ii) a resonance range capturing a self-limiting peak response when varying flow velocities, and (iii) the interrelationships of empirical coefficients and system fluid-structure parameters enabling the associated function applicability to a wide parametric range. Such a reduced-order model has been satisfactorily used for predicting steady flow VIV of long flexible cylinders with multi DOF responses (Zanganeh and Srinil, 2016).

Figure 1 displays a two-dimensional spring idealization of a flexibly mounted circular cylinder horizontally placed in a uniform planar sinusoidal flow whose the time-varying horizontal velocity function U is assumed as

$$U = U_m \sin(\omega_w \tilde{t}) \quad (4)$$

where ω_w is the angular frequency of the oscillatory flow ($\omega_w = 2\pi f_w$) and $\tilde{t}$ denotes the dimensional time. This flow function has been widely used as the most common assumption in experimental (Lipsett and Williamson, 1994; Sumer and Fredsoe, 2006) and numerical (Zhao, 2013) studies. A direction of the instantaneous relative velocity of the incoming flow with respect to the cylinder (V_{rel}) is described by a dynamic angle θ representing an effective direction of fluctuating drag (i.e. aligned with V_{rel}) and lift (perpendicular to V_{rel}) forces. Accordingly,

$$V_{rel} = \sqrt{(U - \dot{X})^2 + \dot{Y}^2} \quad (5)$$

in which a dot denotes differentiation with respect to $\tilde{t}$. In this study, the total drag F_D and lift F_L hydrodynamic forces per unit length may be expressed as

$$F_D = \frac{1}{2} \rho D V_{rel}^2 (C_{DM} + C_D), \quad F_L = \frac{1}{2} \rho D V_{rel}^2 C_L, \quad (6)$$

where ρ is the fluid density, C_D and C_L are the fluctuating drag and lift coefficients associated with the vortex shedding, and C_{DM} is the Morison drag coefficient obtained from a stationary cylinder in an oscillatory flow. The constant C_{DM} is dependent on KC and Re (Sarpkaya and Isaacson, 1981) whose the forcing effect is, nevertheless, time-dependent and may be amplified by V_{rel} . The amplitudes of C_D and C_L may be modulated as the cylinder oscillates, depending on response amplitudes, frequencies and fluid-structure parameters. Such features will be accounted for by the wake oscillators. If the vortex shedding, lift force and transverse response are neglected, the combination of Eq. (5) and the first expression in Eq. (6) entails the Morison drag force $\rho D C_{DM} (U - \dot{X}) |U - \dot{X}| / 2$ accounting for the body oscillation (Sumer and Fredsoe, 2006). Note that Lipsett and Williamson (1994) also presented a numerical fluid model as in Eq. (6), but they neglected C_D from their model and further postulated the lift coefficient variation through a harmonic function with a predefined phase difference between the lift force and the flow, despite the fact that such a relative phase is unknown in practice.

Following a mathematical derivation of a two-directional spring-mass system described in the Appendix of Srinil et al. (2018), F_L and F_D may be projected onto the X and Y coordinates as $F_L \sin \theta + F_D \cos \theta$ and $F_L \cos \theta - F_D \sin \theta$, respectively, with actual $\sin \theta = \dot{Y} / V_{rel}$ and $\cos \theta = (U - \dot{X}) / V_{rel}$ relations. The acceleration force in the X direction consists of two components: one being associated with the pressure gradient (Froude-Krylov) causing the fluid acceleration ($\dot{U}$) independent of the cylinder motion, and the other depending on the cylinder acceleration relative to the flow ($\ddot{X} - \dot{U}$). The acceleration force in the Y direction only accounts for the latter component ($\ddot{Y}$). As a result, the direction of the cylinder acceleration with respect to the flow is not necessarily the same as V_{rel} direction, depending on the magnitude of $\sqrt{(\ddot{X} - \dot{U})^2 + \ddot{Y}^2}$. In the same way as the projection of lift and drag forces, the X and Y acceleration forces may be resolved and approximated as $C_M \rho A_s \dot{U} - C_a \rho A_s \ddot{X}$ and $-C_a \rho A_s \ddot{Y}$, respectively (Williamson, 1991). Consequently, by combining all the hydrodynamic force components, the geometrically nonlinear coupled oscillators governing in-line and cross-flow vibrations of an elastically mounted circular cylinder in an oscillatory flow may be expressed in a dimensional form as

$$m_s \ddot{X} + C_s \dot{X} + K_s (X + \alpha_x^* X^3 + \beta_x^* X Y^2) = F_L \frac{\dot{Y}}{V_{rel}} + F_D \frac{(U - \dot{X})}{V_{rel}} + C_M \rho A_s \dot{U} - C_a \rho A_s \ddot{X}, \quad (7)$$

$$m_s \ddot{Y} + C_s \dot{Y} + K_s (Y + \alpha_y^* Y^3 + \beta_y^* Y X^2) = F_L \frac{(U - \dot{X})}{V_{rel}} - F_D \frac{\dot{Y}}{V_{rel}} - C_a \rho A_s \ddot{Y}, \quad (8)$$

where m_s , A_s , C_s and K_s are the structural mass, outer cross-sectional area, viscous damping and spring stiffness coefficients, respectively, and C_a is the added mass coefficient of the oscillating cylinder in still water ($C_a=1$ for a circular cylinder according to Blevins (1990)). The inertia force F_I is $C_M \rho A_s \dot{U}$ in which C_M is the inertia coefficient of the stationary cylinder in oscillatory flow ($C_M \neq C_a + 1$). The actual inertia or added mass coefficient of the oscillating cylinder in oscillatory flow is unknown *a priori*, depending on the response amplitude, frequency, relative motion and VIV effect. Herein, C_M is considered, which is dependent on KC and Re (Sarpkaya, 2010). Both C_a and C_M assumptions have been applied by Lipsett and Williamson (1994). If the cylinder is stationary in an oscillatory flow, the Morison's equation, comprising the in-line drag and inertia forces in the absence of vortex shedding, is $\rho D C_{DM} U |U| / 2 + \rho A_s C_M \dot{U}$ (Morison et al., 1950). According to Srinil et al. (2018), the geometrically nonlinear coefficients α_x^* , β_x^* , α_y^* and β_y^* are functions of the spring length. These arbitrary quantities may be treated as empirically tuned coefficients to capture relevant VIV features. The effects of C_{DM} and C_M on the prediction will be discussed via model calibration in Section 3.

With respect to the modelling of the vortex-shedding effect, C_D and C_L fluctuations may be represented through the wake oscillator variables $p=2C_D/C_{D0}$ (Srinil and Zanganeh, 2012) and $q=2C_L/C_{L0}$ (Facchinetti et al., 2004), in which C_{D0} and C_{L0} are the associated oscillating drag and lift coefficients of a stationary cylinder in oscillatory flow. The time variation of p and q may be modelled through the van der Pol wake oscillators as

$$\ddot{p} + \varepsilon_x \omega_v (p^2 - 1) \dot{p} + \omega_v^2 p = F_{cx}, \quad (9)$$

$$\ddot{q} + \varepsilon_y \omega_v (q^2 - 1) \dot{q} + \omega_v^2 q = F_{cy}, \quad (10)$$

in which ε_x and ε_y are the empirical damping coefficients, F_{cx} and F_{cy} are the cylinder (displacement, velocity or acceleration) feedback terms simulating the fluid-cylinder interaction effect, and ω_v is the fundamental angular frequency of the lift force associated with the vortex shedding ($\omega_v = 2\pi f_v$). Herein, ω_v is applied to both the drag (Eq. 9) and lift (Eq. 10) force oscillators since recent numerical simulations of 2-DOF VIV in oscillatory flows by Zhao (2013) suggest a common primary response frequency in both cross-flow and in-line directions, in addition to the flow frequency (f_w). This feature is different from a steady flow VIV case where a 2:1 (drag-lift) frequency ratio has been applied to the wake oscillators simulating a figure-of-eight X - Y trajectory (Srinil and Zanganeh, 2012). In addition to f_v and f_w , the higher-order frequency components in both x - y responses have been observed by Zhao (2013) as well as Lipsett and Williamson (1994). By accounting for the relative velocities between the flow and the cylinder, such an appearance of multi-frequency response is accounted for through the multiplication of f_v -dependent (p , q) and f_w -dependent (U) variables (i.e. F_L and F_D terms) in Eqs. (7) and (8). By assigning f_v equivalently in both drag-lift oscillators, the in-line response frequency would contain the primary component f_w (due to oscillatory flow) and the high-order components (due to VIV), one of which potentially matching the dominant cross-flow response frequency, depending on KC and V_r . If a higher-order excitation frequency (e.g. $2f_v$) is specified in the drag wake oscillator, the in-line response would respond with higher-order frequencies considerably differing from those primarily governing the cross-flow response, due to such multiplied and nonlinear terms.

By introducing the dimensionless time ($t = \omega_w \tilde{t}$) and displacements ($x = X/D$, $y = Y/D$), incorporating the wake oscillator (p , q) variables into Eqs. (6)-(8), and accounting for the relationships in Eq. (1), the system of nonlinear equations of coupled motions of the elastically mounted rigid circular cylinder undergoing 2-DOF VIV under oscillatory flow may be expressed in a dimensionless form as

$$\ddot{x} + \lambda (\dot{x}, \dot{y}, t) \dot{x} + (KC/V_r)^2 (x + \alpha_x x^3 + \beta_x xy^2) = f_{wx}(\dot{x}, \dot{y}, t), \quad (11)$$

$$\ddot{p} + \varepsilon_x \text{StKC} (p^2 - 1) \dot{p} + (\text{StKC})^2 p = f_{cx}, \quad (12)$$

$$\ddot{y} + \lambda (\dot{x}, \dot{y}, t) \dot{y} + (KC/V_r)^2 (y + \alpha_y y^3 + \beta_y yx^2) = f_{wy}(\dot{x}, \dot{y}, t), \quad (13)$$

$$\ddot{q} + \varepsilon_y \text{StKC} (q^2 - 1) \dot{q} + (\text{StKC})^2 q = f_{cy}, \quad (14)$$

where a dot now denotes differentiation with respect to t , α_x^* , β_x^* , α_y^* and β_y^* become the dimensionless (dependent on a ratio of D to the spring length) and empirical variables α_x , β_x , α_y and β_y , $f_{cx} = F_{cx}/D\omega_w^2$ and $f_{cy} = F_{cy}/D\omega_w^2$. Following Facchinetti et al. (2004) for a steady flow VIV, one may specify $f_{cx} = \Lambda_x x$, $f_{cx} = \Lambda_x \dot{x}$ or $f_{cx} = \Lambda_x \ddot{x}$ ($f_{cy} = \Lambda_y y$, $f_{cy} = \Lambda_y \dot{y}$ or $f_{cy} = \Lambda_y \ddot{y}$) as a first-order linear in-line (cross-flow) displacement, velocity or acceleration coupling, respectively, with empirical coefficients Λ_x and Λ_y to be identified together with ε_x and ε_y . The fluctuating in-line (f_{wx}) and cross-flow (f_{wy}) hydrodynamic forces read

$$f_{\text{wt}}(\dot{x}, \dot{y}, t) = (2\pi M_L q \dot{y} + M_{DM} KC \sin(t) + M_D p (KC \sin(t) - 2\pi \dot{x})) f_h(\dot{x}, \dot{y}, t) + C_M \frac{KC}{8\mu} \cos(t), \quad (15)$$

$$f_{\text{vy}}(\dot{x}, \dot{y}, t) = (M_L q (KC \sin(t) - 2\pi \dot{x}) - 2\pi M_D p \dot{y}) f_h(\dot{x}, \dot{y}, t). \quad (16)$$

The dimensionless quantities (M_L, M_D, M_{DM}), the mass parameter (μ), the time-varying damping (λ) and the higher-order nonlinear effect of the relative velocities $f_h(\dot{x}, \dot{y}, t)$ are defined, respectively, as

$$\begin{aligned} M_L &= \frac{C_{L0}}{16\pi^2 \mu}, \quad M_D = \frac{C_{D0}}{16\pi^2 \mu}, \quad M_{DM} = \frac{C_{DM}}{8\pi^2 \mu}, \quad \mu = \pi(m^* + C_a)/4, \\ \lambda(\dot{x}, \dot{y}, t) &= 2\xi \left(\frac{KC}{V_r} \right) + \frac{C_{DM}}{4\pi\mu} f_h(\dot{x}, \dot{y}, t), \\ f_h(\dot{x}, \dot{y}, t) &= \sqrt{(KC \sin(t) - 2\pi \dot{x})^2 + (2\pi \dot{y})^2}, \end{aligned} \quad (17)$$

where $\xi = C_s / 2\sqrt{K_s(m_s + C_a \rho A_s)}$ and m^* is the mass ratio with $m^* = 4m_s / \rho \pi D^2$. As $C_a = 1$ is herein assumed for a circular cylinder, ξ is equivalent to the cylinder damping ratio in still water. In both X - Y directions, ξ , m^* and f_h are assumed to be constants. It is worth remarking that both Eqs. (11) and (13) are highly nonlinear and non-homogeneous (explicitly time-dependent), containing a mixture of cosine and sine functions in the X direction. A multiplication of sine terms in Eqs. (15) and (16) potentially leads to multiple competing frequencies. If the flow is steady and uniform (i.e. U is a constant), the system of equations of 2-DOF motions presented in Srinil et al. (2018) will be recovered for steady flow VIV.

Based on Eq. (2), $\text{StKC} = n$ in Eqs. (12) and (14). For stationary cylinders, Sarpkaya (1976) suggested that $n = 2, 4$ and 8 for $KC = 10, 20$ and 40 , respectively, enabling $\text{St} = 0.2$. Sarpkaya (2010) further noted that the lift and drag forces become increasingly important as $KC > 5$ and 15 , respectively, whereas the inertia force dominates when $KC < 10$. For 1-DOF cross-flow VIV in oscillatory flows, Sumer and Fredsoe (1988) suggested, based on Eq. (3), that $N_y = 2, 4$ -5 and 8 -10 for $KC = 10, 20$ and 40 , respectively, signifying a similar trend of n and N_y and a variable N_y at a higher KC for the oscillating cylinder. For 2-DOF VIV, Zhao (2013) reported, through 2-D numerical simulations, the dependence of N_y on both KC and V_r when the latter is varied. For $KC = 10$, $N_y = 2$ in the range of $V_r = 5$ -13. For $KC = 20$, $N_y = 5$ at $V_r = 4$, $N_y = 4$ at $V_r = 5$ and $N_y = 3$ at $V_r = 8$ -13. For $KC = 40$, $N_y = 8$ at $V_r = 5$, $N_y = 7$ at $V_r = 6$, $N_y = 6$ at $V_r = 8$, $N_y = 5$ at $V_r = 10$ -20 and $N_y = 4$ at $V_r = 26$ -30. The x - y trajectories associated with these V_r are repeatable whereas they become quite chaotically irregular during the N_y mode transition. The effects of specifying n and empirical coefficients on VIV prediction will be discussed in Section 3. The highly nonlinear Eqs. (11)-(14) can be numerically integrated using a fourth-order Runge-Kutta scheme with an adaptive time step, and $x = y = \dot{x} = \dot{y} = \dot{p} = \dot{q} = 0$, $p=2$ and $q=2$ as initial conditions. Steady-state responses of about 80 dimensionless flow cycles will be analyzed to present cylinder response amplitudes and frequencies.

3. Calibration and Selection of Empirical Coefficients

For the system of Eqs. (11)-(17), there are several empirical coefficients to be identified among the hydrodynamics

($C_M, C_{DM}, C_a, C_{L0}, C_{D0}, St$), structure ($\alpha_x, \beta_x, \alpha_y, \beta_y$) and wake oscillator ($\varepsilon_x, A_x, \varepsilon_y, A_y$) coefficients, depending on the key input fluid-structure parameters (m^*, ξ, KC, V_r, Re). Recall also that $StKC$ may be equivalent to n and specified a priori; alternatively, St and KC may be individually specified, with the latter being varied so that a resonance between the two sets of oscillators may be achieved. For the present wake-cylinder oscillators, it would be impractical to determine a universal set of empirically tuned coefficients (up to 14 variables) for predicting any cylinder VIV in oscillatory flow, considering that experimental data in 2-DOF VIV cases are still scarce. For model calibration, numerical results from the computational fluid dynamics (CFD) studies of Zhao (2013) are herein considered for 2-DOF VIV of a circular cylinder subject to a two-dimensional planar oscillatory flow with $KC = 10, 20$ and 40 . In such CFD work, $m^* = 1.62$, $\xi = 0.012$ and $Re/V_r = 308$, and the associated Re was varied from 308 to 9240 in the range of $1 < V_r < 30$ in which U_m and f_w were simultaneously varied for a constant KC, f_n and D . These parameters and scenarios will be considered as model inputs in the following simulations, unless stated otherwise.

Based on preliminary sensitivity studies as in Srinil et al. (2018) and calibration of x - y response amplitudes versus CFD results of Zhao (2013), it has been deduced that α_x and β_y play the most influential role in predicting 2-DOF VIV in oscillatory flows. This is because α_x and β_y govern the in-line displacement affecting the nonlinear stiffness x^3 term in Eq. (11) for the in-line direction and yx^2 term in Eq. (13) for the cross-flow direction. The in-line amplitude increases and becomes much greater than the cross-flow counterpart as KC and/or V_r increases (Zhao, 2013). Hence, α_x and β_y may be specifically tuned for amplitude calibration purpose. As for the wake-cylinder coupling terms in Eqs. (12) and (14), $f_{cx} = \Lambda_x \ddot{x}$ and $f_{cy} = \Lambda_y \ddot{y}$ may be applied as in the steady flow VIV case (Facchinetti et al. 2004), see also discussion later on. Herein, based on $308 < Re < 9240$ (Zhao 2013), we specify $C_M = 1$, $C_{DM} = 2$ and $C_a = 1$ (Sarpkaya, 2010). Following Srinil et al. (2018), $C_{L0} = 0.3$, $C_{D0} = 0.2$, $\beta_x = \alpha_y = 0.7$, $\varepsilon_x = 0.3$, $A_x = 12$ and $\varepsilon_y = 0.00234e^{0.228m^*}$ whereas $A_y = 38$ based on the present calibration which accounts for the tuning of the two most influential variables α_x and β_y in order to predict cross-flow and in-line amplitudes and associated qualitative features (including a multi-peak cross-flow response and a monotonically increasing in-line response) as V_r is increased for the assigned input parameters (n, KC, m^*, ξ). A MATLAB solver (e.g. ‘fminsearch’) has been applied to minimize the error function (i.e. the root-mean-square deviation of the wake oscillator predictions from the associated CFD results), and to optimize the obtained results based on solving the system of Eqs. (11)-(17) with multiple variables. This calibration has enabled us to identify a single set of best-fit empirical variables for both wake-cylinder oscillators applicable to the whole considered V_r range. Consequently, new empirical expressions capturing the KC and m^* dependence may be introduced as

$$\alpha_x = 0.46e^{-0.082KC}, \quad (18)$$

$$\beta_y = 2.22e^{-0.032KC} \times 1.416e^{-0.215m^*} = 3.143e^{-(0.032KC+0.215m^*)}. \quad (19)$$

As KC or m^* is increased, α_x and β_y decrease through the above expressions, diminishing the effect of cylinder nonlinear stiffness, because the resulting values of the associated $\alpha_x x^3$ (Eq. 11) and $\beta_y yx^2$ (Eq. 13) terms decrease.

The associated trend of α_x or β_y as a function of KC, as well as β_y as a function of m^* , is shown in Fig. 2a whereas a contour plot of β_y as function of both KC and m^* is shown in Fig. 2b. A decrease of α_x would yield a greater in-line response whereas a decrease of β_y would yield a decrease of cross-flow response particularly in a high V_r range (> 15) as this would be expected for a higher m^* for 2-DOF VIV. If Eq. (19) is only dependent on KC, the predicted cross-flow response will become qualitatively erroneous when varying m^* .

Figure 3 compares x - y response amplitudes (A_x/D , A_y/D) based on the wake oscillator and CFD (Zhao, 2013) models, by also accounting for the effect of specified n . For KC = 10, 20 and 40, n might be equal to 2, 4 and 8 (Sarpkaya, 1976) and specified in the range of 2-4, 4-6 and 8-10, respectively, according to an observation of oscillatory flow around the stationary cylinder in the literature (Sumer and Fredsoe, 2006). For in-line responses, CFD results in Fig. 3 show that the maximum A_x/D occurs at a higher V_r as KC is increased. At KC = 10, A_x/D response (Fig. 3a) reveals a resonance peak at $V_r \approx 13$ -14 captured by both models despite having different maximum amplitudes. At KC = 20, CFD results of A_x/D (Fig. 3c) reveal a peculiar peak at $V_r \approx 18$ -19 which may be due to the coupling with A_y/D (Fig. 3d). However, this feature was not reported in Zhao (2013). The wake oscillator model does not admit such a peak but predicts how A_x/D increases with V_r as in the CFD study and other experimental tests (Williamson, 1985; Bearman and Mackwood, 1991). Greater amplitude differences are noticed in Fig. 3c. Any other tuning based on a trial of different sets of empirical wake oscillator variables (ε_x , A_x , ε_y , A_y) was unsatisfactory for this KC = 20, since an error function could not be minimized during calibration to achieve simultaneously the coupled in-line and cross-flow response predictions. Nevertheless, such A_x/D discrepancies at KC=10 and 20 are reduced when increasing KC to 40 as shown in Fig. 3e exhibiting a monotonic trend and greater response as V_r is increased ($A_x/D \approx 6$ at $V_r = 30$). While A_x/D values predicted by wake oscillators are slightly sensitive to the n change at lower KC = 10 and 20, the predicted A_x/D becomes more sensitive at higher KC = 40 for $V_r < 18$.

For cross-flow responses, CFD results in Fig. 3b (KC = 10), d (KC = 20) and f (KC = 40) show that A_y/D responses relatively fluctuate, exhibiting multiple peaks ($A_y/D \approx 0.7$ -1) as V_r is increased, and suggesting several lock-in events. At low KC = 10, A_y/D significantly diminishes for $V_r \geq 17$ leading to a primarily 1-DOF in-line response with a nearly constant A_x/D (Fig. 3a). For higher KC, both A_y/D and A_x/D come into play across the considered V_r range. These trends, which are quite sensitive to the n change, are also captured by the wake oscillators. Nevertheless, considerable quantitative errors can be noticed for $n = 3$ -4 in Fig. 3a (KC=10), $n = 5$ -6 in Fig. 3b (KC=20) and all n in Fig. 3c (KC=40). Since we use the maximum A_y/D and overall 2-DOF VIV features (averaged A_x/D and A_y/D trends) across $1 < V_r < 30$ as the governing calibration criteria, the most suitable qualitative and quantitative comparisons between wake oscillator and CFD models occur when $n = 2, 4$ and 8 for KC = 10, 20 and 40, respectively, in agreement with what is suggested by Sarpkaya (1976) for such a low Re range.

Figure 4 illustrates the effect of cylinder stiffness nonlinearities (i.e. linear vs. nonlinear oscillators) as well as the effect of wake-cylinder (acceleration, velocity and displacement) coupling terms on numerical predictions in comparison with CFD results. By considering x - y linear oscillators ($\alpha_x = \beta_x = \alpha_y = \beta_y = 0$) for KC = 10, the predicted

A_x/D peak in Fig. 4a is slightly shifted to the left due to the omitted cubic-type hardening stiffness whereas the predicted A_y/D response in Fig. 4b becomes relatively negligible when V_r exceeds 10, which is a substantially lower value than 17, which corresponds to the CFD approach. The multi-peak A_y/D feature disappears with the linear model. Such qualitative and quantitative errors would become greater for higher KC. As for the cylinder coupling or feedback terms in Eqs. (12) and (14), the exemplified in-line response at $KC = 10$ in Fig. 4c appears to be almost identical when applying either the cylinder displacement coupling ($f_{cx} = \Lambda_x x, f_{cy} = \Lambda_y y$), velocity coupling ($f_{cx} = \Lambda_x \dot{x}, f_{cy} = \Lambda_y \dot{y}$) or acceleration coupling ($f_{cx} = \Lambda_x \ddot{x}, f_{cy} = \Lambda_y \ddot{y}$) terms. This implies a negligible effect of the coupling type on A_x/D . On the other hand, A_y/D responses are much more affected by the wake-cylinder coupling choice as displayed in Fig. 4d, e and f for $KC = 10, 20$ and 40 , respectively. In all KC cases and based on the proposed set of empirical coefficients, the displacement coupling terms entail the worst prediction with considerable errors when compared with CFD results. Indeed, such different effects in applying the coupling terms reflect the distinct natures of in-line vs. cross-flow responses, dependent on KC and V_r . For oscillatory flow, the cylinder coupling type in the in-line wake oscillator has been found to play a minor or even negligible role in affecting the feature of in-line response amplitudes because the in-line response is primarily governed by the oscillatory flow frequency. This is different from the associated cross-flow response which is primarily governed by the vortex-induced lift force and associated excitation frequency. Nevertheless, cross-flow and in-line responses are dynamically coupled through the nonlinear terms of hydrodynamic ($q\dot{y}, p\dot{x}, q\ddot{x}, p\ddot{y}$), geometric (xy^2, yx^2) and relative velocities (f_h, λ). By accounting for the same and consistent wake-cylinder coupling type in both drag and lift wake oscillators, the prediction results suggest how the acceleration coupling terms are most suitable for oscillatory flow VIV as in steady flow VIV cases (Facchinetti et al., 2004; Srinil et al., 2018).

By accounting for the stiffness nonlinearities and acceleration coupling terms, and assigning the fixed values $C_a = 1$ and $C_{D0} = 0.2$, Figure 5 illustrates the effect of varying hydrodynamic coefficients governing the inertia (C_M), Morison drag (C_{DM}) and oscillatory lift (C_{L0}) forces on the wake oscillator prediction in comparison with CFD results. For $KC = 10$, A_x/D (Fig. 5a) and A_y/D (Fig. 5b) responses appear to be qualitatively similar (i.e. with the bend-to-right (A_x/D) and multi-peak (A_y/D) features) but quantitatively different with $C_M = 0.5, 1$ and 2 . Since C_M term only appears in Eq. (15) governing the in-line loading, the changes in both x - y responses as C_M is varied suggest a two-dimensional dynamic coupling captured by the model. As for C_{DM} which governs the in-line loading in Eq. (15) as well as the x - y hydrodynamic damping contribution in Eq. (17), the compared A_x/D and A_y/D responses are shown in Fig. 5c and d, respectively, for $KC = 40$. Results indicate a greater effect of varying C_{DM} on A_x/D than A_y/D . That is, A_x/D tends to increase by a faster rate when increasing C_{DM} from 1.5 to 2.5 and increasing V_r from 5 to 30 whereas the maximum A_y/D remains quantitatively comparable. Note that it was not feasible to capture the CFD-based parabolic A_y/D response shape for $10 < V_r < 25$ in Fig. 5d by using a single set of empirical coefficients when varying V_r . A second set of empirical coefficients may be introduced for this range. The effect of C_{L0} on A_x/D (Fig. 5e) and A_y/D (Fig. 5f) is next discussed for $KC = 40$, recalling that C_{L0} affects both x - y responses through M_L in Eqs. (15) and (16). According to Sarpkaya (2010), the maximum C_{L0} is in the range

0.3-1.5 for $KC = 40$, depending on β (Re). By way of tuning examples, $C_{L0} = 0.3, 1.5$ and $1.5/\sqrt{2}$ (root-mean squared value) are employed and compared. It is seen that the C_{L0} variation has a significant quantitative effect on the predicted x - y responses. Based on the present wake oscillator calibration, $C_{L0} = 0.3$ appears to be the most suitable tuning coefficient, in conjunction with the recommended $C_M = 1$, $C_{DM} = 2$, $C_a = 1$ and $C_{D0} = 0.2$. These empirical values and those described in the above second paragraph (Eqs. 18 and 19) will be used. In the Appendix, application of the present 2-DOF wake oscillator model and calibrated empirical coefficients to predict 1-DOF cross-flow or in-line VIV responses is further demonstrated and discussed in comparison with experimental results. In the following, numerical predictions of 2-DOF VIV predictions in oscillatory flows are presented.

4. Oscillatory Flow VIV Prediction Characteristics

Parametric investigations to predict 2-DOF VIV responses in oscillatory flows are now carried out for a wide range of system parameters (KC , V_r , m^* , f_r and Re). As the presented wake oscillators have not been derived from the Navier-Stokes equations describing actual fluid mechanics, the numerical predictions – which have been calibrated with specific CFD results – need to be further validated with new future experimental tests to justify the model validity and identify its limitations. Nevertheless, some prediction insights can be gained in regard to 2-DOF response amplitudes, time histories, oscillation frequencies and motion trajectories, which allow one to realize the feasibility of applying wake oscillators to predict oscillatory flow VIV. In the following, response frequencies and amplitudes based on several model solutions are presented and discussed.

4.1 Multi-Frequency Responses

With $m^* = 1.62$, $\zeta = 0.012$ and $V_r = 7.5$, Fig. 6 presents in-line and cross-flow time histories for $KC = 10$ ($n = 2$), 20 ($n = 4$) and 40 ($n = 8$) whereas Fig. 7 presents the associated phase plane (displacement-velocity) portraits and x - y trajectories (about 16 cycles), see also Fig. 3 for their response amplitudes. At low $KC = 10$ ($KC/V_r = f_n/f_w = f_r = 1.33$), the in-line response in Fig. 6a appears to have a zero mean value and a single harmonic frequency associated with the flow excitation whereas the cross-flow response in Fig. 6b signals the fluctuating maximum/minimum amplitude components as well as competing frequencies. The in-line response is indeed driven by f_w through the Morison drag (C_{DM}) and inertia (C_M) forces in Eq. (15). This observation is in qualitative agreement with the study of Anagnostopoulos and Iliadis (1998). Accordingly, the x phase plane plot in Fig. 7a appears to be a nearly circular closed orbit. In contrast, the cross-flow response is governed by multiple nonlinear/linear forcing terms and frequencies (i.e. $StKC = f_v/f_w = 2$ vs. $f_n/f_w = 1.33$) involving wake oscillators in Eq. (16). This leads to the y multiple orbital phase plane in Fig. 7b. By denoting f_x and f_y as the dimensional x and y oscillation frequencies, $f_x/f_w = N_x = 1$ in Fig. 6a whereas the dominant $f_y/f_w = 1.43$ in Fig. 6b suggesting a quasi-periodic response. Hence, the non-integer x - y frequency ratio entails irregular or non-repeatable x - y trajectories in Fig. 7c. As KC is increased to 20 ($f_r = 2.67$), the in-line response in Fig. 6c begins to be influenced by the wake oscillator frequency ($StKC = 4$) leading to the distorted amplitudes and non-zero mean values. Therefore, the associated x phase plane orbit is modulated, non-circular and asymmetric in Fig. 7d. Highly modulated y responses with greater broadband frequency contents are expected as shown in Figs. 6d and 7e, due to the enhanced nonlinearities. By further

increasing KC to 40 ($KC/V_r = f_r = 5.33$), the highly modulated x and intermittent y responses are clearly visible in Figs. 6e, 6f, 7g and 7h, suggesting a chaotic-like VIV. Accordingly, x - y trajectories become highly irregular in Fig. 7i exhibiting the symmetric y vs. asymmetric x (with a downstream drift) trajectories response with respect to the initial equilibrium at $x=y=0$. These features have also been experimentally observed by Lipsett and Williamson (1994) for high $f_r > 4$ values.

Figure 8 presents the normalized x - y response frequencies ($f_x/f_w, f_y/f_w$) of the oscillating cylinder subject to oscillatory flows based on the Fast Fourier Transform (FFT) analysis, in association with the amplitudes in Fig. 3. Herein, a primary or dominant oscillation frequency corresponds to a highest peak with a normalized spectral amplitude of unity in FFT plots. In all $KC = 10, 20$ and 40 cases, the dominant f_x/f_w values in Fig. 8a, c and e appear to be equal to unity (i.e. $N_x = 1$, see also Eq. 3) as V_r is increased. This is expected, agreeing with CFD results, indicating the governing sinusoidal flow excitation for x responses. Owing to the coupling of oscillatory lift-drag forces, there are several in-line response frequency peaks with variable amplitudes of the power spectral density as illustrated in Fig. 6a ($KC = 10$), c ($KC = 20$) and e ($KC = 40$). The f_x/f_w components of the first three peaks, ranked by their spectral density amplitudes, are summarized in Fig. 9a, c and e. For cross-flow response, f_y/f_w patterns and amplitudes vary substantially with V_r as well as KC . This reflects the dependence on both V_r and KC as suggested by Zhao (2013), Bearman and Mackwood (1991). At $KC = 10$, f_y/f_w values decrease monotonically as V_r is increased until A_y/D becomes trivial (Fig. 3b) as $V_r > 15$ in Fig. 8b. Since the predicted cross-flow responses become highly modulated when increasing KC (Figs. 6 and 7), f_y/f_w plots for $KC = 20$ (Fig. 8d) and 40 (Fig. 8f) reveal several peaks with variable amplitudes. This echoes the effect of system coupling and nonlinearities through the higher-order term f_h in Eqs. (15) and (16) governing the flow-cylinder relative velocities. In all KC cases, f_y/f_w values appear to be mostly non-integers. Figures 9b, d and f show the variation of the dominant f_y/f_w versus V_r for all KC , in comparison with CFD results (Zhao, 2013). Qualitatively speaking, the predicted f_y/f_w values appear to increase (decrease) with KC (V_r) as in the CFD study. However, discrepancies between the two approaches are noticed, increasing with KC . A satisfactory agreement may be justified in the limited range of $13 < V_r < 20$ for $KC = 10$, $7 < V_r < 13$ and $20 < V_r < 26$ for $KC = 20$. The present wake oscillator model is unable to capture actual integer N_y values of f_y/f_w , leading to non-repeatable x - y trajectories as shown in Fig. 7. In other words, N_y is not necessarily equal to the specified n for each KC , suggesting how the oscillation frequencies are nonlinearly amplitude-dependent.

By computing $f_y/f_n = N_y(V_r/KC)$ as in Eq. (3) based on the obtained primary f_y/f_w or N_y which may be integer or non-integer (Fig. 9), Figure 10 plots f_y/f_n versus V_r in comparison with CFD results and the reference lines with a perfect integer N_y . It is also worth referring to the experimental study of cross-flow only VIV by Sumer and Fredsoe (1988) who reported that, for $KC=10$, $N_y = 2$ throughout the V_r range; for $KC=20$, the primary f_y/f_n follows the lines of $N_y=4, 3$ and 2 in the approximated range of $2 < V_r < 6$, $6.5 < V_r < 8.5$ and $8.5 < V_r < 15$, respectively. For the 2-DOF VIV study of Zhao (2013), the primary f_y/f_n for $KC = 10$ follows the lines of $N_y = 2$ ($V_r < 14$) and 1 ($V_r \geq 14$); the primary f_y/f_n for $KC = 20$ follows the lines of $N_y=4, 3$ and 1 and then jumps up to $N_y = 2$ in the

approximate range $4 < V_r < 7$, $8 < V_r < 13$, $14 < V_r < 19$ and $20 < V_r < 30$, respectively, as in Fig. 10b. Notwithstanding a slight difference of $m^*\xi$, experimental (1-DOF) and numerical CFD (2-DOF) results do not perfectly share the same cross-flow response characteristics. Results from the wake oscillator model capture a gradual decreasing f_y/f_n shift – instead of a rapid downward or upward f_y/f_n jump in CFD plots – along N lines as V_r is increased. Both CFD and wake oscillator results in Fig. 10a and b reveal $N_y = 1$ for $KC = 10$ and 20 at some V_r ranges, implying that the cross-flow response is governed by the inline oscillatory excitation frequency f_w .

4.2 Bi-Parametric Response Contours

In previous Sections 3 and 4.1, the predicted 2-DOF VIV responses in oscillatory flows in the case of varying V_r (U_m and f_w) are based on a fixed KC (n), m^* and ξ , with the purpose of empirical tuning and comparison of response amplitudes and frequencies with CFD results (Zhao, 2013). Here, the effects of system fluid-structure parameters in wider ranges are further parametrically explored in order to better understand the prediction capability of the present wake oscillators and chosen empirical coefficients.

Figure 11 presents contour plots of A_x/D and A_y/D as functions of both m^* [0.1-10] and V_r [0.1-30] for $KC = 10$ ($n = 2$), 20 ($n = 4$) and 40 ($n = 8$), while $\xi = 0.012$ as in Fig. 3 whose $m^* = 1.62$. For $KC = 10$, A_x/D peaks in Fig. 11a appear to decrease and slightly shift to a lower V_r as m^* is increased, maintaining a linear resonance-like feature around $10 < V_r < 15$. The multi-peak feature of the associated A_y/D response in Fig. 11b is greatly affected by the m^* change, disappearing as $m^* \approx 6$ leading to a considerably reduced range of appreciable A_y/D centered around $V_r \approx 10$. For $KC = 20$, there is a range of $m^* > 6$ in which A_x/D peaks take place within the considered V_r range as shown in Fig. 11c. The associated A_y/D response in Fig. 11d reveals two distinct m^*-V_r boundaries separated around $V_r \approx 10$, in which comparable peak A_y/D values are found around $V_r \approx 7$ and 15 . Again, such a multi-peak feature disappears as m^* is increased to about 8 being greater than 6 in the lower $KC = 10$ case in Fig. 11a. For $KC = 40$, overall A_x/D responses in Fig. 11e increase with V_r for all m^* , without showing a peak in the considered V_r range. For A_y/D responses, the two m^*-V_r boundaries, shown in Fig. 11d for $KC = 20$, tend to merge and result in a plateau amplitude response in a large V_r range in Fig. 11f. This is due to the amplified responses as KC is increased in the low $m^* < 6$ range. However, the A_y/D response becomes limited in a small V_r range as $m^* > 8$, for which its peak is centered around $V_r = 5$ resembling that occurring in a steady flow VIV. This implies that, for higher m^* and higher KC cases where the drag is increasingly dominated, A_y/D response behaves similarly to VIV in steady flow. For the considered m^* range, the predicted maximum A_y/D does not necessarily correspond to the lowest m^* case for higher $KC = 20$ and 40 . This suggests a bi-parametric m^*-KC dependence. Overall, the maximum $A_x/D \approx 2.5, 3.5$ and 6 times the maximum A_y/D (≤ 1) for $KC=10, 20$ and 40 , respectively.

In contrast to Fig. 11 where KC as well as n is fixed while varying V_r , Figure 12 presents contour plots of A_x/D and A_y/D as functions of both KC [10-60] and V_r [0.1-30] for a given $m^* = 1.62$ and 4 while $\xi = 0.012$. U_m is the only parameter being varied in this scenario. Following Sarpkaya (1976), it may be assumed that $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11 for $10 \leq KC < 15$, $15 \leq KC < 20$, $20 \leq KC < 25$, $25 \leq KC < 30$, $30 \leq KC < 35$, $35 \leq KC <$

40, $40 \leq KC < 45$, $45 \leq KC < 50$, $50 \leq KC < 55$ and $55 \leq KC < 60$, respectively. For lower $m^* = 1.62$, the A_x/D response in Fig. 12a increases as V_r is increased for a given KC. However, A_x/D peaks appear to be more sensitive to the KC as well as n change when $V_r > 15$ than when $V_r < 15$. For higher $m^* = 4$, A_x/D peaks in Fig. 12c appear to be largely independent of KC for a given V_r . These features are distinct from the associated A_y/D responses in Fig. 12b ($m^*=1.62$) and 12d ($m^*=4$) where variations in both V_r and KC affect A_y/D peaks in a stepwise fashion because of varying n . The lower m^* case produces the overall greater A_y/D responses as expected. As in Fig. 11, results in Fig. 12d suggest that A_y/D peaks occur around $V_r \approx 5$ as KC (n) and m^* are both increased. Although the variable n affects the wake oscillators in Eqs. (12) and (14), the maximum A_y/D is still limited to be less than 1.2.

Next, instead of directly specifying an integer n (as $n = \text{StKC}$) regardless of actual St, a parametric study may be carried out by assigning and keeping St fixed. In so doing, we specify $\text{St} = 0.2$ (Sumer and Fredsoe, 2006). Since previous experimental (Lipsett and Williamson, 1994) and CFD (Zhao, 2013) studies highlighted the oscillatory flow VIV dependence on both KC and V_r , one may assign and vary the frequency ratio $f_r = f_n/f_w = \text{KC}/V_r$ (Eq. 2) and the associated KC through StKC in Eqs. (12) and (14). The aim is to determine if a nonlinear resonance between the two sets of wake-cylinder nonlinear oscillators might be achieved. By specifying the ranges of $1 < f_r < 4$ and $0.5 < V_r < 30$, the associated KC variation ($\text{KC} = f_r V_r$) from 0.75 to 120 is shown in Fig. 13 whereas contour plots of the predicted A_x/D and A_y/D are displayed in Fig. 14 for $m^* = 1.62$ (a, b) and 4 (c, d). It is seen that A_x/D responses in both Fig. 14a and 14c increase as both V_r and KC are increased for a given f_r , the latter having a greater effect on the system with lower $m^* = 1.6$. For $m^* = 4$, A_x/D response appears to be relatively independent of varying f_r : this feature is similar to the varying n case in Fig. 12c vs. 12a. On the other hand, A_y/D responses in both Fig. 14b and 14d appear to be strongly dependent on f_r , V_r , KC and m^* ; their features are qualitatively and quantitatively different from those in Fig. 12b and 12d. The f_r - V_r boundary for the maximum A_y/D peaks is found to be greater and wider in the case of lower m^* as shown in Fig. 14b. In association with Fig. 14, a table in Fig. 13 summarizes A_y/D peak values, associated V_r and KC for each specified f_r and m^* . It is seen that the averaged maximum $A_y/D \approx 1$ and 0.68 occurs in the range of $16 < V_r < 21$ and $6 < V_r < 10$ for $m^* = 1.62$ and 4, respectively, suggesting a strong influence of m^* in affecting A_y/D peaks and shifting the critical V_r range, consistently with results in Fig. 11. The corresponding KC increases such that $\text{KC}/V_r \approx f_r$.

Finally, the combined effects of Re and KC on the prediction of 2-DOF VIV responses in oscillatory flows are studied. For a specific $\beta = \text{Re}/\text{KC} = 1107$, the hydrodynamic drag (C_{DM}), lift (C_{L0}) and inertia (C_M) coefficients are plotted versus KC, as an example, in Fig. 15a based on data of Sarpkaya (1976). It is seen that C_{DM} and C_{L0} (C_M) increase (decrease) with increasing KC as well as Re for the inertia-dominated system ($\text{KC} < 10$). After reaching a peak, both C_{DM} and C_{L0} decrease when further increasing KC towards the drag-dominated region ($\text{KC} > 10$). In contrast, the associated C_M increases until approaching a limiting value of about 1.5. Such inertia-drag transitional behaviour is also applicable to other lower and higher β (Sarpkaya, 1976). To demonstrate the dependence on KC and Re (through a fixed $\beta = 1107$), the following exponential (exp) fitting functions associated with Fig. 15a for $6 < \text{KC} < 80$ may be introduced and incorporated into the simulation model.

$$\begin{aligned}
C_{DM} &= 0.43 \exp\left[-\left((KC-13.44)/6.47\right)^2\right] + 1734 \exp\left[-\left((KC+3907)/1460\right)^2\right] + 0.34 \exp\left[-\left((KC-14.05)/16.95\right)^2\right], \\
C_{L0} &= 1.39 \exp\left[-\left((KC-11.64)/6.48\right)^2\right] + 1.04 \exp\left[-\left((KC-17.44)/14.69\right)^2\right] + 7.04 \times 10^{13} \exp\left[-\left((KC+3196)/569.5\right)^2\right], \\
C_M &= 1.40 \exp\left[-\left((KC-5.65)/3.25\right)^2\right] + 0.71 \exp\left[-\left((KC-30.39)/27.60\right)^2\right] + 1.40 \exp\left[-\left((KC-87.28)/59.65\right)^2\right].
\end{aligned} \tag{20}$$

For a given $1.5 < f_r < 4$ range, the associated V_r can be specified and varied through $V_r = KC/f_r$ for a given $6 < KC < 80$ ($5535 < Re < 88560$) range, as shown in Fig. 15b. The minimum and maximum V_r is about 3.33 and 53.33, respectively. By assigning $m^*=1.62$, $\xi=0.012$, accounting for the StKC variation in Eqs. (12) and (14) with $St=0.2$, functions in above Eq (20) for a fixed $\beta=1107$, and keeping other empirical coefficients unchanged as before, the predicted response contours of A_x/D and A_y/D are presented in Fig. 15c and d, respectively. For in-line responses, the maximum A_x/D increases with increasing KC (Re) for a given f_r whereas it decreases with increasing f_r for a given KC since V_r is decreased. For cross-flow responses, the maximum A_y/D values strongly depend on f_r (V_r) and KC (Re) whose contour plots display a complex modulated response map with multi-peak features when varying either KC or f_r . The absolute maximum A_y/D (up to ≈ 2.5) occurs in the intermediate KC and high f_r (low V_r) regions. In comparison with results – which disregard the Re dependence of hydrodynamic coefficients in the prediction model – in Figs. 12 and 14 in which $308 < Re < 9240$ and $0 < V_r < 30$, the maximum $A_x/D \approx 7$ in the range of $V_r < 30$ occurs at $KC \approx 40$ and $f_r \approx 1.5$ ($V_r \approx 26.7$) in Fig. 15c, while the maximum $A_y/D \approx 2.5$ in the range of $V_r < 30$ occurs at $KC \approx 37$ and $f_r \approx 3.9$ ($V_r \approx 9.5$) in Fig. 15d. The associated greater $Re = 44280$ ($KC=40$) and 40959 ($KC=37$) values entail greater x - y responses in Fig. 15c and d when accounting for both Re and KC effects in the prediction model than those in Figs. 12a, 12b, 14a and 14b (for the same range of $10 < KC < 60$) when accounting for only the KC effect in the prediction model. Such trend of increasing responses with increasing Re has been observed in steady flow VIV cases, see, e.g., Blevins and Coughran (2009). Nevertheless, there is a need to justify such oscillatory flow VIV predictions with associated experimental tests or CFD simulations which are currently scarce for a low m^* system in a high Re flow regime.

5. Conclusions

A time-domain simulation model based on nonlinear wake-cylinder oscillators to predict combined cross-flow and in-line VIV of an elastically mounted circular cylinder in planar and oscillatory flow has been presented. This approach is interestingly new since wake oscillators have typically been applied to steady flow VIV predictions. Herein, the time-varying relative flow-cylinder velocities and accelerations have been accounted for in formulating the coupled hydrodynamic lift, drag and inertia forces with explicit time-dependent and hybrid trigonometric terms leading to two-dimensional cylinder oscillations excited by oscillatory flow. Depending on KC incorporating the flow maximum velocity and excitation frequency, a set of model empirical coefficients have been calibrated by tuning response amplitudes with the limited CFD simulation data in the literature for a low mass-damping system subject to specific $KC=10, 20$ and 40 , $0 < V_r < 30$ and $Re/V_r=308$. Dual wake oscillators with cylinder acceleration coupling terms are recommended. Empirical functions depending on KC and mass ratio have been introduced and applied to parametrically investigate the influence of system parameters on response prediction characteristics.

Three feasible model solutions have been investigated in cases of varying KC and V_r , including (i) the assumed and fixed number of lift force oscillations per flow cycle ($n = StKC = \text{constant}$) depending on the KC range, (ii) the assumed St and specified cylinder-to-flow frequency ratio f_r ($f_r = KC/V_r$) and V_r range, and (iii) the specified Stokes parameter β for given KC ($\beta = Re/KC$) and f_r ranges. In the latter case, hydrodynamic coefficients governing the drag, lift and inertia forces are varied as function of KC and Re based on experimental data of oscillatory flows past circular cylinders. These solution scenarios enable insights into a variety of predicted response boundaries with bi-parametric ($m^*-V_r, KC-V_r, f_r-V_r, f_r-KC$) contours of cross-flow and in-line amplitudes and their maximum/minimum values which may be useful for future comparisons and verifications by alternative numerical approaches and experiments. Overall, some qualitative features of oscillatory flow VIV have been captured by wake oscillators, including the increasing in-line response due primarily to the oscillatory flow frequency, the multi-peak cross-flow amplitude responses with fluctuating time histories due to multi-frequency components, the gradual shifts of cylinder oscillation-to-flow frequency ratios, and irregular motion trajectories due to the multi-directional flow velocities and multiple competing frequencies. The predicted dominant vibration frequencies in the in-line direction follow the sinusoidal flow excitation whereas those in the cross-flow direction are strongly dependent on KC , V_r , system frequency ratios, mass ratio and nonlinearities. A main limitation of the present wake oscillators lies in the inability to capture the repetitive x - y trajectories experimentally observed in some KC - V_r cases. This may call for a further development of three-dimensional and higher-order wake oscillators accounting for the cylinder spanwise dependence and the orbital flow motion effect.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRedit authorship contribution statement

Pierre-Adrien Opinel: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing - original draft, Visualization. **Narakorn Srinil:** Conceptualization, Methodology, Resources, Formal analysis, Writing - original draft, review & editing, Supervision.

Appendix: Prediction of 1-DOF VIV response in oscillatory flow

Based on the present model in Section 2 and the calibrated empirical coefficients in Section 3, the system of Eqs. (11) and (12) or the system of Eqs. (13) and (14) is now applied to predict a 1-DOF in-line or cross-flow VIV response, respectively, in comparison with some experimental results in the literature as shown in Fig. 16.

For in-line response, experimental data of Williamson (1985) are considered for which $m^* = 6.5$, $\xi = 0.02$ and $V_r/KC = 0.87$, and those of Bearman and Mackwood (1991) are also considered for which $m^* = 4.7$, $0.0007 \leq \xi \leq 0.06$ (this range was experimentally given), and $V_r/KC = 1/1.83 \approx 0.546$. The tuned values of $C_M = 1$, $C_{DM} = 2$, $C_a = 1$, $C_{D0} = 0.2$, $\beta_x = 0.7$, $\varepsilon_x = 0.3$, $A_x = 12$ and α_x in Eq. (18) are employed in the simulation model. In the case of varying KC , we assign $n=2$ for $KC \leq 15$, $n=3$ for $KC \leq 20$, $n=4$ for $KC \leq 25$, $n=5$ for $KC \leq 30$, $n=6$ for $KC \leq$

35, $n=7$ for $KC \leq 40$, $n=8$ for $KC \leq 45$, $n=9$ for $KC \leq 50$, $n=10$ for $KC \leq 55$, and $n=11$ for higher KC . Numerical and experimental results are compared in Fig. 16a for the maximum x amplitudes (A_x/D) and 16b for the root-mean-squared x amplitudes ($A_{x,rms}/D$), exhibiting a good agreement in terms of the response values and the monotonically increasing trend as KC is varied. The ζ value is seen to have a negligible effect on the predicted response in Fig. 16b. Hence, the set of in-line wake-structure oscillator coefficients from the 2-DOF VIV calibration may be applicable to the 1-DOF in-line VIV prediction.

For cross-flow response, experimental results of Sumer and Fredsoe (1988) are considered for which $m^* = 1.8$, $\zeta = 0.043$ and $KC = 20$. The tuned values of $C_M=1$, $C_{DM} = 2$, $C_a = 1$, $C_{L0} = 0.3$, $\alpha_y = 0.7$, $\varepsilon_y=0.00234e^{0.228m^*}$ and β_y in Eq. (19), together with $n = 4$, are employed in the simulation model. By considering $A_y = 38$ deduced from the 2-DOF VIV calibration, Fig. 16c shows large discrepancies between numerical and experimental results in 1-DOF cross-flow amplitudes (A_y/D) when varying V_r although the multi-peak feature is noticeable. Hence, the set of cross-flow wake-structure oscillator coefficients from the 2-DOF VIV calibration may not be directly applicable to the 1-DOF cross-flow VIV prediction. A further attempt has been carried out by tuning A_y governing the lift force oscillator to match the experimentally observed maximum responses and associated peaks at $V_r \approx 5.33$, 7.92, 11.85 and 12.36. Accordingly, by keeping other empirical values unchanged, it is found that $A_y=2$ for $V_r \leq 5$, $A_y=18$ for $5 < V_r \leq 5.5$, $A_y=2$ for $5.5 < V_r \leq 7$, $A_y=32$ for $7 < V_r \leq 8.5$, $A_y=16$ for $8.5 < V_r \leq 11$, $A_y=34$ for $11 < V_r \leq 13$ and $A_y=20$ for $13 < V_r \leq 16$, leading to an improved prediction as shown in Fig. 16d. This variation in empirical A_y values reflects a difficulty in tuning only one empirical variable (as others are fixed) and in applying a unique set of empirical variables to predict VIV in the whole considered V_r range. This is known as the key limiting capability of the wake oscillator. To overcome this challenge, a machine learning algorithm together with new experimental data and controlled parameters (m^* , ζ , KC , V_r , Re) may be implemented. This is subject to our future research.

References

- Anagnostopoulos, P., Iliadis, G., 1998. Numerical study of the flow pattern and the in-line response of a flexible cylinder in an oscillating stream. *Journal of Fluids and Structures* 12, 225-258.
- Bai, X., Qin, W., 2014. Using vortex strength wake oscillator in modelling of vortex induced vibrations in two degrees of freedom. *European Journal of Mechanics - B/Fluids* 48, 165-173.
- Bearman, P.W., Graham, J.M.R., Obasaju, E.D., 1984. A model equation for the transverse forces on cylinders in oscillatory flows. *Applied Ocean Research* 6, 166-172.
- Bearman, P.W., Mackwood, P.R., 1991. Non-linear vibration characteristics of a cylinder in an oscillating water flow, *The 5th Conference on Flow-Induced Vibrations*, pp. 21-31.
- Blevins, R.D., 1990. *Flow-Induced Vibration*. Van Nostrand Reinhold.
- Blevins, R.D., Coughran, C.S., 2009. Experimental investigation of vortex-induced vibration in one and two dimensions with variable mass, damping, and Reynolds number. *Journal of Fluids Engineering* 131.
- Facchinetti, M.L., De Langre, E., Biolley, F., 2004. Coupling of structure and wake oscillators in vortex-induced

- vibrations. *Journal of Fluids and structures* 19, 123-140.
- Fu, B., Zou, L., Wan, D., 2018. Numerical study of vortex-induced vibrations of a flexible cylinder in an oscillatory flow. *Journal of Fluids and Structures* 77, 170-181.
- Fu, S., Wang, J., Baarholm, R., Wu, J., Larsen, C.M., 2014. Features of vortex-induced vibration in oscillatory flow. *Journal of Offshore Mechanics and Arctic Engineering* 136, 011801.
- Gao, Y., Zou, L., Zong, Z., Takagi, S., Kang, Y., 2019. Numerical prediction of vortex-induced vibrations of a long flexible cylinder in uniform and linear shear flows using a wake oscillator model. *Ocean Engineering* 171, 157-171.
- Hartlen, R.T., Currie, I.G., 1970. Lift-oscillator model of vortex-induced vibration. *Journal of the Engineering Mechanics Division* 96, 577-591.
- Hayashi, K., 1984. *The Non-Linear Vortex-Excited Vibration of A Vertical Cylinder In Waves*. University of Liverpool.
- Justesen, P., 1991. A numerical study of oscillating flow around a circular cylinder. *Journal of Fluid Mechanics* 222, 157-196.
- Kozakiewicz, A., Sumer, B.M., Fredsøe, J., Hansen, E.A., 1997. Vortex regimes around a freely vibrating cylinder in oscillatory flow. *International Journal of Offshore and Polar Engineering* 7, 94-103.
- Lipsett, A.W., Williamson, I.D., 1994. Response of a cylinder in oscillatory flow. *Journal of Fluids and Structures* 8, 681-709.
- Low, Y.M., Srinil, N., 2016. VIV fatigue reliability analysis of marine risers with uncertainties in the wake oscillator model. *Engineering Structures* 106, 96-108.
- Morison, J.R., Johnson, J.W., Schaaf, S.A., 1950. The force exerted by surface waves on piles. *Journal of Petroleum Technology* 2, 149-154.
- Obasaju, E.D., Bearman, P.W., Graham, J.M.R., 1988. A study of forces, circulation and vortex patterns around a circular cylinder in oscillating flow. *Journal of Fluid Mechanics* 196, 467-494.
- Sarpkaya, T., 1976. *Vortex shedding and resistance in harmonic flow about smooth and rough circular at high Reynolds numbers*. Naval Postgraduate School, Monterey CA.
- Sarpkaya, T., 1979. Vortex-induced oscillations: a selective review. *Journal of Applied Mechanics* 46, 241-258.
- Sarpkaya, T., 2010. *Wave Forces on Offshore Structures*. Cambridge University Press.
- Sarpkaya, T., Isaacson, M., 1981. *Mechanics of Wave Forces on Offshore Structures*. Van Nostrand Reinhold Company.
- Srinil, N., 2010. Multi-mode interactions in vortex-induced vibrations of flexible curved/straight structures with geometric nonlinearities. *Journal of Fluids and Structures* 26, 1098-1122.
- Srinil, N., 2011. Analysis and prediction of vortex-induced vibrations of variable-tension vertical risers in linearly sheared currents. *Applied Ocean Research* 33, 41-53.
- Srinil, N., Opinel, P.-A., Tagliaferri, F., 2018. Empirical sensitivity of two-dimensional nonlinear wake-cylinder

oscillators in cross-flow/in-line vortex-induced vibrations. *Journal of Fluids and Structures* 83, 310-338.

Srinil, N., Wiercigroch, M., O'Brien, P., 2009. Reduced-order modelling of vortex-induced vibration of catenary riser. *Ocean Engineering* 36, 1404-1414.

Srinil, N., Zanganeh, H., 2012. Modelling of coupled cross-flow/in-line vortex-induced vibrations using double Duffing and van der Pol oscillators. *Ocean Engineering* 53, 83-97.

Sumer, B.M., Fredsoe, J., 1988. Transverse vibrations of an elastically mounted cylinder exposed to an oscillating flow. *Journal of Offshore Mechanics and Arctic Engineering* 110, 387-394.

Sumer, B.M., Fredsoe, J., 1989. Effect of Reynolds number on vibrations of cylinders. *Journal of Offshore Mechanics and Arctic Engineering* 111, 131-137.

Sumer, B.M., Fredsoe, J., 2006. *Hydrodynamics Around Cylindrical Structures*. World scientific.

Thorsen, M.J., Sævik, S., Larsen, C.M., 2016. Time domain simulation of vortex-induced vibrations in stationary and oscillating flows. *Journal of Fluids and Structures* 61, 1-19.

Ulveseter, J.V., Thorsen, M.J., Sævik, S., Larsen, C.M., 2018. Time domain simulation of riser VIV in current and irregular waves. *Marine Structures* 60, 241-260.

Wang, J., Fu, S., Baarholm, R., 2018. Evaluation of vortex-induced vibration of a steel catenary riser in steady current and vessel motion-induced oscillatory current. *Journal of Fluids and Structures* 82, 412-431.

Williamson, C.H.K., 1985. In-line response of a cylinder in oscillatory flow. *Applied Ocean Research* 7, 97-106.

Williamson, C.H.K., Govardhan, R., 2004. Vortex-induced vibrations. *Annual Review of Fluid Mechanics* 36, 413-455.

Williamson, I.D., 1991. *Response of Cylinders to Oscillatory Flow*, Department of Mechanical Engineering. University of Alberta, p. 108.

Xu, W.-H., Zeng, X.-H., Wu, Y.-X., 2008. High aspect ratio (L/D) riser VIV prediction using wake oscillator model. *Ocean Engineering* 35, 1769-1774.

Yang, W., Ai, Z., Zhang, X., Chang, X., Gou, R., 2018. Nonlinear dynamics of three-dimensional vortex-induced vibration prediction model for a flexible fluid-conveying pipe. *International Journal of Mechanical Sciences* 138-139, 99-109.

Zanganeh, H., Srinil, N., 2014. Characterization of variable hydrodynamic coefficients and maximum responses in two-dimensional vortex-induced vibrations with dual resonances. *Journal of Vibration and Acoustics* 136, 051010.

Zanganeh, H., Srinil, N., 2016. Three-dimensional VIV prediction model for a long flexible cylinder with axial dynamics and mean drag magnifications. *Journal of Fluids and Structures* 66, 127-146.

Zhao, M., 2013. Numerical investigation of two-degree-of-freedom vortex-induced vibration of a circular cylinder in oscillatory flow. *Journal of Fluids and Structures* 39, 41-59.

Highlights

- Nonlinear wake oscillators are applied to two-degree-of-freedom VIV in oscillatory flows.
- Empirical coefficients and functions are proposed through calibration and sensitivity studies.
- Effects of Keulegan-Carpenter, reduced velocity, frequency and mass ratios are investigated.
- Feasible numerical solutions and associated bi-parametric response contours are presented.
- Model dependence of hydrodynamic coefficients on the Reynolds number is discussed.

Figure 1. Schematic view of an elastically mounted circular cylinder in oscillatory flow, instantaneous direction (θ) of relative flow-cylinder velocities and hydrodynamic force components.

Figure 2. Empirical functions: (a) α_x (solid line) and β_y (dashed line) dependent on KC for a given $m^*=1.62$, and β_y (dotted-dashed line) dependent on m^* for a given $KC=10$; (b) β_y dependent on both KC and m^* .

Figure 3. Comparisons of A_x/D and A_y/D predicted by wake oscillators with varying n for different KC versus CFD results of Zhao (2013).

Figure 4. Comparisons of A_x/D and A_y/D predicted by wake oscillators for different KC versus CFD results of Zhao (2013): (a, b) cylinder linear vs. nonlinear stiffness models, (c-f) cylinder displacement, velocity vs. acceleration coupling terms in wake oscillators (Eqs. (12) and (14)).

Figure 5. Comparisons of A_x/D and A_y/D predicted by wake oscillators for different KC versus CFD results of Zhao (2013), with effects of varying (a, b) C_M , (c, d) C_{DM} and (e, f) C_{L0} .

Figure 6. Illustrative in-line and cross-flow response time histories predicted by wake oscillators and associated FFT spectra at $V_r = 7.5$ for different KC.

Figure 7. Illustrative in-line and cross-flow response phase portraits and x - y trajectories predicted by wake oscillators, associated with Fig. 6 at $V_r = 7.5$ for different KC.

Figure 8. In-line and cross-flow oscillation frequency spectra predicted by wake oscillators in case of varying V_r for different KC.

Figure 9. Extracted in-line (first three dominant) and cross-flow (primary) oscillation frequency components predicted by wake oscillators in the case of varying V_r for different KC, in comparison with CFD results (Zhao, 2013).

Figure 10. Dominant cross-flow oscillation frequencies normalized by natural frequency in comparison with CFD results of Zhao (2013) in the case of varying V_r for different KC. Reference lines of N integers are also plotted.

Figure 11. Contour plots of A_x/D and A_y/D as functions of V_r and m^* predicted by wake oscillators for a given $n = 2, 4$ and 8 for $KC = 10, 20$ and 40 , respectively.

Figure 12. Contour plots of A_x/D and A_y/D as functions of V_r and KC predicted by wake oscillators for a variable range of n and for different m^* cases.

m^*	f_r	Maximum cross-flow responses		
		V_r	KC	A_y/D
1.62	1.5	20.5	31	0.99
	2	18.5	37	0.99
	2.5	19.0	48	1.00
	3	16.5	50	1.02
	3.5	19.0	67	0.99
	4	19.5	78	1.02
4	1.5	9.5	14	0.66
	2	8.0	16	0.66
	2.5	7.5	19	0.64
	3	7.0	21	0.67
	3.5	6.5	23	0.68
	4	6.5	26	0.70

Figure 13. Variations of KC as functions of V_r and f_r , together with a summary of predicted maximum cross-flow responses, associated V_r and KC, extracted from Fig. 14 for different m^* cases.

Figure 14. Contour plots of A_x/D and A_y/D as functions of f_r and V_r predicted by wake oscillators in the case of varying KC as in Fig. 13 for different m^* .

Figure 15. (a) Hydrodynamic coefficient data (symbols) and associated fitting curves (solid lines) as function of KC for a given $\beta = \text{Re}/\text{KC} = 1107$, (b) variations of V_r as function of f_r and KC, and associated contour plots of (c) A_x/D and (d) A_y/D predicted by wake oscillators for $m^* = 1.62$.

Figure 16. Comparisons of 1-DOF (a, b) in-line and (c, d) cross-flow VIV predictions by wake-oscillators versus experimental results in the literature. Results in (a)-(c) are based on empirical coefficients deduced from 2-DOF VIV model whereas results in (d) are based on new best-fit A_y values for 1-DOF VIV model.